Journal of Computational and Applied Mechanics, Vol. 1., No. 1., (2000), pp. 71–79

PRINCIPLE OF COMPLEMENTARY VIRTUAL WORK AND THE RIEMANN-CHRISTOFFEL CURVATURE TENSOR AS COMPATIBILITY CONDITION

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[Received: June 21, 1999]

Abstract. Due to the deformation of a solid body its metric tensor changes. In this paper the Riemann-Christoffel curvature tensor, considered as the compatibility field equation of the nonlinear theory of deformation and written in terms of the metric tensor of the deformed body, is derived from the principle of complementary virtual work.

Keywords: Riemann-Christoffel curvature tensor, principle of complementary virtual work

1. Introduction

The deformation tensors of a solid body are uniquely defined by its displacement field. Otherwise, when the deformation tensors are known, a single-valued continuous displacement field (without rigid body motions) can be derived only in the case when the deformation tensors satisfy the compatibility conditions. Compatibility conditions consist of compatibility field equations and compatibility boundary conditions. In this paper the compatibility field equations are investigated only.

In the infinitesimal theory of deformation, the compatibility field equation is equivalent to the vanishing of the Saint-Venant compatibility tensor. In the nonlinear deformation theory, the compatibility condition is usually expressed by the requirement that the metric tensor of the deformed body be the metric tensor of a Euclidean space (note that the metric tensor of the deformed body is the Green deformation tensor in the reference configuration and the Cauchy deformation tensor in the current configuration). This means that vanishing of the Riemann-Christoffel curvature tensor written in terms of the metric tensor of the deformed body is equivalent to the compatibility field equation.

Both the tensorial Saint-Venant equation and the zero-valued Riemann-Christoffel curvature tensor have six scalar equations. These six equations are not independent of each other. The problem of necessary and sufficient compatibility conditions arises from this fact. A partial solution for this problem was given by Washizu [1]. In the framework of the classical elasticity theory, the necessary and sufficient compatibility conditions were given by Grycz [2], the compatibility field equations and compatibility boundary conditions were derived from the principle of virtual work by the author, see Kozák [3,4].

The papers [5,6] by Bertóti established compatibility field equations and boundary conditions of the first kind in the linear theory of elasticity. For micropolar case and within the framework of the linear theory Kozák-Szeidl [7] determined the necessary and sufficient conditions the strains should meet to be compatible.

This paper derives the Riemann-Christoffel curvature tensor as the compatibility field equation of the nonlinear theory of deformation using the principle of complementary virtual work. The necessary and sufficient conditions of compatibility in the nonlinear theory of deformation are not investigated here. In this respect we refer to an earlier work of the author Kozák [8].

In the following we assume that the volume of the body is simple-connected and bounded by a closed smooth single surface . Both the invariant (symbolic) and indicial (tensorial) notation of tensor calculus will be used. When indicial notation is used, the covariant derivative of a tensor as well as the partial derivative of a two-point tensor will be denoted by a semicolon followed by an index in the subscript, whereas the total covariant derivative of a two-point tensor will be denoted by a colon followed by an index in the subscript.

2. Coordinate systems. Deformation gradients

2.1. Let the spatial point, the position vector, the spatial coordinates and the base vectors be denoted as follows:

– in the reference coordinate system (in the reference configuration of the body):

$$P^\circ,\ \mathbf{r}^\circ,\ x^{\circ k},\ \mathbf{g}_{k^\circ},\ \mathbf{g}^{l^\circ}, \qquad \mathbf{g}_{k^\circ} = \frac{\partial \mathbf{r}^\circ}{\partial x^{\circ k}}$$

– in the spatial coordinate system (in the current configuration of the body):

$$P,\ \mathbf{r},\ x^p,\ \mathbf{g}_p,\ \mathbf{g}^q, \qquad \mathbf{g}_p = \frac{\partial \mathbf{r}}{\partial x^p}\,.$$

