Journal of Computational and Applied Mechanics, Vol. 4., No. 2., (2003), pp. 119–127

ON SOME RELATIONSHIPS OF SPHERICAL KINEMATICS

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Dedicated to the memory of Professor István Sályi
(October 15, 1924 - October 17, 2001)

[Received: June 19, 2002]

Abstract. In this paper, some relations are developed for the spherical motion of a rigid body. Results are formulated in four theorems which describe a few geometrical properties of spherical motion using the geometrical data of the fixed and moving axode cones. An example illustrates the application of the formulae derived.

Mathematical Subject Classification: 70B10
Keywords: rigid body, spherical motion, curvature, axode, fixed and moving cones

1. **Introduction**

The term spherical motion is used when the rigid body moves around a fixed point. The spherical motion is equivalent to the moving axode cone C_m rolling without slipping over the fixed axode cone C_f. The instanteneous axis of rotation is the line of contact between these cones. The common apexes of the axode cones C_m and C_f is the fix point O [2, 3, 6].

The intersection of the axode cones C_m and C_f with the sphere whose center is point O and the radius is R are the moving polode c_m and the fixed polode c_f, respectively [2, 3, 6]. The common point of the curves c_f and c_m is denoted by P, and the common tangent unit vector of curves c_m and c_f at point P is indicated by $\mathbf{t}$. The angular velocity vector $\boldsymbol{\omega} = \omega\mathbf{e}$ describes the instantaneous motion of the rigid body considered. Here, $\mathbf{e} = \overrightarrow{OP}/R$ – see Figure 1.

The contact point P of the curves c_m and c_f moves along the curve c_f in the frame of the fixed polod cone C_f. This motion has the velocity

$$\mathbf{u}_f = u_p\left(s_f\right)\mathbf{t}\left(s_f\right) \qquad s_f = s_f\left(\tau\right), \tag{1.1}$$

and acceleration

$$\mathbf{w}_f = \dot{u}_p\mathbf{t} + \Gamma_f u_p^2\mathbf{n}_f. \tag{1.2}$$

The contact point P of the curves c_m and c_f moves along the curve c_m in the frame of the rigid body considered. The moving polode cone C_m is attached to the moving rigid body. The motion of point P on the curve c_m is characterized by its velocity and acceleration, which are as follows

$$\mathbf{u}_m = u_p\left(s_m\right)\mathbf{t}\left(s_m\right) \qquad s_m = s_m\left(\tau\right), \tag{1.3}$$

$$\mathbf{w}_m = \dot{u}_p\mathbf{t} + \Gamma_m u_p^2 \mathbf{n}_m\,. \tag{1.4}$$

Here,

- τ denotes the time;
- s_m and s_f are arc coordinates defined on the moving and fixed polode c_m and c_f, respectively;
- Γ_m and Γ_f are the curvatures at point P of curves c_m and c_f, respectively;
- $\mathbf{n}_m$ and $\mathbf{n}_f$ are the principal normal vectors at point P to the curves c_m and c_f, respectively;
- over dot denotes derivation with respect to time.

The consequence of pure rolling is that

$$\dot{s}_f = \dot{s}_m = u_p, \qquad \ddot{s}_f = \ddot{s}_m = \dot{u}_p\,. \tag{1.5}$$

Starting from the equation

$$\boldsymbol{\omega} = \omega\mathbf{e} = \frac{\omega}{R}\overrightarrow{OP}$$

and using the definition of angular acceleration [1, 5]

$$\boldsymbol{\varepsilon} = \left(\frac{d\boldsymbol{\omega}}{d\tau}\right)_f,$$

where the symbol $\left(\frac{d}{d\tau}\right)_f$ denotes the time derivative computed in the fixed frame we get for the body angular acceleration the formula

$$\boldsymbol{\varepsilon} = \dot{\omega}\mathbf{e} + \frac{\omega u_p}{R}\mathbf{t}\,. \tag{1.6}$$

The fixed frame is attached to the base (fixed polode cone) and the moving frame is attached to the moving rigid body (moving polode cone). Point A of the moving rigid body instantaneously coincides with point P. Using the fundamental relationships of relative motion in connection with point P we can write

$$\mathbf{u}_f = \mathbf{v}_A + \mathbf{u}_m, \tag{1.7}$$

$$\mathbf{w}_f = \mathbf{a}_A + 2\boldsymbol{\omega} \times \mathbf{u}_m + \mathbf{w}_m. \tag{1.8}$$

Here, $\mathbf{v}_A$ is the velocity of point A and $\mathbf{a}_A$ is the acceleration of point A. $\mathbf{u}_f, \mathbf{w}_f, \mathbf{a}_A, \boldsymbol{\omega}$ are taken in the fixed frame and $\mathbf{u}_m, \mathbf{w}_m$ are regarded in the moving frame in equation 1.8), the vectorial product of two vectors is denoted by cross.

