Journal of Computational and Applied Mechanics, Vol. 6., No. 2., (2005), pp. 175–182

A RECIPROCAL THEOREM FOR STEADY-STATE HEAT CONDUCTION PROBLEMS

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[Received: October 8, 2004]

Dedicated to István Páczelt on the occasion of his 65th birthday

Abstract. This paper presents a reciprocal theorem for steady-state heat conduction problems. Some examples illustrate the applications of the reciprocal relation formulated. The method applied is based on the analogy which exists between linear elasticity and heat conduction.

Mathematical Subject Classification: 80A20
Keywords: heat conduction, reciprocal theorem, steady-state, thermal energy

1. **Introduction**

Consider a 3D solid body B occupying a closed and limited region $\bar{V}$ for which the steady-state heat condition is defined. The set of inner points V is denoted by V and the set of points on the boundary of $\bar{V}$ is denoted by ∂V, $\bar{V} = V \cup \partial V$. Point P of $\bar{V}$ is indicated by the vector $\vec{OP} = \mathbf{p} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z$ in a given orthogonal Cartesian coordinate system $Oxyz$ with the unit vectors $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$. The volume element in V is denoted by $\mathrm{d}v$ and the surface element defined on ∂V is $\mathrm{d}a$.

The temperature difference field [7, 1] in the body $\bar{V}$ is denoted by $T = T(x, y, z)$. Following Wojnar [7]and the thermal intensity vector field is introduced by the definition

$$\mathbf{t} = -\nabla T\,, \tag{1}$$

where

$$\nabla = \frac{\partial}{\partial x}\mathbf{e}_x + \frac{\partial}{\partial y}\mathbf{e}_y + \frac{\partial}{\partial z}\mathbf{e}_z \tag{2}$$

is the gradient operator [4, 5]. The field equations of the steady-state heat conduction problem are the heat balance equation [1, 6]

$$-\nabla \cdot \mathbf{q} + R = 0\,, \qquad \text{in } V \tag{3}$$

and the Fourier law of heat conduction [6, 7], which takes the form

$$\mathbf{q} = \mathbf{K} \cdot \mathbf{t}\,, \tag{4}$$

and the thermal intensity vector–temperature difference field relation (1). In equations (3), (4) the dot denotes the scalar product according to Malvern [5] and Lurje [4] and in equation (4) $\mathbf{K} = \mathbf{K}(x, y, z)$ is the heat conductivity tensor field, which is symmetric and positive definit [1, 7]. The distributed heat source in B is denoted by $R = R(x, y, z)$. On the boundary surface ∂V the heat flux q is defined at every regular points of ∂V as

$$q = \mathbf{q}(x, y, z) \cdot \mathbf{n} \qquad (x, y, z) \in \partial V \,, \tag{5}$$

where $\mathbf{n}$ is the outward unit normal vector to ∂B at point (x, y, z).

We say that an ordered array $s = [T, \mathbf{t}, \mathbf{q}]$ is an admissible state if T, $\mathbf{t}$ and $\mathbf{q}$ are sufficiently smooth in $\bar{V}$ and they satisfy equations (1) and (4). The admissible state corresponds to internal heat source R and boundary surface heat flux q if equations (3) and (5) are satisfied. The ordered array of R and q is denoted by p as $p = [R, q]$.

2. Reciprocal theorem

Theorem 1. *Let $s = [T, \mathbf{t}, \mathbf{q}]$ and $\tilde{s} = [\tilde{T}, \tilde{\mathbf{t}}, \tilde{\mathbf{q}}]$ be two admissible states of the stationary heat conduction in body B corresponding to the internal heat sources and surface heat fluxes $p = [R, q]$ and $\tilde{p} = [\tilde{R}, \tilde{q}]$, respectively, then we have*

$$\begin{aligned}\int_V \mathbf{t} \cdot \tilde{\mathbf{q}} \, \mathrm{d}v &= \int_V \tilde{\mathbf{t}} \cdot \mathbf{q} \, \mathrm{d}v \\ &= -\int_{\partial V} T\tilde{q} \, \mathrm{d}a + \int_V T\tilde{R} \, \mathrm{d}v = -\int_{\partial V} \tilde{T}q \, \mathrm{d}a + \int_V \tilde{T}R \, \mathrm{d}v \,. \end{aligned} \tag{6}$$