In the course of deformation the arbitrary point $\hat{P}$ of the body, moves from the space point $P^\circ(x^{\circ 1}, x^{\circ 2}, x^{\circ 3})$ to the space point $P(x^1, x^2, x^3)$. The trajectory of the point $\hat{P}$ is determined by the motion:

$$x^p = x^p(x^{\circ 1}, x^{\circ 2}, x^{\circ 3}; t), \qquad J = \det\left|\frac{\partial x^p}{\partial x^{\circ k}}\right| > 0\,. \tag{2.1}$$

The inverse motion is given by

$$x^{\circ k} = x^{\circ k}(x^1, x^2, x^3; t)\,. \tag{2.2}$$

Using material coordinate system, let the point, the coordinates and the base vectors be denoted as follows

– in the reference configuration of the body:

$$\hat{P},\ X^{\circ K} = X^K,\ \mathbf{G}_{K^\circ},\ \mathbf{G}^{L^\circ}, \qquad \mathbf{G}_{K^\circ} = \frac{\partial \mathbf{r}^\circ}{\partial X^{\circ K}} = \frac{\partial x^{\circ l}}{\partial X^{\circ K}}\mathbf{g}_{l^\circ}$$

– in the current configuration of the body:

$$\hat{P},\ X^P = X^{\circ P},\ \mathbf{G}_P,\ \mathbf{G}^Q, \qquad \mathbf{G}_P = \frac{\partial \mathbf{r}}{\partial X^P} = \frac{\partial x^q}{\partial X^P}\mathbf{g}_q\,.$$

2.2. At arbitrary time t the direct and inverse mappings are given by

$$d\mathbf{r} = \boldsymbol{F}\cdot d\mathbf{r}^\circ, \qquad d\mathbf{r}^\circ = \boldsymbol{F}^{-1}\cdot d\mathbf{r} \tag{2.3}$$

respectively, where $\mathbf{F}$ is the deformation gradient:

$$\boldsymbol{F}{=}F^p{}_{l^\circ}\mathbf{g}_p\mathbf{g}^{l^\circ} = \frac{\partial x^p}{\partial x^{l^\circ}}\mathbf{g}_p\mathbf{g}^{l^\circ} = \mathbf{G}_Q\mathbf{G}^{Q^\circ} \tag{2.4}$$

and $\boldsymbol{F}^{-1}$ is the inverse deformation gradient:

$$\boldsymbol{F}^{-1}{=}\left(F^{-1}\right)^{k^\circ}{}_q\,\mathbf{g}_{k^\circ}\mathbf{g}^q = \frac{\partial x^{k^\circ}}{\partial x^g}\mathbf{g}_{k^\circ}\mathbf{g}^q = \mathbf{G}_{Q^\circ}\mathbf{G}^Q\,. \tag{2.5}$$

$\boldsymbol{F}$ and $\boldsymbol{F}^{-1}$ are two-point-tensors.

In the reference configuration they can be written as

$$F_{k^\circ l^\circ} = g_{k^\circ l^\circ} + u_{k^\circ;l^\circ} = g_{k^\circ p}F^p{}_{l^\circ} = g_{k^\circ p}\left(\delta^p_{l^\circ} + u^p{}_{;l^\circ}\right) \tag{2.6}$$

and in the current configuration as

$$\left(F^{-1}\right)_{pq} = g_{pq} - u_{p;q} = g_{pk^\circ}\left(F^{-1}\right)^{k^\circ}{}_q = g_{pk^\circ}\left(\delta^{k^\circ}_q - u^{k^\circ}{}_{;q}\right) \tag{2.7}$$

where $g_{k^\circ p} = g_{pk^\circ}$ is a shifter, $\mathbf{u}^\circ = u_{k^\circ}\mathbf{g}^{k^\circ} = u_p\mathbf{g}^p = \mathbf{u}$ is the displacement vector.

3. The principle of complementary virtual work

3.1. We assume that on the surface part (A_t) loads, on the surface part (A_u) displacements are prescribed and $(A_t) = (A_t)\cup(A_u)$ is the whole surface of the body. In addition we assume that the variation of the Cauchy stress tensor satisfies the following conditions:

$$\delta S^{pq}{}_{;q} = 0 \quad \text{and} \quad \delta S^{pq} = \delta S^{qp}, \qquad x\in(B) \tag{3.1}$$

$$\delta S^{pq} n_q \mathrm{d}A = 0 \qquad x\in(A_t) \tag{3.2}$$

where n_q is the normal unit vector to (A_t).

