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ON TRANSVERSE VIBRATIONS OF BELTS

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Abstract. The paper examines the transverse vibrations of belts and the impact of the vibration modes on unstable speed ranges. Supposing large deformations, it produces a more general non-linear motion of equation for the vibrations, which may be suitable for the examination of further linear and non-linear vibrations. It is shown that in the course of the belt motion, parametrically excited non-linear vibrations develop. The parametrical excitation is caused by the change in length of the belts resulting from the eccentricity of one of the belt pulleys. Next the paper examines the impact of vibration modes developing during the transverse vibrations of the belts on the main instability range. A first approximation of a closed form is developed for the main instability ranges of transverse vibrations. It is shown that the instability ranges belonging to the higher vibration modes become wider and tend to move towards higher numbers of revolutions.

Mathematical Subject Classification: 73A05
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1. Introduction

Belt drives are extensively used in mechanical engineering practice for the transmission of moments and power between axles located far away from each other. Its widespread application – in the automobile industry, a number of branches of the light industry, general engineering and machine tool industry, etc. – can be explained by its inexpensive realisation, quiet operation, easy mounting, favourable vibration damping, and last but not least by its good efficiency. The theory of belt drive design has been known and applied in engineering practice for a long time. Today renowned belt manufacturers support graphical dimensioning of belts based on diagrams. The basis of these selection and dimensioning procedures is provided by strength calculations.

In applications requiring higher accuracy – for example the main and feed drives of machine tools – it is not sufficient to dimension the particular machine elements, in particular belts, exclusively in terms of strength. In such cases it is also essential to apply a knowledge of vibrations that will facilitate the solution or elimination of dynamic problems in the design phase. When designing belt drives, basically two kinds of dynamic tasks are to be solved. One of them is an examination of the problems arising from the longitudinal vibrations of the belts. The other is an examination of the transverse vibrations of the belts. It is known both from the literature and from practical experience that at a certain running speed the belts lose their stability and develop transverse vibrations. These vibrations exert a detrimental influence on the life of the belt and in some cases on that of the machine, and – in the case of machine tools – may exert a non-desirable effect on the machining process and the manufacturing accuracy. Therefore it is expedient and important to determine in the design phase the instability ranges where the non-desirable vibrations mentioned above may develop. The following is a stability analysis of transverse vibrations.

2. The system of equations of motion of a single belt

2.1. **The mechanical model.** On the basis of the understanding of the literature it is expedient in the analysis of certain types of vibrations arising in the application of belt drives to consider the non-linear material properties of the belt (cf. e.g. [3]). One possible way to do so is to approximate the characteristic curve of the belt with a third degree polynomial. Accordingly, the material law applying to the belt is supposed to have the following form

$$\sigma_x = E\varepsilon_x + \beta\varepsilon_x^3, \tag{2.1}$$

where σ_x is the tensile stress arising in the belt, ε_x is the strain in direction x, E and β are material constants, which have to be determined by means of measurements. In the derivation of the equations of motion the following are supposed to hold:

- the belt moves only in plane xz according to Figure 1,
- only the force stretching the belt acts on the belt,
- the cross-sectional area of the belt is constant, its material properties do not change along the axis x,
- in the beginning the internal damping of the belt is neglected,
- the effects of the belt separating from and being stretched on the discs are neglected in accordance with [4].

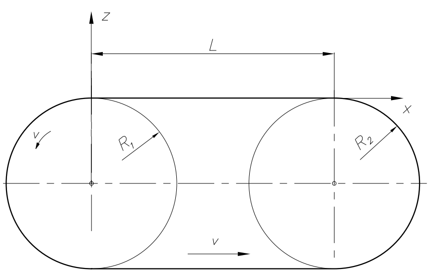

Figure 1. Mechanical model of the drive

As usual, the line connecting the centres of gravity of the cross-sectional areas is called the centre line of the belt. The displacement of the point with abscissa x of the belt centre line in direction x is denoted by $u\,(x, t)$, and that in direction z is denoted by $w\,(x, t)$. In accordance with our supposition, the displacement in direction y is zero, therefore the strain of the center line according to [1, 3] is approximated by

$$\varepsilon_{x0} = \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right]. \tag{2.2}$$

On the basis of experience $\left(\frac{\partial u}{\partial x}\right)^2$ in (2.2) may be negligible as related to the very small $\frac{\partial u}{\partial x}$, but $\frac{\partial w}{\partial x}$ may be large as compared with $\frac{\partial u}{\partial x}$. Therefore on the basis of [1] the approximation

$$\varepsilon_{x0} = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \tag{2.3}$$

is used. If the curvature of the center line is approximated by $\frac{\partial^2 w}{\partial x^2}$, then the axial strain of an arbitrary fibre in the belt can be written in the form

$$\varepsilon_{x} = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - z\frac{\partial^2 w}{\partial x^2}\,. \tag{2.4}$$

