Journal of Computational and Applied Mechanics, Vol. 2., No. 2., (2001) pp. 273–278

A LAGRANGIAN FOR THE LARGE DEFLECTIONS OF A RHOMBIC PLATE

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[Received: May 18, 2001]

Abstract. The inverse problem of the variational calculus is discussed in the present paper. We shall show step by step how to find a Lagrangian for the large deflections of a rhombic plate from the nonlinear partial differential equation proposed by Banerjee.

Mathematical Subject Classification: 74K20,74B20
Keywords: Variational representation, skew plate

1. Introduction

It is well known from the literature [1] that a system of differential equations has a variational representation if it is self-adjoint, but it is very difficult to identify the variational model in a traditional way. If the system of equations is not self-adjoint there is no simple way to find an equivalent variational model.

For example, first let us consider the following equation

$$\frac{u''}{u'} + a = 0, \quad u'(x) \neq 0, \tag{1.1}$$

which is clearly self-adjoint [1]. As we have just mentioned, it is difficult to find an equivalent variational model in the traditional way. By applying the semi-inverse method [2, 3, 4, 5], however, we can easily obtain the corresponding variational functional.

Let us assume that the Lagrangian of equation (1.1) can be expressed as

$$L(x, u, u') = u' \ln u' + F(x, u) , \tag{1.2}$$

where F is an unknown function to be determined. Therefore we obtain the following Euler equation

$$-(\ln u')' - \left(\frac{u'}{u'}\right)' + \frac{\partial F}{\partial u} = 0 , \tag{1.3}$$

or

$$-\frac{u''}{u'} + \frac{\partial F}{\partial u} = 0 . \tag{1.4}$$

If we set

$$\frac{\partial F}{\partial u} = -a , \tag{1.5}$$

then equation (1.4) coincides with the original equation (1.1). It follows from equation (1.5) that the unknown function F has the form

$$F = -au\,. \tag{1.6}$$

Consequently, we have obtained the following functional for equation (1.1):

$$J\,(u) = \int \left(u' \ln u' - au\right)\,\mathrm{d}x\,. \tag{1.7}$$

Howerer, if equation (1.1) is written in the form

$$u'' + au' = 0\,, \tag{1.8}$$

then it is clearly not self-adjoint.

According to He's semi-inverse method [2, 3, 4, 5], a Lagrangian assumes the form

$$L\,(x, u, u') = F\,(x, u)\,u'^2, \tag{1.9}$$

where F is an unknown function. The corresponding Euler equation can be written as

$$\frac{\partial F}{\partial u}u'^2 - 2\,(Fu')' = 0\,, \tag{1.10}$$

from where by performing the derivation we have

$$\frac{\partial F}{\partial u}u'^2 - 2\left(\frac{\partial F}{\partial u}u'^2 + \frac{\partial F}{\partial x}u' + Fu''\right) = 0\,, \tag{1.11}$$

and

$$u'' + \frac{\partial F}{F\partial x}u' + \frac{\partial F}{2F\partial u}u'^2 = 0\,. \tag{1.12}$$

If we assume that

$$\frac{\partial F}{F\partial x} = a \qquad \text{and} \qquad \frac{\partial F}{2F\partial u} = 0, \tag{1.13}$$

then equation (1.12) coincides with equation (1.8). In addition it immediately follows from equations (1.13) that the unknown functional has the form

$$F = Ce^{ax}, \tag{1.14}$$

where C is a nonzero constant. In other words, the variational representation for equation (1.8) can be expressed as

$$J\,(u) = \int Ce^{ax}u'^2\,\mathrm{d}x. \tag{1.15}$$

In the present paper, we shall propose a straightforward approach to the inverse problem of the calculus of variations, and seek a Lagrangian for the differential equation which describes the large deflections of rhombic plates [6]. We should remark that we shall neglect the question of boundary conditions.

2. Mathematical formulae for small displacement theory

Consider a rhombic plate made of an elastic, isotropic material and having a uniform thickness h. Let the size of each side of the skew plate be sufficiently large compared to h. The origin of the rectangular Cartesian coordinate system (x, y) is located at one of the corners of the skew plate.

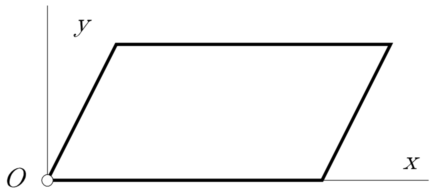

Figure 1. Skew plate

Following Banerjee's hypothesis [7], the differential equation which describes the large deflection of plates is a complex nonlinear 4th order partial differential equation [6,7]:

$$\nabla^4 w - \frac{12A}{h^2}\left(w_{xx} + v w_{yy}\right) - \\ - \frac{6\lambda}{h^2}\left(3w_{xx}w_x^2 + 3w_{yy}w_y^2 + w_{xx}w_y^2 + w_{yy}w_x^2 + 4w_{xy}w_xw_y\right) = \frac{q}{D}, \quad (2.1)$$

where E is the modulus of elasticity, D is the flexural rigidity, h is the thickness of the plate, q is the load intensity, v is the Poisson ratio of the plate material, $\lambda = v^2$, A is a constant, w is the deflection normal to the middle plane of the plate.

