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COMPATIBILITY PATHS OF AN INFINITELY DEGENERATE MECHANISM

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Abstract. The kinematic determinacy of a bar-and-joint mechanism is dependent on the topology and the metric properties of the structure. Special arrangements can result in singularities, such as bifurcation points on the compatibility paths. The analysis of a special four-bar linkage yields an infinitely degenerate bifurcation point. Modifications in plane and in space are investigated in order to produce the most general perturbations of the system. The compatibility path can be perturbed to have a shape of arbitrary finite order or to result in compatible states in an arbitrary finite number.

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1. **Introduction**

A mechanism consisting of rigid bars with a prescribed length and a given topology is called a mechanism with a single degree-of-freedom if it typically has compatible positions where, applying a suitable displacement to a suitable element, the displacement of the other elements can be uniquely determined. Compatibility paths form a set of points which belong to compatible positions in the space of the state variables chosen to define the position of the mechanism. The compatibility paths usually consist of lines which can intersect one another (bifurcation points).

Bifurcation points arise from special geometric configurations. If a mechanism is created with a special geometry, it may have certain positions where the number of instantaneous kinematic degrees-of-freedom increases. At these points the mechanism can change shape and continue its motion along a different path.

Bifurcations of compatibility paths have been studied by several researchers. Tarnai [1] and Litvin [2] have shown mechanisms producing asymmetric bifurcations. Lengyel and You [3] discussed further examples and compared this phenomenon to the well-known equilibrium bifurcations of elastic structures. They made further examinations with the aid of the elementary catastrophe theory [4]. Their method was based on the analogy between equilibrium and compatibility equations.

In paper [5] we proposed a general classification of the points of the compatibility paths. With the aid of an energy function we showed a connection between singularities and catastrophe types in Thom's theorem [6]. A catastrophe point is a singularity of the system where a small change of the parameter(s) may cause significantly different behavior according to the perturbation. Consequently the number of solutions of the governing equations may change, i.e. in this case the number of possible compatible states. Hence at catastrophe points one may apply small perturbations to the original layout of the system so that a different number of states is obtained [5]. The maximum number of the compatible configurations forms the basis of the classification of the singular points.

In paper [5] we also showed a degenerate planar mechanism that produced infinite compatible positions at the critical value of the control parameter. This paper examines the mechanism and presents a perturbation that reduces the degeneracy to arbitrary finite order. The layout of the paper is as follows. The second part introduces the basic structure. The third part defines the geometrical modifications of the system and the mathematical formulation is given in the fourth.

2. Basic structure

Consider a special four-bar linkage with all bars having unit length shown in Figure 1a. The compatibility paths of the mechanism consist of straight lines plotted in the coordinate system

$$(\alpha,\, \beta)$$

as shown in Figure 1b [4, 5]. Each value of β corresponds to a certain position of node B on the perimeter of a unit circle centered at O_B. Geometrically possible positions

Figure 1. Four-bar mechanism (a) basic structure (b) compatibility path

of node A must be at a unit distance from both O_A and B, and hence compatible configurations of the mechanism are obtained at the intersections of a unit circle centered at O_A (set S_1) and another one centered at B (set S_2).

The behaviour of the points of the compatibility paths is determined by the properties of the intersections, which can be further studied by examining the transversality of the sets [7]. Let X and Y be affine subspaces of $\mathbb{R}^n$ of dimensions s, t, respectively, where $\mathbb{R}$ denotes the set of real numbers. They meet transversely if either their intersection $X \cap Y$ is empty or its dimension is $s + t - n$ (if this number is non-negative). Two submanifolds of $\mathbb{R}^n$ meet transversely at a given point provided either they do not meet, or their tangent affine hyperplanes meet transversely.