Proof. The validity of equation (6) follows from the equations

$$\begin{gathered}\int_V \mathbf{t} \cdot \tilde{\mathbf{q}} \, \mathrm{d}v = \int_V \mathbf{t} \cdot \mathbf{K} \cdot \tilde{\mathbf{t}} \, \mathrm{d}v \,, \int_V \tilde{\mathbf{t}} \cdot \mathbf{q} \, \mathrm{d}v = \int_V \tilde{\mathbf{t}} \cdot \mathbf{K} \cdot \mathbf{t} \, \mathrm{d}v \,, \\ \int_V \tilde{\mathbf{t}} \cdot \mathbf{K} \cdot \mathbf{t} \, \mathrm{d}v = \int_V \mathbf{t} \cdot \mathbf{K} \cdot \tilde{\mathbf{t}} \, \mathrm{d}v \,, \end{gathered} \tag{7}$$

$$\begin{aligned}&\int_V \tilde{\mathbf{t}} \cdot \mathbf{K} \cdot \mathbf{t} \, \mathrm{d}v = \int_V \tilde{\mathbf{q}} \cdot \mathbf{t} \, \mathrm{d}v = -\int_V \tilde{\mathbf{q}} \cdot \nabla T \, \mathrm{d}v = \\ &= -\int_V \nabla \cdot (\tilde{\mathbf{q}}T) \, \mathrm{d}v + \int_V T\nabla \cdot \tilde{\mathbf{q}} \, \mathrm{d}v = -\int_{\partial V} \mathbf{n} \cdot \tilde{\mathbf{q}}T \, \mathrm{d}a + \int_V T\tilde{R} \, \mathrm{d}v = \\ &= -\int_{\partial V} T\tilde{q} \, \mathrm{d}v + \int_V T\tilde{R} \, \mathrm{d}v \,. \end{aligned} \tag{8}$$

Here, the rule for derivation of a product function and the divergence theorem have been used. □

3. Energy theorems

In [7], Wojnar introduced the thermal energy U corresponding to a continuous thermal intensity field $\mathbf{t}$ defined on $\bar{V}$ by

$$U\{\mathbf{t}\} = \frac{1}{2} \int_V \mathbf{t} \cdot \mathbf{K} \cdot \mathbf{t} \, \mathrm{d}v \,. \tag{9}$$

Theorem 2. *Let $s = [T, \mathbf{t}, \mathbf{q}]$ and $\tilde{s} = [\tilde{T}, \tilde{\mathbf{t}}, \tilde{\mathbf{q}}]$ be admissible states corresponding to internal heat sources and boundary surface heat fluxes $p = [R, q]$ and $\tilde{p} = [\tilde{R}, \tilde{q}]$, respectively. Then*

$$U\{\mathbf{t}\} \leq U\{\tilde{\mathbf{t}}\} \tag{10}$$

provided

$$-\int_{\partial V} T(\tilde{q} - q)\ \mathrm{d}a + \int_V T(\tilde{R} - R)\ \mathrm{d}v \geq 0\ , \tag{11}$$

or

$$-\int_{\partial V} q(\tilde{T} - T)\ \mathrm{d}a + \int_V R(\tilde{T} - T)\ \mathrm{d}v \geq 0\ . \tag{12}$$

Thus, if ∂V_1 and ∂V_2 are complementary surface segments of ∂V $(\partial V = \partial V_1 \cup \partial V_2$, $\partial V_1 \cap \partial V_2 = \{\emptyset\})$, then we have

$$\left.\begin{array}{rcr} T = \tilde{T} & \mathit{on} & \partial V_1 \\ q = 0 & \mathit{on} & \partial V_2 \\ R = 0 & \mathit{in} & V \end{array}\right\} \Rightarrow \qquad U\{\mathbf{t}\} \leq U\{\tilde{\mathbf{t}}\}\ , \tag{13}$$

or

$$\left.\begin{array}{rcr} T = 0 & \mathit{on} & \partial V_1 \\ q = \tilde{q} & \mathit{on} & \partial V_2 \\ R = \tilde{R} & \mathit{in} & V \end{array}\right\} \Rightarrow \qquad U\{\mathbf{t}\} \leq U\{\tilde{\mathbf{t}}\}\ . \tag{14}$$

