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RELATIONS FOR THE TORSION OF NONHOMOGENEOUS CYLINDRICAL BARS

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Abstract. The present paper refers to the torsion of cylindrical bars, the cross section of which is a simple or multiply connected plane domain. Examples illustrate the application of the Bai-Shield's identity. A lower bound relation is presented for the greatest shearing stress developed in the twisted cylindrical bar and an upper bound relation is proven for the plastic limit torque. Three types of the upper bound formulae are derived for the torsional rigidity of nonhomogeneous isotropic elastic bars.

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1. Introduction

Consider a bar bounded by a cylindrical surface (”side-surface”) and two planes (”end cross sections”), normal to the side surface. For greater generality, it is assumed that the bar under consideration may contain longitudinal cylindrical cavities so that the cross-section of the bar may be multiply connected. Further assumptions are that there are no body forces present, that the side surface of the bar is free from external stresses and that given forces (satisfying the equilibrium conditions of the body as whole) are shearing stresses applied to the end cross sections of the bar. We also suppose that the bar is composed of a material which is homogeneous in the axial direction.

A three-dimensional rectangular Cartesian coordinate system (x, y, z) will be used. The axis Oz is directed parallel to the generators of the side surface and the plane Oxy is chosen to coincide with the "lower" end of the bar. The "upper" end of the bar will then have the coordinate $z = L$, where L is the length of the bar.

Following Bai and Shield [1], we suppose that τ_{xz} and τ_{yz} are the only nonzero stresses in the whole bar. In this case the equilibrium conditions can be formulated as [5-6, 9]:

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = 0 \quad \text{in } A \quad , \tag{1.1}$$

$$\tau_{xz} n_x + \tau_{yz} n_y = 0 \quad \text{on } c \quad , \tag{1.2}$$

where A is the cross-section of the cylindrical bar, c is the boundary of A and n_x, n_y

are the components of the outward unit normal to the curve c.

It follows from equations (1.1) and (1.2) that [1–2, 5]

$$X = \int_A \tau_{xz} dA = 0 \qquad \text{and} \qquad Y = \int_A \tau_{yz} dA = 0\,, \tag{1.3}$$

that is, there are no transverse forces on the cross-section of the bar.

The only moment acting on a cross-section is a twisting moment T given by

$$T = \int\limits_A (x\tau_{yz} - y\tau_{xz}) dA\,. \tag{1.4}$$

Bai and Shield proved [1] that each of the rectangular components of shearing stress provides one half of the twisting moment

$$\int_A x\tau_{yz}\mathrm{d}A = -\int_A y\tau_{xz}\mathrm{d}A = \frac{T}{2}\,. \tag{1.5}$$

Equation (1.5) is valid both for simply connected cross-sections and and for multiply connected ones. It is also independent of any material properties provided that the material properties depend on the cross-sectional coordinates x, y only.

2. Lower bound for the shearing stress

Let τ be the greatest shearing stress in cross-section A of a cylindrical bar subjected to a twisting moment T. We have

$$\tau = \max\sqrt{\tau_{xz}^2 + \tau_{yz}^2} \quad , \quad (x, y) \in \overline{A} = A \cup c \quad . \tag{2.1}$$

Regrading equation (1.5) as a point of departure and using the Schwarz inequality we can write that

$$\int_A x^2\mathrm{d}A \int_A \tau_{yz}^2\mathrm{d}A \geq \frac{T^2}{4} \qquad \text{and} \qquad \int_A y^2\mathrm{d}A \int_A \tau_{xz}^2\mathrm{d}A \geq \frac{T^2}{4}\,. \tag{2.2}$$

A combination of inequality

$$\int_A (\tau_{xz}^2 + \tau_{yz}^2)\mathrm{d}A \leq \tau^2 A \tag{2.3}$$

with inequalities $(2.2)_{1,2}$ results in the following lower bound

$$\frac{\tau}{T} \geq \sqrt{\frac{I_x + I_y}{4 I_x I_y A}}\,, \tag{2.4}$$

where I_x and I_y are the second moments of the cross-section about the axes x and y, respectively, and A is the area of the cross-section.

3. Upper bound for the limit plastic torque

Let us assume that the material of the cylindrical body is elastic-perfectly plastic. In the case of fully plastic torsion we have

$$\tau_{xz}^2+\tau_{yz}^2=\tau_0^2 \quad \text{in } A\cup c\,, \tag{3.1}$$

where $\tau_0=\tau_0(x,y)$ is the yield stress in pure shear, which may depend on the cross-sectional coordinates x,y. Let T_0 be the plastic torque of the cross section [4,7]. The constant A_0 is defined by

$$A_0=\int_A \tau_0^2 \mathrm{d}A\,. \tag{3.2}$$

Making use of equations $(2.2)_{1,2}$ and the Huber-v. Mises-Hencky yield condition (3.1) we obtain the following upper bound

$$T_0\le\sqrt{\frac{4I_xI_yA_0}{I_x+I_y}}\,. \tag{3.3}$$

Remarks to relations (2.4) and (3.3)

R1. Relation (2.4) is an equality for a thin-walled circular tube with constant thickness.

R2. Relation (3.3) is also an equality for a *homogeneous* thin-walled circular tube with constant thickness.

R3. It can be proved that relation (2.4) leads to the best lower bound for τ_o and formula (3.3) gives the sharpest upper bound for T_o if the axes x,y are principal centroidal axes of the cross-section [3].