2.2. **Equations of motion.** The equations of motion are derived by means of the Hamilton principle. Therefore the following can be written

$$\delta \int_{t=t_1}^{t_2} (W - T)\,\mathrm{d}t = 0\,.$$

After the calculations detailed in Appendix A the following equations of motion are obtained:

$$\begin{aligned} &\varrho A \tfrac{\partial^2 u}{\partial t^2} - \tfrac{\partial}{\partial x}\Big\{AE\Big[\tfrac{\partial u}{\partial x} + \tfrac{1}{2}\big(\tfrac{\partial w}{\partial x}\big)^2\Big] + \beta\Big\{A\Big[\tfrac{\partial u}{\partial x} + \tfrac{1}{2}\big(\tfrac{\partial w}{\partial x}\big)^2\Big]^3 + \\ &+3I_y\Big[\tfrac{\partial u}{\partial x} + \tfrac{1}{2}\big(\tfrac{\partial w}{\partial x}\big)^2\Big]\cdot\Big(\tfrac{\partial^2 w}{\partial x^2}\Big)^2 - 2I_{3y}\Big[\tfrac{\partial u}{\partial x} + \tfrac{1}{2}\big(\tfrac{\partial w}{\partial x}\big)^2\Big]\cdot\Big(\tfrac{\partial^2 w}{\partial x^2}\Big)^3\Big\}\Big\} = 0 \end{aligned} \tag{2.5}$$

$$\begin{aligned} &\varrho A \tfrac{\partial^2 w}{\partial t^2} - \tfrac{\partial}{\partial x}\Big\{\tfrac{\partial w}{\partial x}\Big\{AE\Big[\tfrac{\partial u}{\partial x} + \tfrac{1}{2}\big(\tfrac{\partial w}{\partial x}\big)^2\Big] + \beta\Big\{A\Big[\tfrac{\partial u}{\partial x} + \tfrac{1}{2}\big(\tfrac{\partial w}{\partial x}\big)^2\Big]^3 + \\ &+3I_y\Big[\tfrac{\partial u}{\partial x} + \tfrac{1}{2}\big(\tfrac{\partial w}{\partial x}\big)^2\Big]\cdot\Big(\tfrac{\partial^2 w}{\partial x}\Big)^2 - 2I_{3y}\Big[\tfrac{\partial u}{\partial x} + \tfrac{1}{2}\big(\tfrac{\partial w}{\partial x}\big)^2\Big]\cdot\Big(\tfrac{\partial^2 w}{\partial x^2}\Big)^3\Big\}\Big\}\Big\} + \\ &+\tfrac{\partial}{\partial x^2}\Big\{\tfrac{\partial^2 w}{\partial x^2}\Big\{I_yE + \beta\Big\{3I_y\Big[\tfrac{\partial u}{\partial x} + \tfrac{1}{2}\big(\tfrac{\partial w}{\partial x}\big)^2\Big]^2\cdot\Big(\tfrac{\partial^2 w}{\partial x^2}\Big) - \\ &-3I_{3y}\Big[\tfrac{\partial u}{\partial x} + \tfrac{1}{2}\big(\tfrac{\partial w}{\partial x}\big)^2\Big]^2\cdot\Big(\tfrac{\partial^2 w}{\partial x^2}\Big)^2 + I_{4y}\big(\tfrac{\partial w}{\partial x}\big)^3\Big\}\Big\}\Big\} = 0\,, \end{aligned} \tag{2.6}$$

(cf. [5]) where ϱ is the density, A is the cross-sectional area, I_y is the moment of inertia of the cross-section calculated for the axis y, $I_{ny}, n = 3, 4$ are the higher order moments of the cross-sectional area, L is the length of the belt between the belt pulleys, E is the linear part of the modulus of elasticity, β is the non-linear part of the modulus of elasticity. Equations of motions (2.5) and (2.6) describe the general motion of a single belt. When supplemented with the right fitting and boundary conditions, they are suitable for performing general dynamic analyses. Later on the above equations of motion (2.5) and (2.6) are regarded as our starting point for further research.

3. **Analysis of transverse vibrations**

When analyzing transverse vibrations, the non-linear partial differential equation system (2.5) and (2.6) is used as the starting point. Their accurate solution, suitable for engineering work, is not known yet. The method to be presented, based on Kirchhoff [2] and Kauderer [1], was used by Faragó [3] and Patkó [5] for belts as follows. In order to produce simpler equations of motion, the suppositions in [2] were used in (2.5), (2.6) according to which in the expression of the kinetic energy the coordinate $\left(\frac{\partial u}{\partial t}\right)$ in direction x of the velocity vector of the belt element performing the transverse vibration may be neglected beside the component $\left(\frac{\partial w}{\partial t}\right)$ in direction z. Thus, instead of (2.5) and (2.6) a simpler partial differential equation system is obtained. Relying on the train of thoughts by Kauderer and on the basis of the measurement results by Faragó, it is acceptable, as a first approximation in an analysis of transverse vibrations, to approximate the function $\sigma_x = \sigma_x(\varepsilon)$ by its linear part. Using the approximations mentioned, the system of the equations of motion (2.5) and (2.6) of the belt can be written in the form

$$\frac{\partial}{\partial x}\left\{AE\left[\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]\right\}=0 \tag{3.1}$$

$$\varrho A\frac{\partial^2 w}{\partial t^2}+\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 w}{\partial x^2}I_yE\right)-\frac{\partial}{\partial x}\left\{\frac{\partial w}{\partial x}\left\{AE\left[\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]\right\}\right\}=0. \tag{3.2}$$