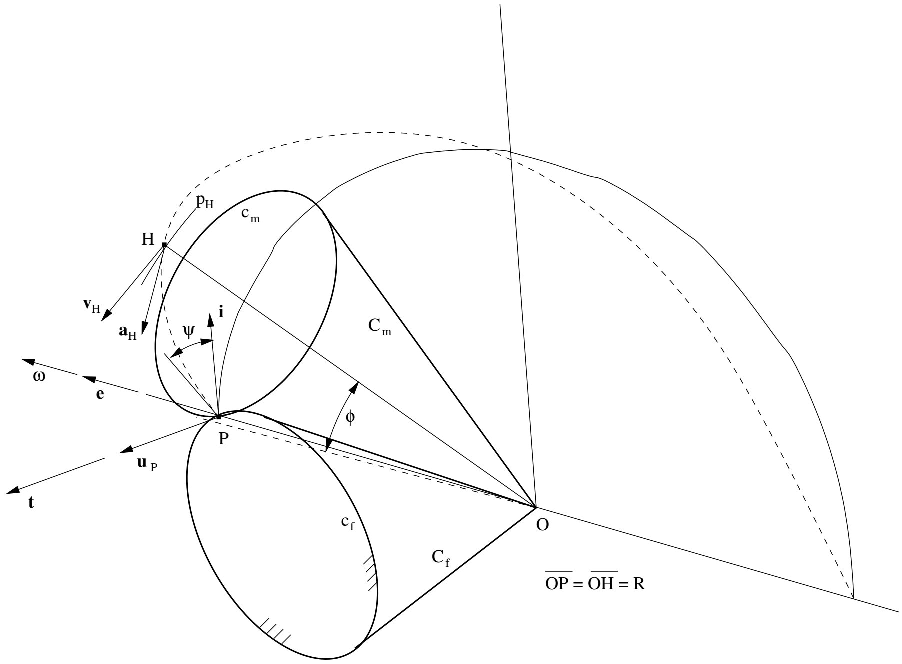

Figure 1. Moving and fixed axode cones

From pure rolling it follows that $\mathbf{v}_A = \mathbf{0}$, $\mathbf{u}_m = \mathbf{u}_f$. The latter statement was mentioned in equation (1.5). The combination of equations (1.2), (1.4), (1.5) with equation (1.8) gives the result

$$\mathbf{a}_A = 2\mathbf{u}_p \times \boldsymbol{\omega} + u_p^2(\Gamma_f \mathbf{n}_f - \Gamma_m \mathbf{n}_m), \tag{1.9}$$

where $\mathbf{u}_p = \mathbf{u}_m = \mathbf{u}_f$ is the velocity of the instantaneous contact point P.

The common tangential unit vector $\mathbf{t}$ of curves c_m and c_f is attached to point P. We know that its time derivative in a fixed frame can be expressed as [1, 3]

$$\left(\frac{d\mathbf{t}}{d\tau}\right)_f = \left(\frac{d\mathbf{t}}{d\tau}\right)_m + \boldsymbol{\omega} \times \mathbf{t}, \tag{1.10}$$

where the symbol $\left(\frac{d}{d\tau}\right)_m$ denotes the time derivative computed in the moving frame. A simple calculation shows that

$$\left(\frac{d\mathbf{t}}{d\tau}\right)_f = \frac{d\mathbf{t}}{ds_f}\dot{s}_f = u_p \Gamma_f \mathbf{n}_f, \tag{1.11}$$

$$\left(\frac{d\mathbf{t}}{d\tau}\right)_m = \frac{d\mathbf{t}}{ds_m}\dot{s}_m = u_p \Gamma_m \mathbf{n}_m. \tag{1.12}$$

Inserting these results into equation (1.10) we obtain

$$\mathbf{t} \times \boldsymbol{\omega} = u_p(\Gamma_m \mathbf{n}_m - \Gamma_f \mathbf{n}_f). \tag{1.13}$$

Using the trivial identity

$$u_p\mathbf{t} \times \boldsymbol{\omega} = \mathbf{u}_p \times \boldsymbol{\omega} = u_p^2(\Gamma_m \mathbf{n}_m - \Gamma_f \mathbf{n}_f)$$

and equations (1.9) and (1.13) we get

$$\mathbf{a}_A = \mathbf{u}_p \times \boldsymbol{\omega}\,. \tag{1.14}$$

Equation (1.14) is in harmony with the result of example 4.8 of the the book [5] by Ginsberg.