Proof. From the definition of thermal energy U it follows that

$$U\{\tilde{\mathbf{t}}\} = U\{\mathbf{t} + (\tilde{\mathbf{t}} - \mathbf{t})\} = U\{\mathbf{t}\} + U\{\tilde{\mathbf{t}} - \mathbf{t}\} + \int_V \mathbf{t} \cdot \mathbf{K} \cdot (\tilde{\mathbf{t}} - \mathbf{t})\ \mathrm{d}v\ . \tag{15}$$

On the other hand the application of Theorem 1 to the admissible states $s = [T, \mathbf{t}, \mathbf{q}]$ and $\hat{s} = [\hat{T} = T - \tilde{T}, \hat{\mathbf{t}} = \mathbf{t} - \tilde{\mathbf{t}}, \hat{\mathbf{q}} = \mathbf{q} - \tilde{\mathbf{q}}]$ were $\hat{p} = [\hat{R} = R - \tilde{R}, \hat{q} = q - \tilde{q}]$ yields

$$\begin{aligned} \int_V \mathbf{t} \cdot \mathbf{K} \cdot (\tilde{\mathbf{t}} - \mathbf{t})\ \mathrm{d}v &= -\int_{\partial V} T(\tilde{q} - q)\ \mathrm{d}a + \int_V T(R - \tilde{R})\ \mathrm{d}v \\ &= -\int_{\partial V} q(\tilde{T} - T)\ \mathrm{d}a + \int_V R(\tilde{T} - T)\ \mathrm{d}v\ . \end{aligned} \tag{16}$$

We have

$$U\{\tilde{\mathbf{t}} - \mathbf{t}\} \geq 0\ , \tag{17}$$

since $\mathbf{K}$ is a positive definite symmetric tensor field. From equations (15), (16) and inequality relation (17) we immediately obtain the statements formulated in Theorem 2. $\square$

Theorem 3. *If the admissible states $s = [T, \mathbf{t}, q]$ and $\tilde{s} = [\tilde{T}, \tilde{\mathbf{t}}, \tilde{q}]$ corresponding to $p = [R, q]$ and $\tilde{p} = [\tilde{R}, \tilde{q}]$ satisfy the following conditions*

$$R = \tilde{R} \qquad \textit{in } V \ , \tag{18}$$

$$T = \textit{arbitrary constant} \qquad \textit{on } \partial V_1 \ , \tag{19}$$

$$q = \tilde{q} \qquad \textit{on } \partial V_2 \ , \tag{20}$$

where ∂V_1 and ∂V_2 are complementary surface segments of ∂V such that $\partial V = \partial V_1 \cup \partial V_2$ and $\partial V_1 \cap \partial V_2 = \{\emptyset\}$, then

$$U\{\mathbf{t}\} \leq U\{\tilde{\mathbf{t}}\} \ . \tag{21}$$

Proof. We have, according to the global heat balance equation,

$$\int_{\partial V_1} q \ \mathrm{d}a = \int_V R \ \mathrm{d}v - \int_{\partial V_2} q \ \mathrm{d}a \ , \qquad \int_{\partial V_1} \tilde{q} \ \mathrm{d}a = \int_V \tilde{R} \ \mathrm{d}v - \int_{\partial V_2} q \ \mathrm{d}a \ . \tag{22}$$

From equations (18), (20) and (22) it follows that

$$\int_{\partial V_1} (\tilde{q} - q) \ \mathrm{d}a = 0 \ . \tag{23}$$

By the use of equations (20) and (23) we can write

$$\int_{\partial V} T(\tilde{q} - q) \ \mathrm{d}a = T \int_{\partial V_1} (\tilde{q} - q) \ \mathrm{d}a + \int_{\partial V_2} T(\tilde{q} - q) \ \mathrm{d}a = 0 \ . \tag{24}$$