Comparing equations (3.1) and (3.2) with (2.3) shows that the elongation of the center line of the belt does not depend on place x, therefore it can only depend on time t. Integrating (3.1) according to variable x gives the form

$$AE\left[\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]=F(t)\,. \tag{3.3}$$

Equation (3.3) is again integrated along length L of the belt from $x=0$ to $x=L$, which gives

$$\frac{AE}{L}\left[u(L,t)-u(0,t)+\frac{1}{2}\int_{x=0}^{L}\left(\frac{\partial w}{\partial x}\right)^2\mathrm{d}x\right]=F(t)\,. \tag{3.4}$$

Thus supposing a constant belt cross-sectional area from (3.2) and (3.4) for the function $w=w(x,t)$ describing the transverse vibrations gives the following integro-differential equation

$$\varrho A\frac{\partial^2 w}{\partial t^2}+I_yE\frac{\partial^4 w}{\partial x^4}-\frac{\partial^2 w}{\partial x^2}\left\{\frac{AE}{L}\left[u(L,t)-u(0,t)+\frac{1}{2}\int_{x=0}^{L}\left(\frac{\partial w}{\partial x}\right)^2\mathrm{d}x\right]\right\}=0. \tag{3.5}$$

There are time-dependent boundary conditions belonging to (3.5). If the coordinate system xyz is taken so that axis x passes through the current points of contact between the belt and the belt plates, then these boundary conditions can be formulated in the following forms

$$w\,(vt, t) = 0$$

and

$$w\,(vt + L, t) = 0,$$

where $v = constant$, the velocity of the belt in the coordinate system xyz. Let us

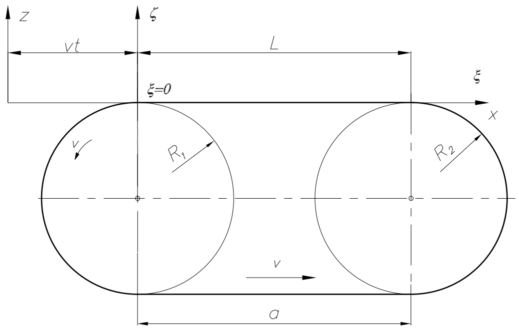

Figure 2. Mechanical model of the drive after transformation of coordinates

attach a coordinate system $\xi\eta\varsigma$ according to Figure 2 to the belt side in motion so that there should only be a translation in direction x between the coordinate systems xyz and $\xi\eta\varsigma$. Let us transform differential equation (3.5) into the coordinate system $\xi\eta\varsigma$ that the time dependence of the boundary conditions will be eliminated. Let us introduce the transformation

$$x = \xi + v \cdot t \tag{3.6}$$

according to Figure 2, thus the equation of motion is transformed into the form

$$\begin{aligned} &\varrho A \frac{\partial^2 w}{\partial t^2} - 2\varrho v A \frac{\partial^2 w}{\partial \xi \partial t} + \varrho A v^2 \frac{\partial^2 w}{\partial \xi^2} + I_y E \frac{\partial^4 w}{\partial \xi^4} - \\ &- \frac{\partial^2 w}{\partial \xi^2} \left\{ \frac{AE}{L} \left[u\,(L, t) - u\,(0, t) + \frac{1}{2} \int_0^L \left(\frac{\partial w}{\partial \xi} \right)^2 \mathrm{d}\xi \right] \right\} = 0 \end{aligned} \tag{3.7}$$

and the boundary conditions are transformed into the time-independent forms

$$w\,(0, t) = 0 \tag{3.8}$$

and

$$w\,(L, t) = 0. \tag{3.9}$$

Equation (3.7) is a non-linear partial integro-differential equation. In the analysis of certain types of non-linear vibrations the Galerkin method is widely used [6]. On the basis of observations and experience the solution of the above integro-differential equation (3.7) is looked for in the following trigonometric series

$$w(\xi,t)=\sum_{n=1}^{\infty} q_n(t)\sin\left(\frac{n\pi}{L}\xi\right), \tag{3.10}$$

the members of which are orthogonal in the interval $[0, L]$ and satisfy the boundary conditions (3.8) and (3.9). Then