2. **Some useful relations**

Theorem 1. The angular velocity of the moving rigid body is determined by the geometry of the polode curves and the speed of the contact point according to the equation

$$\boldsymbol{\omega} = u_p\left(\Gamma_f \mathbf{b}_f - \Gamma_m \mathbf{b}_m\right), \tag{2.1}$$

where $\mathbf{b}_f$ and $\mathbf{b}_m$ are the binormal unit vectors at point P to the curves c_f and c_m.

Proof. The proof of equation (2.1) follows from the equations

$$\boldsymbol{\omega} \cdot \mathbf{t} = \frac{\omega}{R}\mathbf{R} \cdot \frac{d\mathbf{R}}{ds_f} = \frac{\omega}{R}\frac{d}{ds_f}\left(\frac{R^2}{2}\right) = 0,$$

$$\mathbf{t} \times (\boldsymbol{\omega} \times \mathbf{t}) = \omega \mathbf{t}^2 - \mathbf{t}(\boldsymbol{\omega} \cdot \mathbf{t}) = \omega,$$

and the definition of the binormal vector

$$\mathbf{b}_f = \mathbf{t} \times \mathbf{n}_f, \qquad \mathbf{b}_m = \mathbf{t} \times \mathbf{n}_m$$

and the validity of equation (1.13). Here, the dot between two vectors denotes their scalar product.

Theorem 2. The angular acceleration of the moving rigid body is determined by the geometry of the polode curves, the speed of the contact point and the rate of change of the speed of the contact point according to the equation

$$\boldsymbol{\varepsilon} = \dot{u}_p\left(\Gamma_f \mathbf{b}_f - \Gamma_m \mathbf{b}_m\right) + u_p^2\left(\frac{d\Gamma_f}{ds_f}\mathbf{b}_f - \frac{d\Gamma_m}{ds_m}\mathbf{b}_m - \Gamma_f T_f \mathbf{n}_f + \Gamma_m T_m \mathbf{n}_m\right) + \frac{\omega u_p}{R}\mathbf{t}. \tag{2.2}$$

Here, T_f and T_m are the torsions of curves c_f and c_m at point P, respectively.

Proof. Starting from the expression of $\boldsymbol{\omega}$ given by equation (2.1) we obtain

$$\boldsymbol{\varepsilon} = \left(\frac{d\boldsymbol{\omega}}{d\tau}\right)_f = \dot{u}_p\left(\Gamma_f \mathbf{b}_f - \Gamma_m \mathbf{b}_m\right) + u_p^2\left(\frac{d\Gamma_f}{ds_f}\mathbf{b}_f + \Gamma_f\frac{d\mathbf{b}_f}{ds_f} - \frac{d\Gamma_m}{ds_m}\mathbf{b}_m\right) - u_p\Gamma_m\left(\frac{d\mathbf{b}_m}{d\tau}\right)_f. \tag{2.3}$$

Making use of the fundamental equation of relative motion [1, 5]

$$\left(\frac{d\mathbf{b}_m}{d\tau}\right)_f = \left(\frac{d\mathbf{b}_m}{d\tau}\right)_m + \boldsymbol{\omega} \times \mathbf{b}_m, \tag{2.4}$$

and the equations

$$\left(\frac{d\mathbf{b}_m}{d\tau}\right)_m = \left(\frac{d\mathbf{b}_m}{ds_m}\right)\dot{s}_m = \frac{d\mathbf{b}_m}{ds_m}u_p, \tag{2.5}$$

$$\frac{d\mathbf{b}_f}{ds_f} = -T_f \mathbf{n}_f, \qquad \frac{d\mathbf{b}_m}{ds} = -T_m \mathbf{n}_m, \tag{2.6}$$

$$u_p \Gamma_m \boldsymbol{\omega} \times \mathbf{b}_m = -\frac{\omega}{R} \mathbf{u}_p \tag{2.7}$$