The principle of complementary virtual work states that when equation

$$\int\limits_{(B)}\left[g_{pq} - \left(F^{-1}\right)_{pq}\right]\delta S^{pq}\,\mathrm{d}V = \int\limits_{(A_u)}\tilde{u}_p\delta S^{pq}n_q\mathrm{d}A \tag{3.3}$$

holds for any δS^{pq} satisfying (3.1) and (3.2) in the current configuration (B) of the body, where $\tilde{u}_p$ is the prescribed displacement field, then the inverse deformation gradient and the gradient of the displacement vector

$$\left(F^{-1}\right)^{k^\circ}_{\ q} = g^{k^\circ p}\left(F^{-1}\right)_{pq} \qquad \textit{and} \qquad u^{k^\circ}_{\ ;q} = g^{k^\circ p} u_{p;q} \tag{3.4}$$

are kinematically admissible.

3.2. Any tensor δS^{pq} satisfying (3.1) can be derived from a second-order, symmetric, otherwise arbitrary stress function tensor δH_{rs}:

$$\delta S^{pq} = \varepsilon^{prm}\varepsilon^{qsn}\delta H_{rs;mn} \tag{3.5}$$

Inserting (3.5) in (3.5) we obtain:

$$\int\limits_{(B)} \varepsilon^{prm}\varepsilon^{qsn}\left[g_{pq} - \left(F^{-1}\right)_{pq}\right]\delta H_{rs;mn}\,\mathrm{d}V = \int\limits_{(A_u)} \tilde{u}_p\varepsilon^{prm}\varepsilon^{qsn}\delta H_{rs;mn}n_q\mathrm{d}A \tag{3.6}$$

Applying the Gauss-theorem twice on the volume integral, another form of the principle of complementary virtual work is obtained:

$$\int\limits_{(B)} \varepsilon^{prm}\varepsilon^{qsn}\left(F^{-1}\right)_{pq:mn}\delta H_{rs}\,\mathrm{d}V = \int\limits_{(A)} n_n\varepsilon^{prm}\varepsilon^{qsn}\left[g_{pq} - \left(F^{-1}\right)_{pq}\right]\delta H_{rs;m}\,\mathrm{d}A +$$

$$+ \int\limits_{(A)} n_m\varepsilon^{prm}\varepsilon^{qsn}\left(F^{-1}\right)_{pq:n}\delta H_{rs}\,\mathrm{d}A - \int\limits_{(A_u)} \tilde{u}_p n_q\varepsilon^{prm}\varepsilon^{qsn}\delta H_{rs;mn}n_q\mathrm{d}A\,. \tag{3.7}$$

Taking into account that δH_{rs} is arbitrary in the volume of the current configuration of the body, from (3.7) we get the compatibility field equation for the inverse deformation gradient $\boldsymbol{F}^{-1}$:

$$\varepsilon^{prm}\varepsilon^{qsn}\left(F^{-1}\right)_{pq:mn} = 0, \qquad x \in (B)\,. \tag{3.8}$$

As mentioned in the introduction, this paper does not investigate the necessary and sufficient compatibility conditions of the nonlinear theory of deformation, therefore equation (3.7) is used for the derivation of the compatibility field equation (3.8) only.

4. The compatibility field equation and the curvature tensor

4.1. In the following our investigations will be carried out in a material coordinate system, proposed by Lurie [9]. In this case the inverse deformation gradient can be written as

$$\boldsymbol{F}^{-1} = \left(F^{-1}\right)^{K^\circ}_{\ Q}\mathbf{G}_{K^\circ}\mathbf{G}^Q = \mathbf{G}_{Q^\circ}\mathbf{G}^Q \tag{4.1}$$

i.e.,

$$\left[\left(F^{-1}\right)^{K^\circ}_{\ Q}\right] = \left[\frac{\partial X^{\circ K}}{\partial X^Q}\right] = \left[\delta^{K^\circ}_{\ Q}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.2}$$

and

$$\left(F^{-1}\right)_{PQ} = G_{PK^\circ}\left(F^{-1}\right)^{K^\circ}_{\ \ Q}\ , \tag{4.3}$$

where G_{PK° is a shifter.

