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USE OF DISCONTINUITY FUNCTIONS TO OBTAIN THE FREQUENCY EQUATION

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Dedicated to Professor József FARKAS on the occasion of his seventy-fifth birthday

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Abstract. An integration technique based on the use of discontinuous functions has been applied to obtain the natural frequencies of free flexural vibrations in beams. The two examples presented show the logical basis of the method in a detailed form.

Mathematical Subject Classification: 74K10
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1. Introduction

In this paper discontinuity functions are applied to derive the frequency equation for the flexural vibration of light beams. The designations and the definitions of the discontinuity functions are borrowed from the textbook by Gere, J. M. and Timoshenko, S. P. [1]. We will use mainly Table 7-2 of the aforementioned textbook. Two examples illustrate how to derive the frequency equation in the form of a determinant. The sign rules we applied are shown in Figure 1. In the state of free flexural vibration all quantities vary with time in the following form

$$\widetilde{X} = X \sin \omega t \qquad X = v, \varphi, V, M, R, ... \tag{1.1}$$

The factor independent of time in equation (1.1) is referred to as the amplitude of the quantity X.

2. Examples

2.1. **The first problem is that of a light beam with uniform cross-section.** The beam is clamped at its two ends. A mass m is attached to point B and a torsional spring is fixed to point C. In the present problem the Young modulus E of the beam is constant.

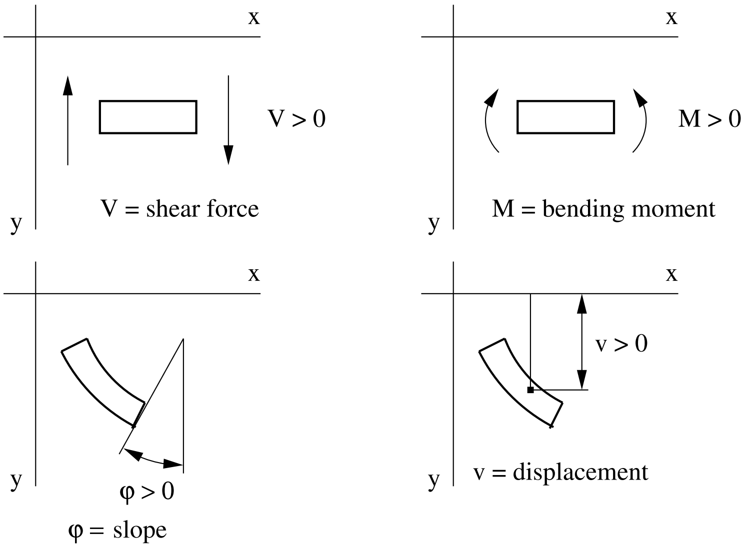

Figure 1. Sign rules for shear, bending, shape and deflection

The light beam with mass and spring is shown in Figure 2, and the free-body diagram of the beam segment AD is given in Figure 3.

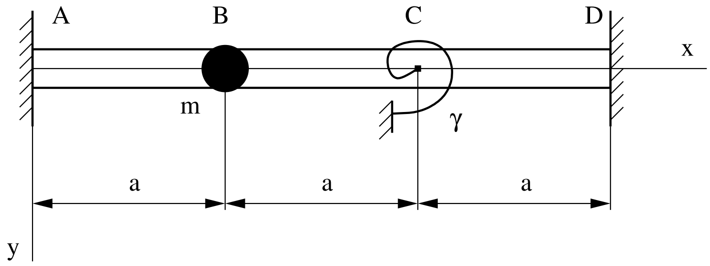

Figure 2. Fixed beam

We shall apply the following designations:

R_A, R_B	amplitudes of the reactions,
M_A, M_B	amplitudes of reaction couples,
$T_B = mv_B\omega^2$	amplitude of the inertia force,
v	displacement amplitude,
ω	eigenfrequency of the free vibrations,
$Q_C = -\varphi_C/\gamma$	amplitude of couple at the torsional spring,
γ_C	amplitude of slope,
γ	spring constant.