we get the proof of formula (2.3). The validity of equation (2.7) can be proved as

$$0 = \frac{d}{ds_m} (\mathbf{R} \cdot \mathbf{t}) = \frac{d\mathbf{R}}{ds_m} \cdot \mathbf{t} + \mathbf{R} \cdot \frac{d\mathbf{t}}{ds_m} = \tag{2.8}$$
$$= 1 + R\Gamma_m \mathbf{e} \cdot \mathbf{n}_m \qquad \text{that is} \qquad \mathbf{e} \cdot \mathbf{n}_m = -\frac{1}{R\Gamma_m}$$

and, on the other hand, we have

$$u_p \Gamma_m \boldsymbol{\omega} \times \mathbf{b}_m = u_p \Gamma_m \omega \mathbf{e} \times (\mathbf{t} \times \mathbf{n}_m) = u_p \Gamma_m \omega \left[\mathbf{t} \left(\mathbf{e} \cdot \mathbf{n}_m \right) - \mathbf{n}_m \left(\mathbf{e} \cdot \mathbf{t} \right) \right] = \tag{2.9}$$
$$= \mathbf{u}_p \Gamma_m \omega \left(\mathbf{e} \cdot \mathbf{n}_m \right).$$

The combination of equation (2.8) with equation (2.9) yields equation (2.7).

Consider point H of the moving rigid body. Let the position vector of point H be

$$\overrightarrow{OH} = R \cos\Phi \mathbf{e} + R \sin\Phi \cos\psi \mathbf{i} + R \sin\Phi \sin\psi \mathbf{t}, \qquad \mathbf{i} = \mathbf{t} \times \mathbf{e}. \tag{2.10}$$

The path of point H in the fixed reference frame lies on a sphere whose radius is R and whose center is point O – Figure 1. In (2.10), Φ is the angle between the lines OP and OH. Let $[\mathbf{t}; OP]$ be the plane whose normal vector is $\mathbf{t}$ and which contains the line OP. The plane $[OPH]$ is determined by points O, P and H. The angle formed by the planes $[\mathbf{t}; OP]$ and $[OPH]$ was denoted by ψ in equation (2.10).

The following theorems describe the relations of the geometrical properties of the path of point H with the moving and fixed polodes c_m and c_f.

Theorem 3. Let c_H be the path curve of point H. Assuming that the instantaneous axis is the line OP, then the following equations hold

$$\mathbf{t}_H = -\mathbf{i} \sin\psi + \mathbf{t} \cos\psi, \tag{2.11}$$

$$\mathbf{N}_H = -\mathbf{e} + \mathbf{i} \left(\cot\Phi \cos\psi - \frac{R}{d_p} \right) + \mathbf{t} \left(\cot\Phi \sin\psi - \frac{R}{d_p} \tan\psi \right), \tag{2.12}$$

$$\mathbf{B}_H = -\mathbf{e} \left(\cot\Phi - \frac{R}{d_p} \frac{1}{\cos\psi} \right) - \mathbf{i} \cos\psi - \mathbf{t} \sin\psi, \tag{2.13}$$

$$\left(\frac{\cos\psi}{\sin^2\Phi} \frac{d_p}{R} - \cot\Phi \right)^2 + 1 = (R\Gamma_H)^2, \tag{2.14}$$

where $\mathbf{t}_H$, $\mathbf{N}_H$ and $\mathbf{B}_H$ are the tangential, principal and binormal vectors to the curve c_H at point H, respectively. $\mathbf{t}_H$ is a unit vector, $\mathbf{N}_H$ and $\mathbf{B}_H$ are not unit vectors. Furthermore, d_p is defined as

$$d_p = \frac{1}{\Gamma_f \cos\alpha_f - \Gamma_m \cos\alpha_m}, \tag{2.15}$$

$$\cos\alpha_f = \mathbf{b}_f \cdot \mathbf{e}, \qquad \cos\alpha_m = \mathbf{b}_m \cdot \mathbf{e} \tag{2.16}$$

and Γ_H denotes the curvature of c_H at point H.