The form of the compatibility field equation (3.8) in the material coordinate system is

$$\varepsilon^{PRM}\varepsilon^{QSN}\left(F^{-1}\right)_{PQ:MN} = 0, \qquad x \in (B)\ . \tag{4.4}$$

To carry out the covariant differentiations in (4.4), the rule for the total covariant differentiation of two-point tensors will be used, taking into account that the total covariant derivative of a shifter is zero. First we obtain:

$$\left(F^{-1}\right)_{PQ:M} = \left[G_{PK^\circ}\left(F^{-1}\right)^{K^\circ}_{\ \ Q}\right]_{:M} = G_{PK^\circ}\left(F^{-1}\right)^{K^\circ}_{\ \ Q:M} \tag{4.5}$$

where

$$\begin{aligned}\left(F^{-1}\right)^{K^\circ}_{\ \ Q:M} &= \left(F^{-1}\right)^{K^\circ}_{\ \ Q;A^\circ}\left(F^{-1}\right)^{A^\circ}_{\ \ M} + \left(F^{-1}\right)^{K^\circ}_{\ \ Q;M} = \\ &= \Gamma^{K^\circ}_{B^\circ A^\circ}\left(F^{-1}\right)^{B^\circ}_{\ \ Q}\left(F^{-1}\right)^{A^\circ}_{\ \ M} - \Gamma^{U}_{QM}\left(F^{-1}\right)^{K^\circ}_{\ \ U}\ .\end{aligned} \tag{4.6}$$

In (4.6) $\Gamma^{K^\circ}_{B^\circ A^\circ}$ and Γ^{U}_{QM} are Christoffel symbols of the second kind.

Following from (4.5) and (4.6) we can write:

$$\left(F^{-1}\right)_{PQ:MN} = G_{PK^\circ}\left(F^{-1}\right)^{K^\circ}_{\ \ Q:MN}\ , \tag{4.7}$$

where

$$\left(F^{-1}\right)^{K^\circ}_{\ \ Q:MN} = \left[\left(F^{-1}\right)^{K^\circ}_{\ \ Q:M}\right]_{;C^\circ}\left(F^{-1}\right)^{C^\circ}_{\ \ N} + \left[\left(F^{-1}\right)^{K^\circ}_{\ \ Q:M}\right]_{;N} \tag{4.8}$$

$$\left[\left(F^{-1}\right)^{K^\circ}_{\ \ Q:M}\right]_{;C^\circ} = \frac{\partial\Gamma^{K^\circ}_{B^\circ A^\circ}}{\partial X^{\circ C}}\left(F^{-1}\right)^{B^\circ}_{\ \ Q}\left(F^{-1}\right)^{A^\circ}_{\ \ M} + \Gamma^{K^\circ}_{C^\circ D^\circ}\left(F^{-1}\right)^{D^\circ}_{\ \ Q:M} \tag{4.9}$$

and

$$\left[\left(F^{-1}\right)^{K^\circ}_{\ \ Q:M}\right]_{;N} = \frac{\partial\Gamma^{U}_{QM}}{\partial X^{N}}\left(F^{-1}\right)^{K^\circ}_{\ \ U} - \Gamma^{V}_{QN}\left(F^{-1}\right)^{K^\circ}_{\ \ V:M} - \Gamma^{W}_{MN}\left(F^{-1}\right)^{D^\circ}_{\ \ Q:W} \tag{4.10}$$