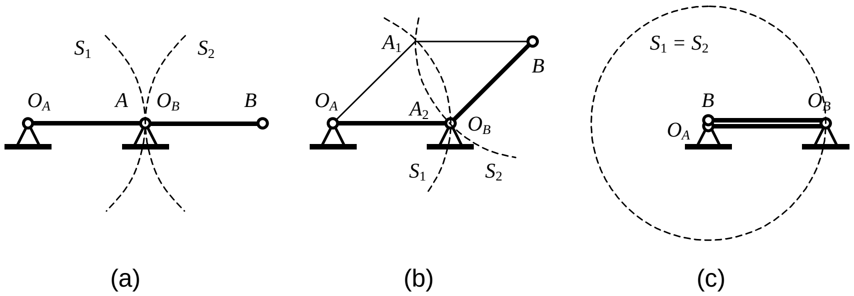

Figure 2. Compatible configurations at different values of parameter β (a) $\beta = 0$ (b) $0 < \beta < \pi$ (c) $\beta = \pi$

At $\beta = 0$ only one compatible configuration exists as the two circles have one common point (see Figure 2a). The tangents are common, hence the intersection is not transverse. At $0 < \beta < \pi$ two intersections are obtained: $\alpha = 0$ and $\alpha = \beta$ (see Figure 2b). Here the two sets meet transversely because the tangents are not parallel. At $\beta = \pi$ node B and O_A coalesce and the two circles become the same (see Figure 2c). This is again a non-transverse intersection. Here all values of α correspond to compatible configurations. Thus at the bifurcation point

$$(\alpha = 0,\ \beta = \pi)$$

there is an infinitely degenerate compatibility path.

A small perturbation of the system modifies the sets S_1 and S_2 and, consequently, the number of compatible positions of the mechanism. The first of the three cases above is non-transverse. A small perturbation of the geometry, such as imperfections of lengths of the bars or of positions of supports, would make the circles detach or intersect at two points transversely (Figure 3a). In the second case the circles meet at two points transversely and the transversality is unchanged by a small perturbation (Figure 3b). In the third case the two circles coalesce and a small perturbation can reduce the number of intersections to two, one, or none (Figure 3c). The three cases are denoted by dashed, dotted and dashdot lines, respectively.

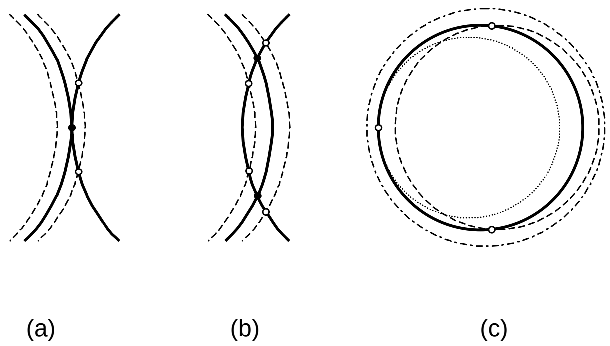

Figure 3. Transversality of intersections of two circles

Planar imperfections can modify the behaviour to produce one or two compatible configurations or none. Our goal is to create a non-transverse intersection that can be modified by a suitable small perturbation so that the number of compatible configurations is any arbitrarily chosen number. In order to obtain this, we introduce a spatial structure.

3. **Spatial configuration**

Consider the mechanism in Figure 1a with unchanged topology but with spherical joints applied at nodes O_A, A and B. Also keep the revolute joint at O_B. The mechanism has two independent degrees-of-freedom by means of these modifications. Sets S_1 and S_2 become unit spheres in the

$$(X, Y, Z)$$

coordinate system shown in Figure 4a. The two spheres typically meet at transverse points forming a circle $C_1 = S_1 \cap S_2$. At point $\beta = \pi$ the two spheres coalesce and all points are non-transverse intersections (Figure 4b): $C_1 = S_1 = S_2$.