$$\ddot{q}_k+\left(\tfrac{k\pi}{L}\right)^2\left\{\left(\tfrac{k\pi}{L}\right)^2\tfrac{I_y}{A}\tfrac{E}{\varrho}-v^2+\tfrac{E}{\varrho L}\left[u(L,t)-u(0,t)+\tfrac{\pi^2}{4L}\sum_{m=1}^{p}\left(m^2q_m^2\right)\right]\right\}q_k=0$$
$$(k=1,2,3,\dots,p) \tag{3.11}$$

is obtained. A detailed presentation of the calculations can be found in Appendix B. Let us introduce the notations

$$Q_k(\xi)=\sin\left(\frac{k\pi}{L}\xi\right)\qquad(k=1,2,3,\dots,p)$$

where the functions $Q_k(\xi)$ are from now on called vibration modes. On the basis of experience (cf. [7]) it can be supposed that the arising vibrations in the first approximation have the property that there is a dominant vibration mode $Q_k(\xi)$ and a dominant frequency belonging to them, beside which the amplitudes $q_m(t)$ belonging to the other $Q_m(\xi)$ $(m\neq k)$ are mostly negligibly small. Based on the above, in (3.11) only one such function $q_k(t)$ considered to be dominant is kept. Thus instead of (3.11) it is sufficient to analyse the differential equation

$$\ddot{q}_k+\left(\tfrac{k\pi}{L}\right)^2\left\{\left(\tfrac{k\pi}{L}\right)^2\tfrac{I_y}{A}\tfrac{E}{\varrho}-v^2+\tfrac{E}{\varrho L}\left[u(L,t)-u(0,t)+\left(\tfrac{k\pi}{2}\right)^2\tfrac{1}{L}q_k^2\right]\right\}q_k=0$$
$$(k=1,2,3,\dots,p)\ . \tag{3.12}$$

4. **Stability analysis**

4.1. **The exciting effect.** In order to investigate the stability of belt, let us linearise (3.12) at $q_{k0}=0$, which gives

$$\ddot{q}_k+\left(\frac{k\pi}{L}\right)^2\left\{\left(\frac{k\pi}{L}\right)^2\frac{I_y}{A}\frac{E}{\varrho}-v^2+\frac{E}{\varrho L}\left[u(L,t)-u(0,t)\right]\right\}q_k=0\ . \tag{4.1}$$

It can be seen from the equation of motion that one possible cause of the transverse vibrations of belts is the longitudinal elongation of the belt, which changes in time. Let us examine the case when the longitudinal displacement of the ends of the belts is caused by the eccentricity of one of the belt pulleys. Let us suppose that in the coordinate system $\xi\eta\zeta$ one end of the belt does not get displaced, that is

$$u(0,t)=0, \tag{4.2}$$

and its other end gets displaced by the value u_0 resulting from the pre-tensioning and the transferred moment, then performs an oscillatory motion described by the function $u_L=e_2\cos(\nu t)$ in direction ξ due to the eccentricity of one of the belt pulleys

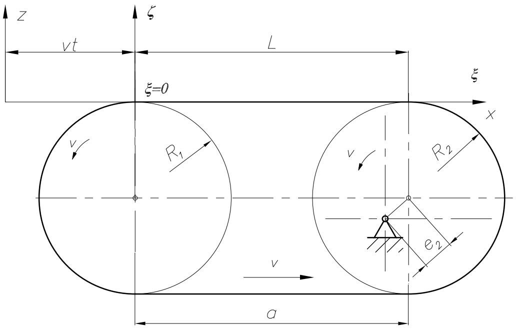

Figure 3. The model of a drive with eccentricity

(in this case the driven one), where e_2 is the eccentricity of the driven belt pulley, ν is its angular velocity, and using them gives

$$u\,(L, t) = u_0 + e_2 \cos{(\nu t)}\,. \tag{4.3}$$

The velocity of the belt can be expressed in terms of the angular velocity of the driving plate denoted by 1 and gives

$$v = R_1 \nu, \tag{4.4}$$

where ν is the angular velocity of the driving plate and R_1 is the radius of the driving plate – see Figure 3. Substituting equations (4.2)-(4.4) into (4.1), let us introduce the dimension-free time coordinate

$$\tau = \frac{1}{2}\nu t\ , \tag{4.5}$$

and we get

$$q_k^{''} + 4\left(\frac{k\pi}{L}\right)^2 \left\{\left(\frac{k\pi}{L}\right)^2 \kappa\,(h)\,\frac{E}{\varrho\nu^2} - R_1^2 + \frac{E}{\varrho\nu^2 L}\left[u\,(L,\tau) - u\,(0,\tau)\right]\right\} q_k = 0$$
$$(k = 1, 2, 3, \ldots, p)\ , \tag{4.6}$$

where the comma denotes differentiation according to τ and $\kappa\,(h) = \sqrt{\frac{I_y}{A}}$ is the inertia-radius. Let us furthermore introduce the notations

$$\lambda_k = 4\left(\tfrac{k\pi}{L}\right)^2 \left\{\tfrac{E}{\varrho\nu^2}\left[\left(\tfrac{k\pi}{L}\right)^2 \kappa\,(h) + \tfrac{u_0}{L}\right] - R_1^2\right\}\ , \tag{4.7}$$

$$\mu_k = -2\left(\tfrac{k\pi}{L}\right)^2 \tfrac{E}{L\varrho\nu^2} e_2\ . \tag{4.8}$$