Proof. The proof of equations (2.11), (2.12), (2.13), (2.14) is based on the definition of $\mathbf{B}_H$, which is

$$\mathbf{B}_H = \mathbf{t}_H \times \mathbf{N}_H,$$

and the following kinematical equations of a particle and a rigid body [1, 5]

$$\mathbf{t}_H = \frac{\mathbf{V}_H}{|\mathbf{V}_H|} = \frac{\boldsymbol{\omega} \times \overrightarrow{OH}}{\left|\boldsymbol{\omega} \times \overrightarrow{OH}\right|},$$

$$\mathbf{a}_H = \boldsymbol{\omega} \times \mathbf{V}_H + \epsilon \times \overrightarrow{OH},$$

$$\mathbf{a}_H = \frac{\mathbf{a}_H \cdot \mathbf{V}_H}{\mathbf{V}_H^2}\mathbf{V}_H + \Gamma_H \mathbf{V}_H^2 \mathbf{n}_H,$$

$$\mathbf{n}_H = \frac{\mathbf{N}_H}{|\mathbf{N}_H|},$$

and the equation

$$d_p = \frac{u_p}{\omega}. \tag{2.17}$$

The validity of equation (2.17) follows from equation (2.1).

Theorem 4. Let T_H be the torsion of curve c_H at point H, and let s be an arc coordinate defined on curve c_H. We have

$$|T_H| = \frac{\left|\left\{\frac{d}{ds}\ell n \frac{\Gamma}{\Gamma_H}\right\}\right|_{s=s_H}}{\left|\frac{\cos\Psi}{\sin^2\Phi}\frac{d_p}{R} - \cot\Phi\right|} \qquad (\Phi \neq 0, \pi)\,. \tag{2.18}$$

Here, $\Gamma = \Gamma(s)$ is the curvature at an arbitrary point of c_H and the position of point H on c_H is given by s_H.

Proof. Using the concept of the osculating sphere in connection with the spherical curve c_H we can write [4, 7, 8]:

$$\left(\frac{1}{\Gamma_H}\right)^2 + \left(\frac{d}{ds}\left(\frac{1}{\Gamma}\right)_{s=s_H}\frac{1}{T_H}\right)^2 = R^2. \tag{2.19}$$

The combination of formula (2.14) with equation (2.19) leads to formula (2.18).

3. Remark on the computation of d_p

This section concentrates on the computation of $\cos\alpha_f$ and $\cos\alpha_m$, which appear in formula (2.15).

Let us consider an arbitrary curve c on the sphere whose radius and center are R and O, respectively. Let $\overrightarrow{OQ} = \varrho = \varrho(s)$ be the equation of curve c, where Q is an arbitrary point of c and s is an arc coordinate defined on c. A repeated differentiation of the equation

$$\varrho^2 = R^2 = \text{constant} \tag{3.1}$$

with respect to s gives

$$\Gamma\varrho \cdot \mathbf{n} + 1 = 0, \tag{3.2}$$

where Γ is the curvature of c at point Q and $\mathbf{n}$ is the principal normal vector of c at point P. Equation (3.2) can be obtained from Meusnier's theorem as well [4, 7, 8].

The application of equation (3.2) to curve c_f at point P yields

$$\mathbf{n}_f \cdot \mathbf{e} = -\frac{1}{R\Gamma_f}. \tag{3.3}$$

The angle formed by the vectors $\mathbf{e}$ and $\mathbf{n}_f$ is denoted by β_f. It is obvious that the angle between the vectors $\mathbf{e}$ and $\mathbf{b}_f$ is

$$\alpha_f = \beta_f \pm \frac{\pi}{2}. \tag{3.4}$$

From equations (3.3) and (3.4) we get

$$\cos\alpha_f = \operatorname{sgn}(\mathbf{b}_f \cdot \mathbf{i}) \sqrt{1 - \frac{1}{(R\Gamma_f)^2}}. \tag{3.5}$$

A similar formula can be derived to obtain the value of $\cos\alpha_m$:

$$\cos\alpha_m = \operatorname{sgn}(\mathbf{b}_m \cdot \mathbf{i}) \sqrt{1 - \frac{1}{(R\Gamma_m)^2}}. \tag{3.6}$$

Here, we remark that the analogue pair of equation (3.6) for the curve c_m was derived in Section 2 (equation (2.8)).

4. **Example**

Figure 2 illustrates a rigid body's circular cone OPH. The point O is fixed and the cone rolls without slipping on the horizontal plane $[\mathbf{i}; OP]$ whose normal vector is $\mathbf{i}$. The fixed axode cone is the "plane $[\mathbf{i}; OP]$" (degenerate cone) and the moving axode cone is the circular cone OPH. Q is the center point of the base circle of the cone OPH. This base circle can be considered as a moving polode curve c_m. The fixed polode curve c_f is a circle in the plane $[\mathbf{i}; OP]$ whose radius is $R = \overline{OP}$ and its center is point O. Our aim is to determine the local geometrical property of the path of point H at the instant shown in Figure 2.