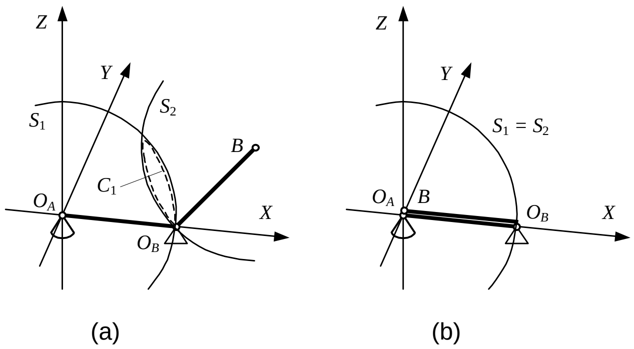

Figure 4. Spatial configuration (a) $0 < \beta < \pi$ (b) $beta = \pi$

In order to obtain a mechanism with a single degree-of-freedom, the number of degree-of-freedom has to be reduced by introducing an additional support. Let now node A be supported by a roller allowing it to move on a given smooth surface

$$S_3 = S_3\,(X,\,Y)$$

(Figure 6). Hence now the motion of node A is restricted to the intersection set $C_2 = S_1 \cap S_3$, i.e. the common points of the sphere S_1 and the surface S_3, which is a curve on the surface of sphere S_1. Compatible positions can be obtained only at points which are on curve C_1 as well because A has to be on S_2, the circle centered at B. The set of compatible positions is now defined as

$$P = C_1 \cap C_2 = S_1 \cap S_2 \cap S_3\ .$$

Now in case of $\beta = \pi$ it is possible to apply appropriate imperfections to the structure so that the set of intersections is reduced to a required finite number. Let

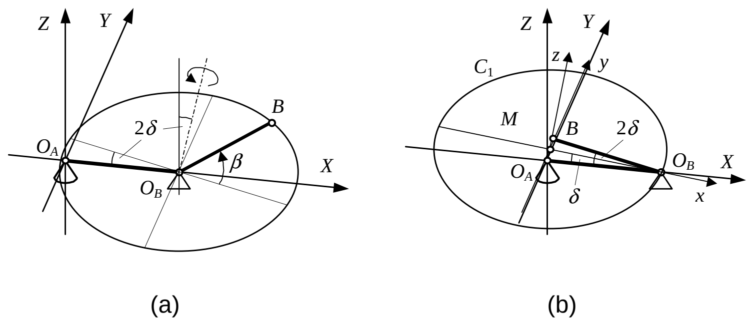

Figure 5. Imperfection of joint O_B (a) tilting angle (b) intersection circle

us do this in a number of steps as follows. First let the axis of rotation of joint O_B be tilted from Z by a small angle 2δ anticlockwise in plane $(X,\,Z)$ as is shown in Figure 5a. Node B at $\beta = \pi$ is out of the plane $(X,\,Y)$:

$$B\,(1 - \cos 2\delta,\, 0,\, \sin 2\delta)\ .$$

Hence the intersection of the two spheres is a circle: $C_1 = S_1 \cap S_2$. It is centered at M, the middle point of section O_AB and is fitting to point O_B (Figure 5b), hence its plane is tilted from $(X,\,Y)$ by angle δ. It is then possible to define the surface S_3 so that C_1 and C_2 meet non-transversely at O_B and they produce a required number of intersections.

In order to do that, let us define the final shape of C_2 in a few steps. First construct S_3 so that C_2 be equal to C_1. Then rotate points $P \in C_2$ around axis MO_B by an angle which is not constant but proportional to the nth order of the sine of the arc length O_BP. It adds an nth order perturbation to circle C_2. The first perturbation term is shown in Figure 7a. If a suitable linear combination of these terms of various orders is applied, then an oscillating curve is obtained which has n intersections with