Thus (4.6) will take the form

$$q_k^{''} + \left(\lambda_k - 2\mu_k \cos{(2\tau)}\right) q_k = 0 \qquad (k = 1, 2, 3, \ldots, p)\,. \tag{4.9}$$

The stability ranges of the above Mathieu-type differential equations are known from the literature [1, 8]. The case $\lambda_k < 0$ is of no importance for practical belt drives. Among the instability ranges what is called the main instability range is the most dangerous, for here even for small μ_k values may arise stability loss in a wide interval

λ_k. The other instability ranges are of smaller significance due to the dampings not taken into account here. Therefore this paper is limited to an analysis of the main instability range. It should be noted here that due to the damping present in the system, but not taken into account now, the sizes of the instability ranges decrease.

4.2. **First approximation of the main instability ranges.** Practical calculations show that the values μ_k in belt drives are small, therefore in the first approximation

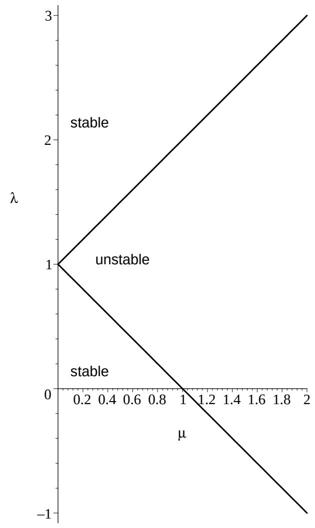

Figure 4. First approximation of the main instability range of the Mathieu equation

it is sufficient to approximate the main instability range by its tangents. Accordingly, the main instability range of (4.9) is approximated – see Figure 4. – in the form

$$\lambda_k = 1 \pm \mu_k \ . \tag{4.10}$$

Substituting the variables (4.7)-(4.8) into (4.10) and solving the expression obtained for angular velocity ν of the belt plates, the relationship

$$\nu = 2k\pi\sqrt{\frac{E\left[k^2\pi^2 I_y + LA\left(u_0 \pm \frac{e_2}{2}\right)\right]}{\varrho A L^2\left(4k^2\pi^2 R_1^2 + L^2\right)}} \tag{4.11}$$

is obtained. It means that the unstable angular velocities are in the region

$$2k\pi\sqrt{\frac{E\left[k^2\pi^2 I_y + LA\left(u_0 - \frac{e_2}{2}\right)\right]}{\varrho A L^2\left(4k^2\pi^2 R_1^2 + L^2\right)}} < \nu < 2k\pi\sqrt{\frac{E\left[k^2\pi^2 I_y + LA\left(u_0 + \frac{e_2}{2}\right)\right]}{\varrho A L^2\left(4k^2\pi^2 R_1^2 + L^2\right)}} \ . \tag{4.12}$$

It describes the first approximation of the main instability range as depending on the further belt parameters. The following is an analysis of how the positions of the main instability ranges change for different vibration patterns.

4.3. **Analysis of the impact of vibration patterns.** It can be seen from equation (4.11) as well as from the Figures that when number k of the vibration pattern being analysed is increased, the instability range moves towards the higher revolution numbers. It can be observed and can also be seen from relationship (4.11) that for higher values of k and for the usual belt parameters the ranges become slightly wider.