Substitution of equations (20) and (24) into relation (11) we obtain that the statement formulated in Theorem 3 is a direct consequence of Theorem 2. □

4. **Mean heat flux vector**

We define the mean heat flux vector $\langle \mathbf{q} \rangle$ corresponding to an admissible state $s = [T, \mathbf{t}, \mathbf{q}]$ and $p = [R, q]$ as

$$\langle \mathbf{q} \rangle = \frac{1}{V} \int_V \mathbf{q} \ \mathrm{d}v \ . \tag{25}$$

Theorem 4. *The mean heat flux vector of the admissible state corresponding to internal heat source field R and surface heat flux q can be expressed as*

$$\langle \mathbf{q} \rangle = \frac{1}{V} \left(\int_{\partial V} \mathbf{p} q \ \mathrm{d}a - \int_V \mathbf{p} R \ \mathrm{d}v \right) \ . \tag{26}$$

Proof. Be

$$\tilde{T} = \boldsymbol{\alpha} \cdot \mathbf{p} \tag{27}$$

in equation (6) where $\boldsymbol{\alpha}$ is a constant vector. A simple computation gives

$$\tilde{R} = -\nabla \cdot \mathbf{K} \cdot \boldsymbol{\alpha} \qquad \text{in } V \ , \tag{28}$$

$$\tilde{q} = -\mathbf{n} \cdot \mathbf{K} \cdot \boldsymbol{\alpha} \qquad \text{on } \partial V \ , \tag{29}$$

$$\int_{\partial V} \tilde{T} q \ \mathrm{d}a - \int_V \tilde{T} R \ \mathrm{d}v = \boldsymbol{\alpha} \cdot \left(\int_{\partial V} \mathbf{p} q \ \mathrm{d}a - \int_V \mathbf{p} R \ \mathrm{d}v \right) \ , \tag{30}$$

$$\int_{\partial V} T\tilde{q}\,\mathrm{d}a - \int_V T\tilde{R}\,\mathrm{d}v = -\int_{\partial V} T\mathbf{n}\cdot\mathbf{K}\cdot\boldsymbol{\alpha}\,\mathrm{d}a + \int_V T\nabla\cdot\mathbf{K}\cdot\boldsymbol{\alpha}\,\mathrm{d}v$$
$$= -\int_{\partial V} T\mathbf{n}\cdot\mathbf{K}\cdot\boldsymbol{\alpha}\,\mathrm{d}a + \int_{\partial V} T\mathbf{n}\cdot\mathbf{K}\cdot\boldsymbol{\alpha}\,\mathrm{d}v + \int_V \boldsymbol{\alpha}\cdot\mathbf{K}\cdot\mathbf{t}\,\mathrm{d}v$$
$$= \boldsymbol{\alpha}\cdot\int_V \mathbf{q}\,\mathrm{d}v\,. \quad (31)$$

In the derivation of equation (31) we have used the rule of differentiation of product function, divergence theorem and equation (4). Combination of equation (30) with equation (31) gives the formula of mean heat flux vector.

We note that formula (26) can be derived only by the use of equations (3) and (5). It is not necessary for $\mathbf{q} = \mathbf{q}(x, y, z)$ in (26) to be the solution of a heat conductance problem [2]. If $\mathbf{q} = \mathbf{q}(x, y, z)$ is a solution of a steady-state heat conduction problem, then it satisfies

$$\nabla\times\mathbf{R}\cdot\mathbf{q} = \mathbf{0} \qquad \text{in } V \qquad (32)$$

where $\mathbf{R}$ is the inverse tensor of $\mathbf{K}$ (the thermal resistivity tensor [7, 1] $\mathbf{R}\cdot\mathbf{K} = \mathbf{1}$, $\mathbf{1}$ is the unit tensor). The cross between two vectors in equation (32) denotes their vectorial product according to Lurje [4] and Malvern [5]. Compatibility conditions for $\mathbf{q}$ given by (32) are obtained from equations (1), (4). □