First we shall consider the biharmonic equation

$$\nabla^4 w = 0\,. \quad (2.2)$$

The Lagrangian of equation (2.2) can be found with ease:

$$L_1(w) = \frac{1}{2}\left(\nabla^2 w\right)^2. \quad (2.3)$$

To proceed, we regard the equation

$$w_{xx} + v w_{yy} = 0\,, \quad (2.4)$$

for which obviously

$$L_2(w) = -\frac{1}{2}\left(w_x^2 + v w_y^2\right) \quad (2.5)$$

is the Lagrangian. Now we consider the Lagrangian

$$L_3(w) = -\frac{1}{2}w_x^2 + w_y^2 \quad (2.6)$$

since the corresponding Euler equation reads

$$\left(w_x w_y^2\right)_x + \left(w_y w_y^2\right)_y = 0 \tag{2.7}$$

or which is the same

$$w_{xx} w_y^2 + w_{yy} w_x^2 + 4 w_x w_y w_{xy} = 0\,. \tag{2.8}$$

It is obvious that the left side of equation (2.8) is involved in equation (2.1).

Now we take the following Lagrangian

$$L_4\left(w\right) = w\left(w_{xx} w_x^2 + w_{yy} w_y^2\right)\,. \tag{2.9}$$

The corresponding Euler equation reads

$$w w_{xx} w_x^2 + \left(w w_x^2\right)_{xx} - 2\left(w w_{xx} w_x\right)_x + w w_{yy} w_y^2 + \left(w w_y^2\right)_{yy} - 2\left(w w_{yy} w_y\right)_y = 0. \tag{2.10}$$

By a simple manipulation, equation.(2.10) can be transformed into the form

$$3 w_{xx} w_x^2 + 3 w_{yy} w_y^2 = 0. \tag{2.11}$$

which is again a part of equation (2.1). Making use of equations (2.3), (2.5) (2.6) and (2.7), we obtain

$$\begin{aligned} L\left(w\right) &= L_1\left(w\right) - \frac{12A}{h^2} L_2\left(w\right) - \frac{6\lambda}{h^2}\left(L_3 + \frac{3}{4} L_4\right) \\ &= \frac{1}{2}\left(\nabla^2 w\right)^2 + \frac{6A}{h^2}\left(w_x^2 + v w_y^2\right) + \frac{3\lambda}{h^2} w_x^2 w_y^2 - \frac{9\lambda}{2h^2} w\left(w_{xx} w_x^2 + w_{yy} w_y^2\right). \end{aligned} \tag{2.12}$$

as the Lagrangian of equation (2.1). It can easily be checked by determining the Euler equation of the functional (2.12) that the former really coincides with equation (2.1).

3. Conclusion

We have found a Lagrangian for the Banerjee equation which describes the large deflections of a rhombic plate. However, the paper has dealt neither with the issue of the boundary conditions nor with the effect the skew angle in the rhombic has on the solutions. As regards the issue how to involve boundary conditions in the model, we refer the reader to paper [8]. At the same time we remark that the singularities due to discontinuous distributions of bending moments can be taken into account by applying the method proposed in the book by Washizu [9].

Acknowledgement. The author is grateful to the unknown reviewer for his helpful comments.

REFERENCES

1. Santilli, R. M.: *Foundations of Theoretical Mechanics I: The Inverse Problem in Newtonian Mechanics*, Springer-Verlag, 1978.

2. He, J. H.: *Semi-inverse method of establishing generalized variational principles for fluid mechanics with emphasis on turbomachinery aerodynamics*, Int. J. Turbo & Jet-Engines, **14**(1), (1997), 23-28.

3. He, J. H.: *Semi-inverse method and generalized variational principles with multivariables in elasticity*, Applied Math. Mech., **21**, (2000), 721-731.

4. He, J. H.: *Generalized Hellinger-Reissner principle*, ASME J. Appl. Mech., **67**, (2000), 326-331.

5. He, J. H.: *A classical variational model for micropolar elastodynamics*, International Journal of Nonlinear Sciences and Numerical Simulation, **1**(2), (2000), 133-138.

6. Ray. A,K., Banerjee, B. and Bhattacharjee, B.: *Large deflection of rhombic plate - A new approach*, International Journal of Nonlinear Mechanics, **27**(6), (1992), 1007-1014.

7. Banerjee, B.: *Large deflections of polygonal plates under non-stationary temperature*, J. Thermal Stresses, 7, 1984, 285-292.

8. He, J. H.: *Coupled variational principles of piezoelectricity*, Int. J. Engineering Sciences, **39**(3), (2000), 323-341.

9. Washizu, K.: *Variational Methods in Elasticity and Plasticity*, (3^{rd} ed.), Pergamon Press, Oxford, 1982.