Using the data given in Figure 2 we can write

$$\mathbf{n}_f = -\mathbf{e}, \quad \mathbf{b}_f = -\mathbf{i}, \quad \Gamma_f = \frac{1}{R},$$
$$\mathbf{n}_m = \mathbf{i}\cos\vartheta - \mathbf{e}\sin\vartheta,$$
$$\mathbf{b}_m = -\mathbf{i}\sin\vartheta - \mathbf{e}\cos\vartheta,$$
$$\Gamma_m = \frac{1}{R\sin\vartheta}, \ \Phi = 2\vartheta, \ \psi = 0.$$
$$\mathbf{b}_f \cdot \mathbf{e} = 0, \ \mathbf{b}_m \cdot \mathbf{e} = -\cos\vartheta,$$
$$d_p = R\tan\vartheta.$$

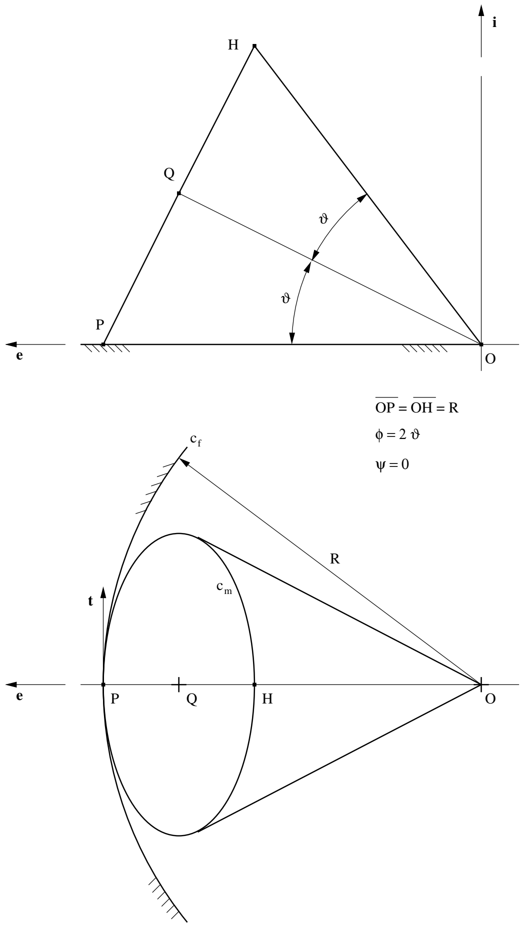

Figure 2. Rolling circular cone

The application of *Theorem 3* to this problem gives

$$\mathbf{t}_H = \mathbf{t},\ \mathbf{n}_H = -\frac{1}{\sqrt{(\cot 2\vartheta)^2 + 1}}\left(\mathbf{e} + \frac{\mathbf{i}}{\sin 2\vartheta}\right),$$

$$\mathbf{b}_H = \frac{1}{\sqrt{(\cot 2\vartheta)^2 + 1}}\left(\frac{\mathbf{e}}{\sin 2\vartheta} - \mathbf{i}\right),$$

$$\Gamma_H = \frac{1}{R}\left[\left(\frac{\lambda^4 + 4\lambda^2 - 1}{4\lambda}\right)^2 + 1\right],\ \lambda = \tan\vartheta.$$

The path of point H is a spherical cycloid and the motion analyzed can be considered a spherical cycloidal motion [2].

5. **Conclusion**

Some relations are derived for the spherical motion of a rigid body. The geometrical properties of spherical motion are expressed in the geometrical data of the fixed and moving axode cones. The approach applied does not use the tools of instantaneous spherical kinematics [2, 3]. The method presented is based on a vectorial approach that one can meet in [1, 5].

Curvature type relations such as (2.1), (2.14) can be considered as a form of Euler-Savary equation for spherical motion. Different forms of the Euler-Savary equation using the terminology and the concept of instantaneous invariants introduced by Bottema are given in [2, 3] for spherical and plane motions.

One example shows how we can use the derived formulas to determine the tangent, principal normal and binormal vectors together with the curvature at a point of path curve in the case of spherical motion.

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