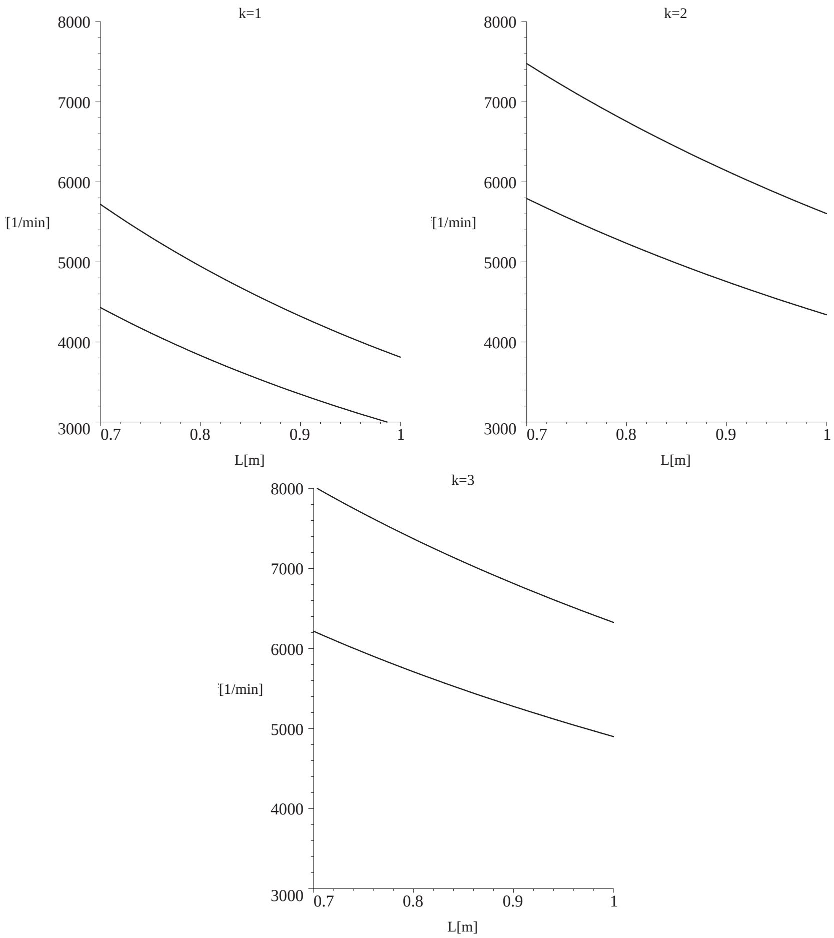

Figure 5. First approximations of the main instability domains for $k = 1, 2, 3$, $e_2 = 3 \cdot 10^{-3}[m]$, $I_y = 3.2 \cdot 10^{-11}[m^4]$, $\varrho = 2 \cdot 10^3[kg/m^3]$, $E = 1.5 \cdot 10^9[N/m^2]$, $A = 2.1 \cdot 10^{-7}[m^2]$, $R_1 = 0.1[m]$, $u_0 = 6 \cdot 10^{-3}[m]$

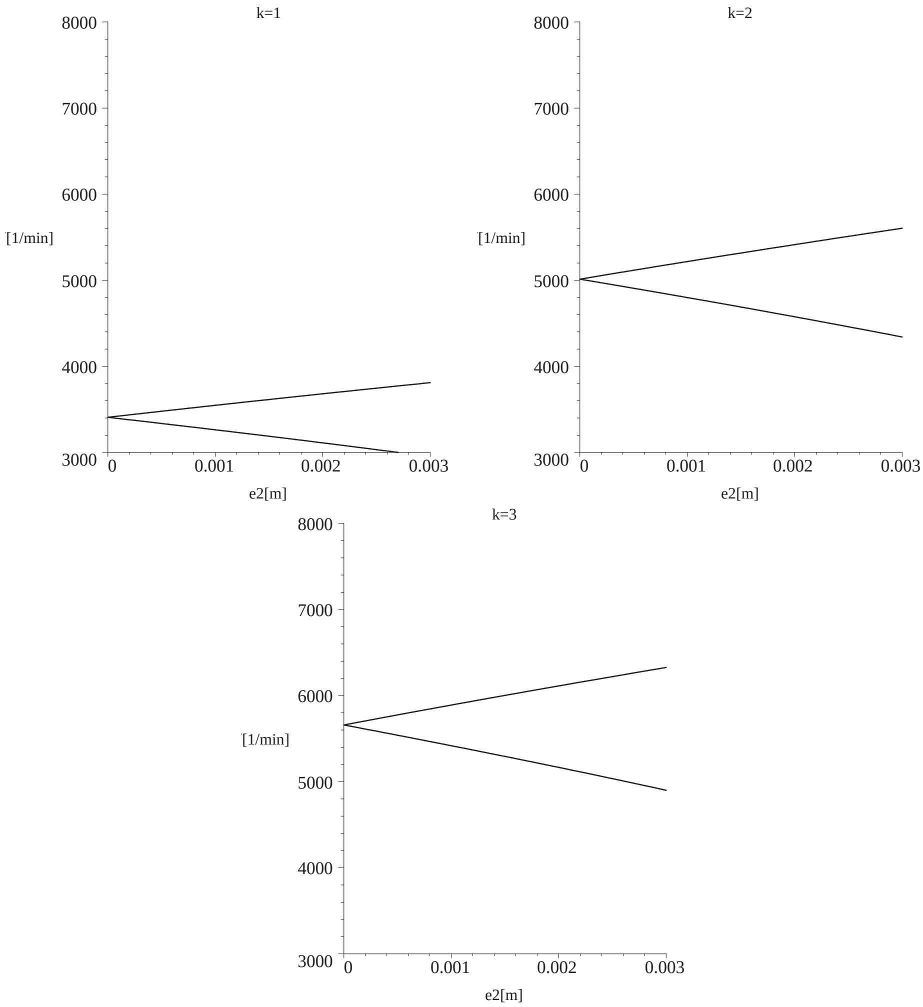

Figure 6. First approximations of the main instability domains for $k = 1, 2, 3,\ L = 1[m], I_y = 3.2 \cdot 10^{-11}[m^4], \varrho = 2 \cdot 10^3[kg/m^3], E = 1.5 \cdot 10^9[N/m^2], A = 2.1 \cdot 10^{-7}[m^2], R_1 = 0.1[m], u_0 = 6 \cdot 10^{-3}[m]$