4. Upper bound for the torsional rigidity

In this section it is assumed that the material of the twisted bar is inhomogeneous isotropic elastic, the equilibrium state of the bar is the pure torsion according to Saint-Venant's theory [3], [5-6]. A consequence of the nonhomogenity is that the shear modulus G may depend on x,y that is $G=G(x,y)$.

Once again we regard equation (1.5) as our point of departure and use the Schwarz inequality. We get

$$\frac{T^2}{4}=\left(\int_A x\tau_{yz}dA\right)^2=\left[\int_A\left(\frac{\tau_{yz}}{\sqrt{G}}\right)\left(x\sqrt{G}\right)dA\right]^2\le\int_A\frac{\tau_{yz}^2}{G}dA\int_A Gx^2dA \quad , \tag{4.1a}$$

$$\frac{T^2}{4}=\left(\int_A y\tau_{xz}dA\right)^2=\left[\int_A\left(\frac{\tau_{xz}}{\sqrt{G}}\right)\left(y\sqrt{G}\right)dA\right]^2\le\int_A\frac{\tau_{xz}^2}{G}dA\int_A Gy^2dA \quad . \tag{4.1b}$$

The strain energy stored in the unit length of the twisted bar [5-7], [9] is given by

$$U = \int_A \frac{\tau_{xz}^2 + \tau_{yz}^2}{2G} dA \quad . \tag{4.2}$$

The torque-twist relation is of the form

$$T = R\vartheta \,, \tag{4.3}$$

where R is the torsional rigidity of the cross-section and ϑ is the rate of twist [5-6], [9].

We shall consider a unit length of the bar. In this case

$$W = \frac{1}{2} T\vartheta = \frac{T^2}{2R} \tag{4.4}$$

is the work done by the twisting moment T. According to the Clapeyron theorem [9] we can write

$$U = \frac{T^2}{2R} \,. \tag{4.5}$$

Combination of equations (4.1a,b) with formula (4.2) gives

$$U \geq T^2 \frac{J_x + J_y}{8 J_x J_y} \,, \tag{4.6}$$

where

$$J_x = \int_A G y^2 dA \qquad \text{and} \qquad J_y = \int_A G x^2 dA \tag{4.7}$$

are the G-weighted second moments of the cross-section about the centroidal axes x and y, respectively. Inserting equations (4.1a,b) into inequality (4.6) we obtain the upper bound

$$R \leq \frac{4 J_x J_y}{J_x + J_y} \,. \tag{4.8}$$

Let J_0 be defined as

$$J_0 = J_x + J_y = \int_A G(x^2 + y^2) dA \,. \tag{4.9}$$

It is clear that

$$(J_x - J_y)^2 = (J_x + J_y)^2 - 4 J_x J_y \geq 0 \tag{4.10}$$

from which we get the lower bound

$$J_0 \geq \frac{4 J_x J_y}{J_x + J_y} \quad . \tag{4.11}$$

Relations (4.8) and (4.11) show that

$$R \leq J_0 \,. \tag{4.12}$$

The upper bound (4.12) is weaker than the upper bound (4.8).

Combination of inequalities (4.1a,b) with the lower bounds

$$2U \geq \int_A \frac{\tau_{xz}^2}{G} dA \qquad \text{and} \qquad 2U \geq \int_A \frac{\tau_{yz}^2}{G} dA$$

and equation (4.5) yields the Grammer type upper bound

$$R \leq \min\{4J_x\,,\, 4J_y\} \tag{4.13}$$

for the torsional rigidity of nonhomogeneous cylindrical bars. This estimation is used mainly for narrow rectangular cross-sections [3].

Remarks to relations (4.8), (4.12) and (4.13):

R1. For a homogeneous bar, estimation (4.8) was first derived by Nicolai [8].

R2. For a homogeneous bar the upper bound (4.12) was deduced from the theory of Saint-Venant by Diaz and Weinstein [2].

R3. It can be proved that inequality (4.12) gives the best upper bound if the origin of the cross-sectional coordinate system is chosen in such a way [3] that the equations

$$\int\limits_A xG(x,y)dA = 0 \qquad \text{and} \qquad \int\limits_A yG(x,y)dA = 0 \tag{4.14}$$

hold.

R4. It is proved in [3] that inequalities (4.8) and (4.13) lead to the best upper bound if the origin and the direction of the axes of the cross-sectional coordinate system x, y are chosen in such a way [3] that equations (4.14) and equation

$$\int\limits_A xyG(x,y)dA = 0 \tag{4.15}$$

are all satisfied.

R5. Relations (4.8) and (4.12) are equalities if the cross-section is bounded by two concentric circles on which

$$x^2 + y^2 = a_1^2 \qquad \text{and} \qquad x^2 + y^2 = a_2^2$$

and the shear modulus depends only on the radial coordinate $r = \sqrt{(x^2 + y^2)}$. Here, a_1 and a_2 are the radii of the boundary circles.

5. Conclusions

Some applications of the Bai-Shield identity for the torsion of a nonhomogeneous cylindrical bar have been presented. A lower bound is derived for the greatest shearing stress developed in twisted cylindrical bars and an upper bound is set up for the plastic limit torque. Three different upper bounds are derived for the torsional rigidity of nonhomogeneous isotropic elastic bars. It is assumed that the material properties of the bar do not depend on the axial coordinate.

All derivations are based on the Bai-Shield identity and the strength of materials' approach makes it possible to avoid the use of variational methods and the application of the procedures known from higher analysis [3].

The formulas derived are also valid for cases when the bar is a composite one made of different homogeneous materials. These bars are compound bars and reinforced bars. Their discontinuities in the material properties do not affect the validity of the bounding formulas presented here.

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