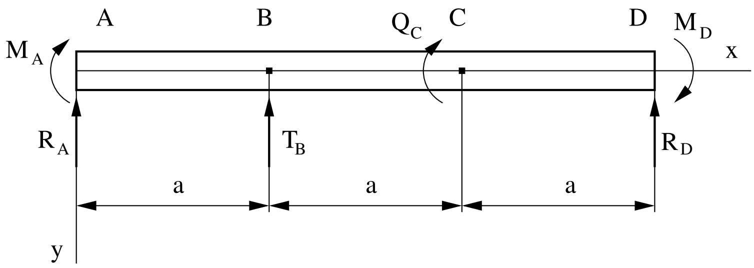

Figure 3. Free-body diagram of the beam segment AD

By using the method of **Clebsch-Macaulay** we obtain the exact expressions for the shear force V and the bending moment M:

$$V = R_A + M_A\langle x\rangle^{-1} + T_B\langle x-a\rangle^0 + Q_C\langle x-2a\rangle^{-1} \quad , \tag{2.1}$$

$$M = R_A x + M_A + T_B\langle x-a\rangle^1 + Q_C\langle x-2a\rangle^0 \quad . \tag{2.2}$$

It is well-known [1] that the bending moment satisfies the equation

$$IEv'' = -M \quad . \tag{2.3}$$

Here prime denotes the derivation with respect to x, i.e., $v'' = \dfrac{d^2v}{dx^2}$.

Combination of equation (2.2) with equation (2.3) yields the slope $\varphi = v'$ and the deflection v:

$$IE\varphi = -\left(R_A\frac{x^2}{2} + M_A x + T_B\frac{\langle x-a\rangle^2}{2} + Q_C\langle x-2a\rangle^1\right) + C \quad , \tag{2.4a}$$

$$IEv = -\left(R_A\frac{x^3}{6} + M_A\frac{x^2}{2} + T_B\frac{\langle x-a\rangle^3}{6} + Q_C\frac{\langle x-2a\rangle^2}{2}\right) + Cx + D \quad . \tag{2.4b}$$

where C and D are constants of integration. From the boundary conditions

$$v(0) = 0\,, \qquad v'(0) = 0\,, \tag{2.5a,b}$$

we get

$$C = 0\,, \qquad D = 0\,. \tag{2.6a,b}$$

The boundary conditions $v(3a) = 0$ and $v'(3a) = 0$ at point D lead to the following equations:

$$4.5a^2R_A + 3aM_A + 2a^2T_B + Q_C a = 0 \quad , \tag{2.7a}$$

$$4.5a^3R_A + 4.5a^2M_A + 1.3333a^3T_B + 0.5a^2Q_C = 0 \quad . \tag{2.7b}$$

In the state of free vibration the relationship between the amplitude of inertia force and the amplitude of displacement at point B is

$$v_B = \frac{T_B}{m\omega^2} \quad . \tag{2.8}$$

From the definition of spring constant γ it follows that

$$\gamma_C = v' = -Q_C\gamma \quad . \tag{2.9}$$

By applying the expressions for the slope and deflection – these are given by equations (2.4a,b) – we can eliminate both v_B and γ_C from equations (2.8), (2.9). These eliminations give

$$0.16666a^3R_A + 0.5a^2M_A + \frac{IE}{m\omega^2}T_B = 0\ , \tag{2.10a}$$

$$2a^2R_A + 2aM_A + 0.5a^2T_B - IE\gamma Q_C = 0\ . \tag{2.10b}$$

We have four equations for the four unknown quantities R_A, M_A, T_B, Q_C. Equations (2.7a,b) and (2.10a,b) form a system of linear equations for the unknown amplitudes R_A, M_A, T_B, Q_C. There exists a non-trivial solution for the system of equations (2.7a,b) and (2.10a,b) if the frequency determinant vanishes:

$$\begin{vmatrix} 4.5a^2 & 3a & 2a^2 & a \\ 4.5a^3 & 4.5a^2 & 1.3333a^3 & 0.5a^2 \\ 0.1666a^2 & 0.5a^2 & IE/m\omega^2 & 0 \\ 2a^2 & 2a & 0.5a^2 & -IE\gamma \end{vmatrix} = 0 \quad . \tag{2.11}$$

2.2. **The second example is that of a simply supported beam with non-uniform cross-section.** In this case the material of the beam is also homogeneous, that is the Young modulus E is constant. The beam with a rigid disc and springs is shown in Figure 4. The spring constants are c and γ and the mass of the disc is m. The second moment of the disc with respect to centroidal axis b is J. The main centroidal axis b passes through point B and is perpendicular to the plane xy.