Now we make the following transformations: first we insert (4.6) in (4.9) and (4.10), then (4.9) and (4.10) in (4.8). After some algebra, we obtain the compatibility field equation (4.4) in the current configuration in terms of material coordinates (with changed dummy indices):

$$\begin{aligned}&\varepsilon^{IMP}\varepsilon^{JNQ}\left(F^{-1}\right)_{PQ:MN} = \\ &= \varepsilon^{IMP}\varepsilon^{JNQ}G_{PK^\circ}\left[\left(\frac{\partial\Gamma^{K^\circ}_{B^\circ A^\circ}}{\partial X^{\circ C}} + \Gamma^{K^\circ}_{C^\circ D^\circ}\Gamma^{D^\circ}_{A^\circ B^\circ}\right)\left(F^{-1}\right)^{A^\circ}_{\ \ M}\left(F^{-1}\right)^{B^\circ}_{\ \ Q}\left(F^{-1}\right)^{C^\circ}_{\ \ N}\right] - \\ &\quad -\varepsilon^{IMP}\varepsilon^{JNQ}G_{PK^\circ}\left[\left(\frac{\partial\Gamma^{U}_{QM}}{\partial X^{N}} + \Gamma^{U}_{NW}\Gamma^{W}_{MQ}\right)\left(F^{-1}\right)^{K^\circ}_{\ \ U}\right]\end{aligned} \tag{4.11}$$

Making use of the components of the inverse deformation gradient given in (4.2), (4.11) can be written in the following form:

$$\varepsilon^{IMP}\varepsilon^{JNQ}\left(F^{-1}\right)_{PQ:MN}=\varepsilon^{IA^\circ P^\circ}\varepsilon^{JC^\circ B^\circ}G_{P^\circ K^\circ}\left(\frac{\partial\Gamma^{K^\circ}_{B^\circ A^\circ}}{\partial X^{\circ C}}+\Gamma^{K^\circ}_{C^\circ D^\circ}\Gamma^{D^\circ}_{A^\circ B^\circ}\right)-$$
$$-\varepsilon^{IMP}\varepsilon^{JNQ}G_{PU^\circ}\left(\frac{\partial\Gamma^{U}_{QM}}{\partial X^{N}}+\Gamma^{U}_{NW}\Gamma^{W}_{MQ}\right). \tag{4.12}$$

4.2. The Riemann-theory states that in order to tensor G_{PK} of the material coordinate system be the metric tensor of a Euclidean space in the current configuration, it is necessary and sufficient that G_{PK} be positive definite and satisfy the following equation:

$$R_{NQM}{}^{A}=-\frac{\partial\Gamma^{A}_{MQ}}{\partial X^{N}}+\frac{\partial\Gamma^{A}_{MN}}{\partial X^{Q}}-\Gamma^{A}_{NB}\Gamma^{B}_{MQ}+\Gamma^{A}_{QB}\Gamma^{B}_{MN}=0\,, \tag{4.13}$$

where $R_{NQM}{}^{A}$ is the Riemann-Christoffel curvature tensor. If tensor G_{PQ} is defined as a metric tensor, just like in our case (see equation (4.21)), positive definiteness of G_{PK} is a priori satisfied and the only condition left to be investigated is the zero-valuedness of the Riemann-Christoffel curvature tensor (4.13).

Instead of the Riemann-Christoffel curvature tensor, the so-called Ricci tensor can also be used. Definition of the Ricci tensor is given by

$$\begin{aligned}A^{IJ}&=\frac{1}{4}\varepsilon^{IMP}\varepsilon^{JNQ}R_{NQMP}=\frac{1}{4}\varepsilon^{IMP}\varepsilon^{JNQ}G_{PA}R_{NQM}{}^{A}=\\&=\frac{1}{4}\varepsilon^{IMP}\varepsilon^{JNQ}G_{PA}\left(-\frac{\partial\Gamma^{A}_{MQ}}{\partial X^{N}}-\Gamma^{A}_{NB}\Gamma^{B}_{MQ}\right)=0\,.\end{aligned} \tag{4.14}$$

Both the Riemann-Christoffel curvature tensor and the Ricci tensor have six independent non-zero components:

$$\begin{aligned}A^{11}&=\frac{1}{G}R_{2323}, & A^{22}&=\frac{1}{G}R_{3131}, & A^{33}&=\frac{1}{G}R_{1212},\\ A^{12}&=\frac{1}{G}R_{2131}, & A^{23}&=\frac{1}{G}R_{3112}, & A^{31}&=\frac{1}{G}R_{1223},\end{aligned} \tag{4.15}$$

$$G=\det\left|G_{PA}\right|\,. \tag{4.16}$$

According to (4.15), zero-valuedness of the Riemann-Christoffel curvature tensor is equivalent with the zero-valuedness of the Ricci tensor.