5. Heat conduction on non-homogeneous curved beam

Consider a curved beam (Figure 1) which is an incomplete torus in the 3D space. A torus-like body is generated by the rotation of a plane figure about axis z whose inner and boundary points are taken from the sets A and ∂A, respectively. The domain $\bar{A} = A\cup\partial A$ is bounded and called the cross-section of curved beam. The curved beam occupies the region $\bar{V} = V\cup\partial V$; $V = \{(r,\varphi,z)|(r,z)\in A,\ 0<\varphi<\alpha\}$, $\partial V = A_1\cup A_2\cup A_3$, $A_i = \{(r,\varphi,z)|(r,z)\in A \quad \varphi=\varphi_i \quad (i=1,2) \quad \varphi_1 = 0, \varphi_2 = \alpha\}$, $A_3 = \{(r,\varphi,z)|(r,z)\in\partial A \quad 0\le\varphi\le\alpha\}$, which is referred to a given cylindrical coordinate system $Or\varphi z$. Unit vectors of the cylindrical coordinate system $Or\varphi z$ are $\mathbf{e}_r = \mathbf{e}_x\cos\varphi + \mathbf{e}_y\sin\varphi$, $\mathbf{e}_\varphi = \mathbf{e}_z\times\mathbf{e}_r$ and $\mathbf{e}_z$. The polar coordinates r and φ are defined as $r = \sqrt{x^2+y^2}$, $\tan\varphi = y/x$. The incomplete torus-like body (curved beam) is isotropic and φ-homogeneous. This means that

$$\mathbf{K} = k(r,z)\ \mathbf{1}\,, \qquad (33)$$

where $k = k(r,z)$ is the thermal conductivity of curved beam (incomplete torus, $0<\alpha<2\pi$), which may depend upon the cross-sectional coordinates r and z. The following boundary-value problem of the steady-state heat conduction is analysed:

$$T(r,0,z) = \vartheta_1(r,z) \qquad \text{on } A_1 \qquad (\vartheta_1(r,z) \text{ is given function on } A_1)\,, \qquad (34)$$
$$T(r,\alpha,z) = \vartheta_2(r,z) \qquad \text{on } A_2 \qquad (\vartheta_2(r,z) \text{ is given function on } A_2)\,, \qquad (35)$$
$$\mathbf{q}\cdot\mathbf{n} = q_3(r,\varphi,z) \qquad \text{on } A_3 \qquad (q_3(r,\varphi,z) \text{ is given function on } A_3)\,, \qquad (36)$$

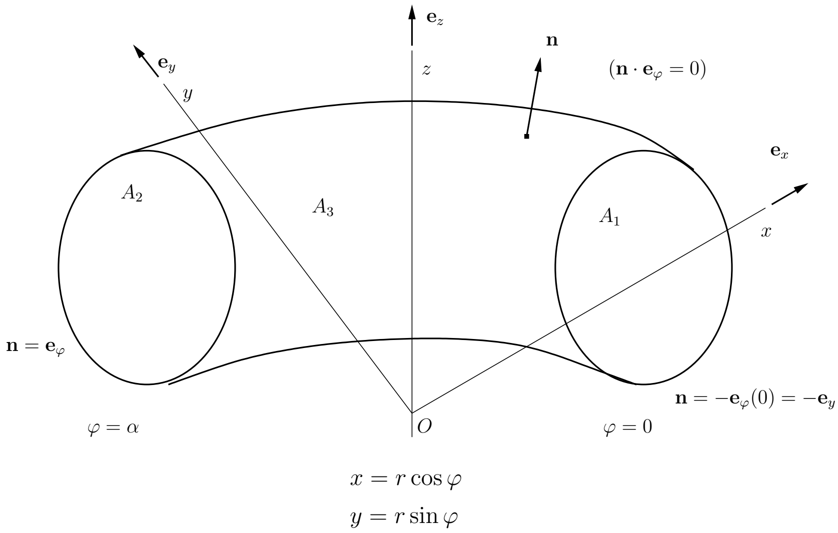

Figure 1. Incomplete non-homogeneous torus (curved beam)