Figure 5 shows the boundaries of the instability ranges calculated from (4.11) with the values $k = 1, 2, 3$. In the diagrams of Figure 5 the instable angular velocity range is drawn versus length L of the belt. In the diagrams of Figure 6 the unstable angular velocity range is drawn versus eccentricity e_2 of the pulley. It can be seen from the Figures that certain revolution number ranges may become dangerous even for different vibrations. Relationship (4.12) lends itself to further noteworthy conclusions. The formula shows what impact the belt parameters exert on the positions and dimensions of unstable ranges.

5. Concluding remarks

The paper has shown more general equations of motion of transverse vibrations of belts than those known from the literature. These are the equations of motion (2.5), (2.6), (3.5) and (3.11), which take into account the non-linear behaviour of the belts and also provide a basis for further research. Based on the above, it can be stated that one possible cause of the transverse vibrations of belts is the eccentricity of the belt pulleys. In that case the transverse vibrations are described by a differential equation system with a non-linear variable coefficient. The stability analysis of the belt has also been presented. The first approximation of the main instability range has been performed versus the angular velocity of the belt pulley. It has been investigated how the main instability domains of the transverse vibrations of a belt change with the different vibration patterns. It was found that the main instability ranges belonging to the higher vibration patterns move towards the higher numbers of revolution and become slightly wider.

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Appendix A. Derivation of the equations of motion

The Hamilton integral is

$$H = \int_{t=t_1}^{t_2} (W - T)\,\mathrm{d}t, \tag{A.1}$$

where W represents the strain work of the belt, and T is its kinetic energy. Let us first determine the strain work resulting from the flexible displacements in the transverse and longitudinal directions of the belt cross-sections. Let U be the strain work of the belt for unit volume. Then, using (2.1)

$$\bar{U} = \int_{\varepsilon_x=0}^{\varepsilon_x} \left(E\varepsilon_x + \beta\varepsilon_x^3\right) \mathrm{d}\varepsilon_x = \frac{1}{2}E\varepsilon_x^2 + \frac{1}{4}\beta\varepsilon_x^4 \tag{A.2}$$

can be written. The strain work of the belt for unit length can be calculated by using (2.4) and (A.2) according to

$$U = \iint_A \bar{U}\mathrm{d}y\mathrm{d}z, \tag{A.3}$$

where A is the cross-section of the belt. Giving details of expression (A.3)

$$\begin{aligned} U = \tfrac{1}{2}E \textstyle\iint_A \left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - z\frac{\partial^2 w}{\partial x^2}\right]^2 \mathrm{d}y\mathrm{d}z + \\ + \tfrac{1}{4}\beta E \textstyle\iint_A \left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - z\frac{\partial^2 w}{\partial x^2}\right]^4 \mathrm{d}y\mathrm{d}z \end{aligned} \tag{A.4}$$

is obtained. When U is known, the strain work accumulated in the belt can be calculated as

$$W = \int_{x=0}^{L} U\mathrm{d}x,$$

which is detailed to give the following integral

$$\begin{aligned} W = \tfrac{1}{2}\textstyle\int_{x=0}^{L} \Bigg\{ E \Bigg\{ A\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]^2 + I_y\left(\frac{\partial^2 w}{\partial x^2}\right)^2 \Bigg\} + \\ + \tfrac{1}{2}\beta \Bigg\{ A\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]^4 + 6I_y\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]^2\left(\frac{\partial^2 w}{\partial x^2}\right)^2 - \\ - 4I_{3y}\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]^2\left(\frac{\partial^2 w}{\partial x^2}\right)^3 + I_{4y}\left(\frac{\partial^2 w}{\partial x^2}\right)^4 \Bigg\}\Bigg\} \mathrm{d}x. \end{aligned} \tag{A.5}$$

In the integration of function $\bar{U}$ the facts that the first moment $S_y = \iint_A z\mathrm{d}y\mathrm{d}z$ of the belt cross-section calculated for its centroidal axis is zero and that the quantity $\iint_A z^i\mathrm{d}y\mathrm{d}z$ was designated $I_{iy}\,(i = 3, 4)$ were made use of.