In the material coordinate system and reference configuration the Ricci tensor reads:

$$\begin{aligned}A^{\circ I^\circ J^\circ}&=\frac{1}{4}\varepsilon^{I^\circ A^\circ P^\circ}\varepsilon^{J^\circ C^\circ B^\circ}R_{C^\circ B^\circ A^\circ P^\circ}=\frac{1}{4}\varepsilon^{I^\circ A^\circ P^\circ}\varepsilon^{J^\circ C^\circ B^\circ}G_{P^\circ K^\circ}R_{C^\circ B^\circ A^\circ}{}^{K^\circ}=\\&=\frac{1}{4}\varepsilon^{I^\circ A^\circ P^\circ}\varepsilon^{J^\circ C^\circ B^\circ}G_{P^\circ K^\circ}\left(-\frac{\partial\Gamma^{K^\circ}_{B^\circ A^\circ}}{\partial X^{\circ C}}-\Gamma^{K^\circ}_{C^\circ D^\circ}\Gamma^{D^\circ}_{A^\circ B^\circ}\right)=0\,.\end{aligned} \tag{4.17}$$

If the space of the reference configuration is Euclidean, then ${R_{C^\circ B^\circ A^\circ}}^{K^\circ} = 0$ and $A^{\circ I^\circ J^\circ} = 0$, and similarly, if the space of the current configuration is Euclidean then ${R_{NQM}}^{A} = 0$ and $A^{IJ} = 0$.

4.3. Comparing the compatibility field equation (4.12) obtained from the principle of complementary virtual work and equation (4.14) for the Ricci tensor in the current configuration as well as equation (4.17) for the Ricci tensor in the reference configuration we obtain:

$$\varepsilon^{IMP}\varepsilon^{JNQ}\left(F^{-1}\right)_{PQ:MN} = -2G^{I}_{I^\circ}G^{J}_{J^\circ}A^{\circ I^\circ J^\circ} + 2A^{IJ} = 0\,. \tag{4.18}$$

4.4. Then, assuming that in the case of the direct mapping the reference configuration is Euclidean, i.e., $A^{\circ I^\circ J^\circ} = 0$, it follows from (4.18) and (4.14) that

$$\varepsilon^{IMP}\varepsilon^{JNQ}\left(F^{-1}\right)_{PQ:MN} = 2A^{IJ} = \frac{1}{4}\varepsilon^{IMP}\varepsilon^{JNQ}G_{PA}{R_{NQM}}^{A} = 0\,, \tag{4.19}$$

$$\varepsilon^{IMP}\varepsilon^{JNQ}G_{PA}\left(\frac{\partial\Gamma^{A}_{MQ}}{\partial X^{N}} + \Gamma^{A}_{NB}\Gamma^{B}_{MQ}\right) = 0\,. \tag{4.20}$$

In other words, the compatibility field equation (4.18) is equivalent to the zero-valuedness of the Ricci, as well as - according to (4.15) - the Riemann-Christoffel curvature tensor in the current configuration, provided they are expressed in terms of the changed metric tensor G_{PA}. Tensor G_{PA} is nothing but the Cauchy deformation tensor:

$$G_{PA} = G_{P^\circ A^\circ} + 2E_{PA} \tag{4.21}$$

where E_{PA} is the Euler-Almansi strain tensor. For equation (4.20) we have to take into consideration that

$$\begin{aligned}\Gamma^{A}_{MQ} &= G^{AS}\Gamma_{MQ,S} = \frac{1}{2}G^{AS}\left[\frac{\partial}{\partial X^{M}}G_{QS} + \frac{\partial}{\partial X^{Q}}G_{MS} - \frac{\partial}{\partial X^{S}}G_{MQ}\right] = \\ &= G^{AS}\left[\Gamma_{M^\circ Q^\circ,S^\circ} + \left(\frac{\partial}{\partial X^{M}}E_{QS} + \frac{\partial}{\partial X^{Q}}E_{MS} - \frac{\partial}{\partial X^{S}}E_{MQ}\right)\right]\end{aligned} \tag{4.22}$$

where $\Gamma_{MQ,S}$ and $\Gamma_{M^\circ Q^\circ,S^\circ}$ are Christoffel symbols of the first kind.