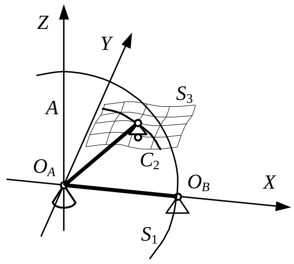

Figure 6. Perturbation of motion A

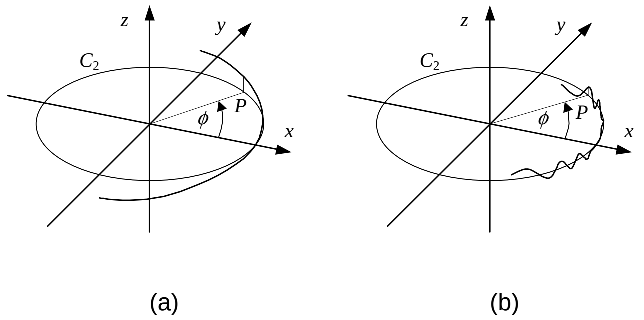

Figure 7. Perturbation (a) single term (b) combination

C_1 in the vicinity of O_B (Figure 7b). If the highest order term is the nth, a maximum of n intersections can be obtained by suitable imperfections. These are the only points referring to compatible configurations of the mechanism.

4. **Transformations**

The mathematical formulation of the principles above is as follows. C_1 is a circle centered at

$$M\left(\left(1-\cos 2\delta\right)/2,\ 0,\ \sin 2\delta/2\right)$$

with radius $r = \cos\delta$. A local

$$(x,\ y,\ z)$$

coordinate system is created so that x and y cover the plane of C_1 and x is aligned with MO_B as shown in Figure 5b. In this local system the points of the circle $c \in C_1$ can be defined as

$$c = \left(r\cos\varphi,\ r\sin\varphi,\ 0\right)^{\mathrm{T}}$$

where angle φ measures the $O_B P$ arc length. An ith order rotational perturbation around axis x mentioned above is given by the transformation matrix:

$$T_i = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\left(p_i \sin^i \varphi\right) & -\sin\left(p_i \sin^i \varphi\right) \\ 0 & \sin\left(p_i \sin^i \varphi\right) & \cos\left(p_i \sin^i \varphi\right) \end{bmatrix}, \tag{1}$$

where p_i is a suitable constant and i is a positive integer number. A linear combination of these terms gives the final shape of the curve C_2 in the local (x, y, z) coordinate system:

$$\mathbf{c}_2^{xyz} = \sum_i \mathbf{T}_i \mathbf{c} = \sum_i \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\left(p_i \sin^i \varphi\right) & -\sin\left(p_i \sin^i \varphi\right) \\ 0 & \sin\left(p_i \sin^i \varphi\right) & \cos\left(p_i \sin^i \varphi\right) \end{bmatrix} \begin{bmatrix} r\cos\varphi \\ r\sin\varphi \\ 0 \end{bmatrix}. \tag{2}$$

In the global coordinate system (X, Y, Z) a rotation and a translation yield the equation

$$\mathbf{c}_2^{XYZ} = \mathbf{T}\mathbf{c}_2^{xyz} + \mathbf{v}\,, \tag{3}$$

where the rotational matrix $\mathbf{T}$ and the translation vector $\mathbf{v}$ define the transformation from (x, y, z) to (X, Y, Z):

$$\mathbf{T} = \begin{bmatrix} \cos\delta & 0 & \sin\delta \\ 0 & 1 & 0 \\ -\sin\delta & 0 & \cos\delta \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} (1-\cos 2\delta)/2 \\ 0 \\ (\sin 2\delta)/2 \end{bmatrix}. \tag{4}$$

5. Conclusions

In this paper we have examined how the compatibility of an infinitely degenerate mechanism can be modified by suitable imperfections of the geometry. Discrete geometric imperfections of the structure, e.g. a constant error of the length of the bars or of the position of the supports can only add a limited (second order) variation to the degenerate compatibility path as not more than two possible compatible positions can be obtained at a certain value of the parameter. Therefore, in order to exploit the potential of the degeneracy of the geometry, a spatial modification has been introduced incorporating a continuum perturbation of the system. Forcing the motion of node A on a smooth continuous surface above the coordinate plane allows us to define a compatibility path of arbitrary shape in the neighborhood of the bifurcation point.

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