Let us now turn to calculating the kinetic energy of the elementary belt. Let the mass of the belt per unit length be denoted by $m_0 = \varrho A$. The velocity of one point of the central line of the belt is calculated according to

$$v_0 = \sqrt{\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2}. \tag{A.6}$$

If the moments of inertia of the individual belt elements are neglected - and thus the kinetic energy resulting from the angular velocities of the revolutions of the cross-sections is also neglected - then the kinetic energy can be written as

$$T = \frac{1}{2}m_0 \int_{x=0}^{L} \left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right] \mathrm{d}x. \tag{A.7}$$

If now the relationships (A.5) and (A.7) are substituted into (A.1), this gives the integral of form

$$H = \int_{t=t_1}^{t_2} \int_{x=0}^{L} F\left(\frac{\partial u}{\partial t}; \frac{\partial u}{\partial x}; \frac{\partial w}{\partial t}; \frac{\partial w}{\partial x}; \frac{\partial^2 w}{\partial x^2}\right) \mathrm{d}x\mathrm{d}t \tag{A.8}$$

from which the Euler-Lagrange equations

$$\frac{\partial}{\partial x}\frac{\partial F}{\partial\left(\frac{\partial u}{\partial x}\right)} + \frac{\partial}{\partial t}\frac{\partial F}{\partial\left(\frac{\partial u}{\partial t}\right)} = 0 \tag{A.9}$$

$$\frac{\partial w}{\partial x}\frac{\partial F}{\partial\left(\frac{\partial w}{\partial x}\right)} - \frac{\partial^2}{\partial x^2}\frac{\partial F}{\partial\ \frac{\partial^2 w}{\partial x^2}} + \frac{\partial}{\partial t}\frac{\partial F}{\partial\left(\frac{\partial w}{\partial t}\right)} = 0 \tag{A.10}$$

can be derived [1] on the basis of the Hamilton variation principle ($\delta H = 0$). Completing the differentiations designated gives the differential equations (2.5) and (2.6) describing the motion of the belt.

Appendix B. Details of the calculations using the Galerkin method

According to (3.7) the equation of motion can be written in the form

$$\varrho A\frac{\partial^2 w}{\partial t^2} - 2\varrho vA\frac{\partial^2 w}{\partial\xi\partial t} + \varrho Av^2\frac{\partial^2 w}{\partial\xi^2} + I_y E\frac{\partial^4 w}{\partial\xi^4} - \\ - \frac{\partial^2 w}{\partial\xi^2}\left\{\frac{AE}{L}\left[u(L,t) - u(0,t) + \frac{1}{2}\int_0^L \left(\frac{\partial w}{\partial\xi}\right)^2 \mathrm{d}\xi\right]\right\} = 0 \tag{B.1}$$

and the boundary conditions are transformed into the time-independent forms

$$w(0,t) = 0 \tag{B.2}$$

and

$$w(L,t) = 0. \tag{B.3}$$

The solution of equation (B.1) is sought following Galerkin's method in the form

$$w(\xi,t) = \sum_{n=1}^{\infty} q_n(t)\sin\left(\frac{n\pi}{L}\xi\right) \tag{B.4}$$

which satisfies the boundary conditions (B.2) and (B.3). Substituting (B.4) in the equation of motion (B.1) results in equation

$$\sum_{n=1}^{\infty}\left\{\left\{\varrho A\ddot{q}_n - \varrho Av^2\frac{n^2\pi^2}{L^2}q_n + I_y E\frac{n^4\pi^4}{L^4}q_n + \left(\frac{n^2\pi^2}{L^2}q_n\right)\left\{\frac{AE}{L}\left[u_L(t) + \right.\right.\right.\right. \\ \left.\left.\left.\left. + \frac{L}{4}\sum_{k=1}^{\infty}\left(\frac{k^2\pi^2}{L^2}q_k^2\right)\right]\right\}\right\}\sin\left(\frac{n\pi}{L}\xi\right)\right\} = 0\,. \tag{B.5}$$

Let us multiply equation (B.5) by the expression $\sin\left(\frac{i\pi}{L}\xi\right)$, then integrate it on the interval $[0, L]$. In accordance with Galerkin's method, the equation

$$\sum_{n=1}^{\infty}\left\{\left\{\varrho A\ddot{q}_n - \varrho A v^2 \frac{n^2\pi^2}{L^2} q_n + I_y E \frac{n^4\pi^4}{L^4} q_n + \left(\frac{n^2\pi^2}{L^2} q_n\right)\left\{\frac{AE}{L}\left[u_L(t) + \right.\right.\right.\right.$$
$$\left.\left.\left.\left.+\frac{L}{4}\sum_{k=1}^{\infty}\left(\frac{k^2\pi^2}{L^2} q_k^2\right)\right]\right\}\right\}\right\}\int_{\xi=0}^{L} \sin\left(\frac{n\pi}{L}\xi\right)\sin\left(\frac{i\pi}{L}\xi\right) \mathrm{d}\xi = 0 \quad \text{(B.6)}$$

can be written. Completing the integrations designated gives equation

$$\varrho A\ddot{q}_i + \frac{i^2\pi^2}{L^2}\left[I_y E \frac{i^2\pi^2}{L^2} - \varrho A v^2 + \frac{AE}{L}\left(u_L(t) + \frac{\pi^2}{4L}\sum_{k=1}^{\infty} k^2 q_k^2\right)\right] q_i = 0. \quad \text{(B.7)}$$