4.5. When, in contrary to the above, the inverse mapping is considered and we assume that the space of the current configuration is Euclidean, i.e., $A^{IJ} = 0$ from (4.18) and (4.17) we have

$$\begin{aligned}&\varepsilon^{IMP}\varepsilon^{JNQ}\left(F^{-1}\right)_{PQ:MN} = -2G^{I}_{I^\circ}G^{J}_{J^\circ}A^{\circ I^\circ J^\circ} = \\ &\qquad = -\frac{1}{2}G^{I}_{I^\circ}G^{J}_{J^\circ}\varepsilon^{I^\circ A^\circ P^\circ}\varepsilon^{J^\circ C^\circ B^\circ}G_{P^\circ K^\circ}{R_{C^\circ B^\circ A^\circ}}^{K^\circ} = 0\end{aligned} \tag{4.23}$$

and

$$\varepsilon^{I^\circ A^\circ P^\circ}\varepsilon^{J^\circ C^\circ B^\circ}G_{P^\circ K^\circ}\left(-\frac{\partial\Gamma^{K^\circ}_{B^\circ A^\circ}}{\partial X^{\circ C}} - \Gamma^{K^\circ}_{C^\circ D^\circ}\Gamma^{D^\circ}_{A^\circ B^\circ}\right) = 0\,. \tag{4.24}$$

Thus, in this case the compatibility field equation (4.18) is equivalent to the zero-valuedness of the Ricci, as well as - according to (4.15) - the Riemann-Christoffel curvature tensor in the reference configuration, provided they are expressed in terms of the changed metric tensor $G_{P^\circ K^\circ}$. Tensor $G_{P^\circ K^\circ}$ is nothing but the Green deformation tensor:

$$G_{P^\circ K^\circ} = G_{PK} - 2E^\circ_{P^\circ K^\circ} \tag{4.25}$$

where $E^\circ_{P^\circ K^\circ}$ is the Green-Lagrange strain tensor. For equation (4.24) we have to take into consideration that

$$\Gamma^{K^\circ}_{B^\circ A^\circ} = G^{K^\circ L^\circ}\Gamma_{A^\circ B^\circ, L^\circ} = \frac{1}{2}G^{K^\circ L^\circ}\left(\frac{\partial}{\partial X^{\circ A}}G_{B^\circ L^\circ} + \frac{\partial}{\partial X^{\circ B}}G_{A^\circ L^\circ} - \frac{\partial}{\partial X^{\circ L}}G_{A^\circ B^\circ}\right) =$$

$$= G^{K^\circ L^\circ}\left[\Gamma_{AB,L} - \left(\frac{\partial}{\partial X^{\circ A}}E^\circ_{B^\circ L^\circ} + \frac{\partial}{\partial X^{\circ B}}E^\circ_{A^\circ L^\circ} - \frac{\partial}{\partial X^{\circ L}}E^\circ_{A^\circ B^\circ}\right)\right] \tag{4.26}$$

5. Conclusions

Applying a material coordinate system it has been pointed out that for solid bodies the compatibility field equation obtained from the principle of complementary virtual work is equivalent to the zero-valuedness of the Riemann-Christoffel curvature tensor:

- in the case of direct mapping the Riemann-Christoffel curvature tensor is expressed, according to (4.19)-(4.22), by the Cauchy deformation tensor defined as the metric tensor of the current configuration,
- in the case of inverse mapping the Riemann-Christoffel curvature tensor is expressed, according to (4.23)-(4.26), by the Green deformation tensor defined as the metric tensor of the reference configuration.

In the above statements, instead of the Riemann-Christoffel curvature tensors the Ricci tensors can equally be used.

Note. This paper is dedicated to I. Páczelt on the occasion of his 60th birthday since it applies one of the proposals of Prof. Lurie in his book 'Theory of elasticity' and I. Páczelt was a graduate student of Prof. Lurie in the years of 1966-1969.

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