furthermore $R = R(r, \varphi, z)$ is specified in V. Our aim is to obtain the values of heat flux resultants Q_1 and Q_2 which are defined as

$$Q_1 = \int_{A_1} \mathbf{q}\cdot\mathbf{n}_1 \,\mathrm{d}a = -\int_{A_1} \mathbf{q}\cdot\mathbf{e}_\varphi \,\mathrm{d}a\ , \tag{37}$$

$$Q_2 = \int_{A_2} \mathbf{q}\cdot\mathbf{n}_2 \,\mathrm{d}a = \int_{A_2} \mathbf{q}\cdot\mathbf{e}_\varphi \,\mathrm{d}a\ . \tag{38}$$

Here, we note (Figure 1) $\mathbf{n} = \mathbf{n}_1 = -\mathbf{e}_\varphi$ on A_1, $\mathbf{n} = \mathbf{n}_2 = \mathbf{e}_\varphi$ on A_2 and $\mathbf{n} = \mathbf{n}_3$ on A_3, $\mathbf{n}_3\cdot\mathbf{e}_\varphi = 0$. The global heat balance equation for the incomplete torus is formulated as

$$Q_1 + Q_2 + Q_3 - Q_v = 0\ , \tag{39}$$

where

$$Q_3 = \int_{A_3} q_3 \,\mathrm{d}a = \int_0^\alpha \left(\oint_{\partial A} r q_3 \,\mathrm{d}\sigma\right) \mathrm{d}\varphi\ , \tag{40}$$

$$Q_v = \int_0^\alpha \left(\int_A rR \,\mathrm{d}A\right) \mathrm{d}\varphi\ . \tag{41}$$

In equation (40), σ is the arc-length defined on the boundary curve of A. The first equation, which we will use to determine the heat flux resultants Q_1 and Q_2, is equation (39) and the second one will be derived from the reciprocal relation (6). Let the state $s = [T, \mathbf{t}, \mathbf{q}]$ be the solution of the heat conduction problem of the curved beam specified by boundary conditions (34), (35), (36) with the prescribed internal heat source $R = R(r, \varphi, z)$. The second state of steady heat conduction for the curved beam shown in Figure 1 is given by the following equations

$$\tilde{T} = C\varphi\ , \qquad \tilde{\mathbf{t}} = -\frac{C}{r}\mathbf{e}_\varphi\ , \qquad \tilde{\mathbf{q}} = -C\frac{\lambda(r,z)}{r}\mathbf{e}_\varphi\ ,$$

$$\tilde{R} = \nabla\cdot\tilde{\mathbf{q}} = 0\ , \qquad \tilde{q} = -C\frac{\lambda(r,z)}{r}\mathbf{e}_\varphi\cdot\mathbf{n} \qquad \text{on } A = A_1\cup A_2\cup A_3\ ,$$

where C is a constant different from zero. It is very easy to show that

$$\int_{\partial V}\tilde{T}q\ \mathrm{d}a - \int_V \tilde{T}R\ \mathrm{d}V = C(I_1 - I_2 + \alpha Q_2)\ , \tag{42}$$

$$\int_{\partial V} T\tilde{q}\ \mathrm{d}a - \int_V T\tilde{R}\ \mathrm{d}V$$
$$= -C\left(\int_{A_2}\frac{\lambda(r,z)}{r}\ \vartheta_2(r,z)\ \mathrm{d}a - \int_{A_1}\frac{\lambda(r,z)}{r}\ \vartheta_1(r,z)\ \mathrm{d}a\right)\ . \tag{43}$$

Here,

$$I_1 = \int_{A_3}\varphi q_3\ \mathrm{d}a = \int_0^\alpha\oint_{\partial A} r\varphi q_3\ \mathrm{d}\sigma\ \mathrm{d}\varphi\ , \tag{44}$$

$$I_2 = \int_V \varphi R\ \mathrm{d}V = \int_0^\alpha\oint_{\partial A} r\varphi R\ \mathrm{d}a\ \mathrm{d}\varphi\ . \tag{45}$$

Substitution of equations (42) and (43) into reciprocal relation (6) yields

$$Q_2 = -\frac{1}{\alpha}\left(\int_{A_2}\frac{\lambda(r,z)}{r}\ \vartheta_2(r,z)\ \mathrm{d}a - \int_{A_1}\frac{\lambda(r,z)}{r}\ \vartheta_1(r,z)\ \mathrm{d}a\right) + \frac{I_2 - I_1}{\alpha}\ . \tag{46}$$

Formula (46) gives the value of heat flux resultant Q_2 without knowing the solution of the corresponding 3D heat conduction problem of the incomplete torus.