If the beam vibrates freely, it is loaded by a force and a couple at point B. This force-couple system arises from the inertia effects and the action of springs. The amplitude of the resultant force at point B is

$$T_B = \left(m\omega^2 - \frac{1}{C}\right)v_B\ . \tag{2.12a}$$

The amplitude of the resultant couple at the same point is

$$Q_B = \left(J\omega^2 - \frac{1}{\gamma}\right)\varphi_B\ . \tag{2.12b}$$

The free-body diagram of the beam AD is shown in Figure 5.

The shear force V and the bending moment M are given by the following formulae:

$$V = R_A + T_B\langle x-a\rangle^0 + Q_B\langle x-a\rangle^{-1}\ , \tag{2.13}$$

$$M = R_Ax + T_B\langle x-a\rangle^1 + Q_B\langle x-a\rangle^0\ . \tag{2.14}$$

The boundary condition for the bending moment M at point D yields

$$M(4a) = 4aR_A + 3aT_B + Q_B = 0\ . \tag{2.15}$$

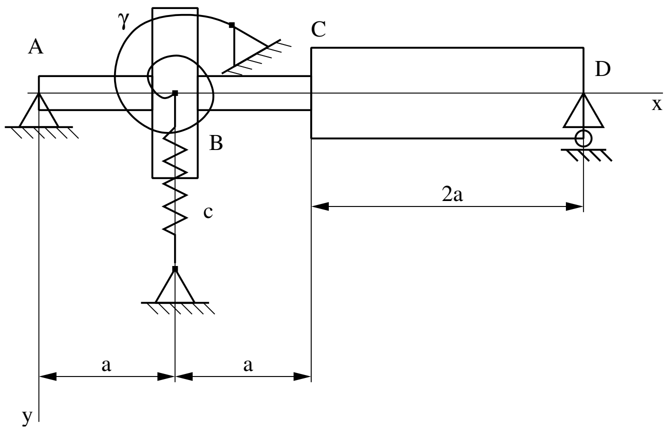

Figure 4. Simply supported beam of non-uniform cross-section

The flexural rigidity is given by the equations

$$EI = EI_0 \ , \qquad 0 \leq x < 2a \ , \tag{2.16a}$$

$$EI = 2EI_0 \ , \qquad 2a < x \leq 4a \ , \tag{2.16b}$$

where E and I_0 are constants.

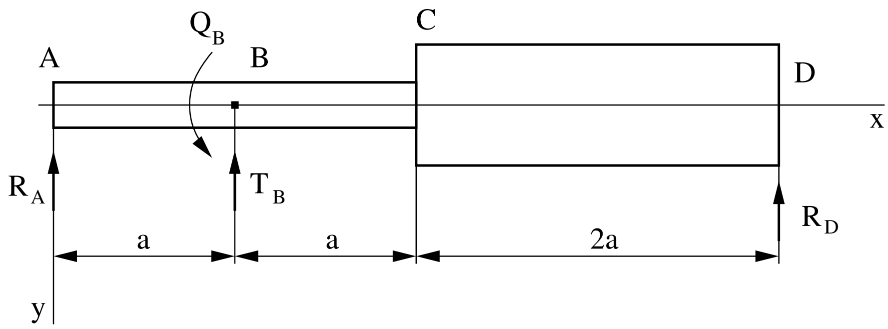

Figure 5. Free-body diagram for the non-uniform beam segment AD

Integrating equation (2.3) we obtain

$$-I_0Ev' = R_A\frac{x^2}{2} + T_B\frac{\langle x-a\rangle^2}{2} + Q_B\langle x-a\rangle^1 + C_1 \qquad 0 \leq x < 2a \ , \tag{2.17a}$$

$$-2I_0Ev' = R_A\frac{x^2}{2} + T_B\frac{\langle x-a\rangle^2}{2} + Q_B\langle x-a\rangle^1 + C_2 \qquad 2a < x \leq 4a \ . \tag{2.17b}$$