6. Mean temperature

Let B be a homogeneous solid sphere. The domain $\bar{V}$ occupied by B is

$$\bar{V} = \{\mathbf{p} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z \mid p^2 - \varrho^2 \le 0\}\ ,$$

where ϱ is the radius of the bounding spherical surface. The following boundary value problem of heat conduction is considered:

$$R(x,y,z) = R(\mathbf{p}) \qquad \text{is prescribed in } V\ , \tag{47a}$$

$$T(x,y,z) = \vartheta(x,y,z) \qquad \text{on } \partial V \qquad (\vartheta(x,y,z) \text{ is given function on } \partial V)\ . \tag{47b}$$

It is obvious that equation (47b) formulates a Dirichlet's type boundary condition. The position vector of a point on the spherical surface ∂V is denoted by $\boldsymbol{\varrho}$. Our purpose is to compute the mean value of the temperature field of a solid sphere without knowing the solution of the boundary value problem determined by the prescriptions mentioned above. We use the reciprocal relation (6). The first admissible state is the solution of the heat conduction problem specified by equations (47a) and (47b). The second admissible state is given by the following equations

$$\tilde{T} = \frac{C}{2}(\varrho^2 - p^2) \,, \qquad \tilde{\mathbf{t}} = C\mathbf{p} \quad \text{on } \bar{V} \,, \tag{48}$$

$$\tilde{\mathbf{q}} = C\mathbf{K}\cdot\mathbf{p} \quad \text{on } \bar{V} \,, \tag{49}$$

$$\tilde{q} = C\; k(\boldsymbol{\varrho}) \quad \text{on } \partial V \qquad \tilde{R} = CK_I \quad \text{in } V \,. \tag{50}$$

Here,

$$k(\boldsymbol{\varrho}) = \frac{\boldsymbol{\varrho}\cdot\mathbf{K}\cdot\boldsymbol{\varrho}}{\varrho} \qquad \text{defined on } \partial V \,, \qquad K_I = \mathbf{K}\cdot\cdot\mathbf{1} \,, \tag{51}$$

K_I is the first scalar invariant of the conductivity tensor and the double dot denotes the double dot product of $\mathbf{K}$ and $\mathbf{1}$ according to Malvern [5] and Lurje [4], and we note that $\mathbf{K}$ is constant tensor. Substitution of the fields of two chosen admissible states into formula (6) gives the result

$$\langle T\rangle = \frac{3}{4\pi K_I \varrho^3}\left[\int_{\partial V} k(\boldsymbol{\varrho})\;\vartheta(\boldsymbol{\varrho})\;\mathrm{d}a + \frac{\varrho^2}{2}\int_V R(\boldsymbol{\varrho})\;\mathrm{d}V - \frac{1}{2}\int_V p^2\;R(\mathbf{p})\;\mathrm{d}V\right] \,. \tag{52}$$

In equation (52), the mean temperature field $\langle T\rangle$ in the sphere is defined as

$$\langle T\rangle = \frac{3}{4\pi\varrho^3}\int_V T(\mathbf{p})\;\mathrm{d}V \,. \tag{53}$$

7. Conclusions

In this paper, a reciprocal theorem is formulated by the use of the analogy which exists between linear elasticity and heat conduction. The formalism of applied analogy follows Wojnar's approach [7]. The theorems proven are analogous to those obtained in linear elasticity theory by Gurtin [3].

Acknowledgement. The support provided by the Hungarian National Research Foundation within the framework of projects T049115 and T037759 is gratefully acknowledged.

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