After a new integration we arrive at the deflection:

$$-I_0Ev = R_A\frac{x^3}{6} + T_B\frac{\langle x-a\rangle^3}{6} + Q_B\frac{\langle x-a\rangle^2}{2} + C_1x + D_1 \qquad 0 \le x < 2a\ , \quad (2.18a)$$

$$-2I_0Ev = R_A\frac{x^3}{6} + T_B\frac{\langle x-a\rangle^3}{6} + Q_B\frac{\langle x-a\rangle^3}{6} + C_2x + D_2 \qquad 0 \le x < 2a\ . \quad (2.18b)$$

By using the displacement boundary condition $v(0) = 0$ we get

$$D_1 = 0 \quad . \qquad (2.19)$$

In the present case we have six unknowns, namely $R_A, T_B, Q_B, C_1, C_2, D_2$. The equations which we have to use to determine the above mentioned quantities are as follows:

$$M(4a) = 0 \qquad , \qquad v(4a) = 0 \quad , \qquad (2.20a,b)$$

$$v'(2a-\varepsilon) = v'(2a+\varepsilon) \qquad \varepsilon \longrightarrow 0 \quad , \qquad (2.20c)$$

$$v(2a-\varepsilon) = v(2a+\varepsilon) \qquad \varepsilon \longrightarrow 0 \quad , \qquad (2.20d)$$

$$v(a) = \frac{T_B}{m\omega^2 - \frac{1}{c}} \qquad , \qquad v'(a) = \frac{Q_B}{J\omega^2 - \frac{1}{\gamma}} \quad . \qquad (2.20e,f)$$

Equations (2.20c,d) are the joint conditions for the solutions which determine the deflection and the slope in the intervals $0 \le x < 2a$ and $2a < x \le 4a$. These solutions can be obtained from the formulae (2.17a,b) and (2.18a,b). The preceding equations form a system of linear equations for the six unknowns $R_A, T_B, Q_B, C_1, C_2, D_2$. From the condition of the existence of a nontrivial solution we get the frequency equation in the form of a determinant:

$$\begin{vmatrix} 4a^2 & 3a & 1 & 0 & 0 & 0 \\ 0.5a^2 & 0 & I_0E/(J\omega^2 - \frac{1}{\gamma}) & 1 & 0 & 0 \\ 0.16666a^3 & I_0E/(m\omega^2 - \frac{1}{c}) & 0 & a & 0 & 0 \\ 10.666a^3 & 4.5a^3 & 4.5a^2 & 0 & 4a & 1 \\ 2a^2 & 0.5a^2 & a & 2 & -1 & 0 \\ 1.3333a^3 & 0.16666a^3 & 0.5a^2 & 2a & -a & 1 \end{vmatrix} = 0 \quad . \qquad (2.21)$$

3. Conclusion

An integration technique for the discontinuous expressions has been applied to obtain the frequency equation of the vibrations in beams. The two examples presented show the logical basis of the method and illustrate well the scheme to be followed in order to get the frequency equation for flexural vibrations. It is shown that this method can also be used if discontinuities arise in the expression of the flexural rigidity EI. The use of **Macaulay's** brackets in an analysis of the beam problems results in a unified method which has a pedagogical value in teaching the elementary theory of beams. As regards the **Macaulay-Clebsch** method, a number of applications can be found in Wittrick's paper [2].

More advanced mathematical methods based on the distribution theory of Schwarz are applied to solve the static bending problems of beams with material, geometric and loading discontinuities in the papers by Reddy, Yavari, Sarkani [3, 4, 5].

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