Mixing angle between $^3P_1$ and $^1P_1$ in HQET

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(Dated: January 24, 2010)

Some claim that there are two independent mixing angles ($\theta = 35.3^\circ$, -54.7°) between $^3P_1$ and $^1P_1$ states of heavy-light mesons in heavy quark symmetric limit, and others claim there is only one ($\theta = 35.3^\circ$). We clarify the difference between these two and suggest which should be adopted. General arguments on the mixing angle between ^3L_L and ^1L_L of heavy-light mesons are given in HQET and a general relation is derived in heavy quark mass limit as well as that including the first order correction in $1/m_Q$.

PACS numbers: 11.30.-j, 12.39.Hg
Keywords: heavy quark effective theory; spectroscopy; heavy quark symmetry

I. INTRODUCTION

The P states have a rich structure because combined with quark spins they form four P states, i.e., $^3P_0$, $^3P_1$, $^3P_2$, and $^1P_1$, and also because there is an interesting feature of mixing between two 1^+ states. The first study on the mixing between $^3P_1$ and $^1P_1$ states in the context of a heavy-light system is given by Rosner[1] and is restudied a few years later in [2] by taking into account more states, D, D_s, B, and B_s mesons. Here we illustrate their idea using notations of [2] in which as well as in [1] they have assumed that dominant interaction between heavy and light quarks is non-relativistic spin-orbit terms, which give mass for a heavy-light meson:

$$\begin{aligned} H_{SO} &= \frac{4}{3}\frac{\alpha_s}{r^3}\frac{\left(\vec{S}_q+\vec{S}_Q\right)\cdot\vec{L}}{m_q m_Q} + \frac{1}{4}\left(\frac{4}{3}\frac{\alpha_s}{r^3}-\frac{b}{r}\right)\left[\left(\frac{1}{m_q^2}+\frac{1}{m_Q^2}\right)\left(\vec{S}_q+\vec{S}_Q\right)\cdot\vec{L}+\left(\frac{1}{m_q^2}-\frac{1}{m_Q^2}\right)\left(\vec{S}_q-\vec{S}_Q\right)\cdot\vec{L}\right] \\ &= H^q_{SO}\vec{S}_q\cdot\vec{L}+H^Q_{SO}\vec{S}_Q\cdot\vec{L}. \end{aligned} \tag{1}$$

Taking the heavy quark mass limit ($m_Q\to\infty$), we are left only with the first term in the second line of Eq. (1). Assuming other interaction terms including kinetic terms give a constant M_0 contribution, then they give the following relation between mass eigenstates and angular momentum eigenfunctions.[2]

$$\begin{pmatrix} M\left(^3P_1\right) \\ M\left(^1P_1\right)\end{pmatrix} = \begin{pmatrix} M_0-\langle H^q_{SO}\rangle & -\sqrt{2}\,\langle H^q_{SO}\rangle \\ -\sqrt{2}\,\langle H^q_{SO}\rangle & M_0\end{pmatrix}\begin{pmatrix} ^3P_1 \\ ^1P_1\end{pmatrix}, \tag{2}$$

which is, as shown later, translated into

$$\begin{pmatrix} \left|j^P=1^+, j_\ell=\frac{1}{2}\right\rangle \\ \left|j^P=1^+, j_\ell=\frac{3}{2}\right\rangle\end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix} \left|^3P_1\right\rangle \\ \left|^1P_1\right\rangle\end{pmatrix}, \tag{3}$$

with two mixing angles,

$$\theta = \arctan\left(\frac{1}{\sqrt{2}}\right) = 35.3^\circ \quad\text{or}\quad \theta = \arctan\left(-\sqrt{2}\right) = -54.7^\circ, \tag{4}$$

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where the left hand side of Eq. (3) is the mass eigenstate and is specified in terms of eigenvalues of a total anguluar momentum $\vec{j}$ of a $Q\bar{q}$ bound state, $\vec{j}_\ell$ which stands for the light quark total angular momentum, $\vec{j}_\ell = \vec{L} + \vec{S}_q$, whose square is conserved in heavy quark symmetric limit, and the parity P. Here $\vec{S}_q$ is a light quark spin. The vector of the right hand side of Eq. (3) denoted as $\left|^{2S+1}L_j\right\rangle$ is specified in terms of eigenvalues of a light quark angular momentum $\vec{L}$, a sum of intrinsic quark spins $\vec{S} = \vec{S}_q + \vec{S}_Q$, and a total angular momentum $\vec{j}$ of the heavy-light meson.

On the other hand, using the heavy quark symmetry we have derived the relation[1] equivalent to Eq. (3),[3, 4]

$$\begin{pmatrix} \left|j^P = 1^+, j_\ell = \frac{1}{2}\right\rangle \\ \left|j^P = 1^+, j_\ell = \frac{3}{2}\right\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \left|^3P_1\right\rangle \\ \left|^1P_1\right\rangle \end{pmatrix}. \tag{5}$$

but with only one mixing angle,

$$\theta = \arctan\left(\frac{1}{\sqrt{2}}\right) = 35.3^\circ. \tag{6}$$

Equations (3) and (5) are equivalent to each other but Eq. (6) is more restrictive than Eq. (4). We would like to solve the origin of this discrepancy and give a reasonable interpretation which should be adopted for the heavy-light mesons.

II. MASS MATRIX BY ROSNER OR GODFREY AND KOKOSKI

The expression of Eq. (2) is very confusing in the sense that 1) there are no eigenstates on the l.h.s. of the equation, and 2) the eigenvalues on the l.h.s., $M(^3P_1)$ and $M(^1P_1)$, are written with explicit arguments $^3P_1$ and $^1P_1$. They have assumed that the upper and lower components on the l.h.s. are dominated by $^3P_1$ and $^1P_1$ from the beginning, respectively.

To better understand Eq. (2), we introduce ket vectors as eigenstates with angular momentum quantum numbers, and an orthogonal matrix, U, to diagonalize the mass matrix. We rewrite Eq. (2) as an eigenvalue equation in an operator form so that everybody is on the same footing.

$$\left(M_0 + H^q_{SO}\vec{L}\cdot\vec{S}_q\right)|\psi\rangle = \lambda\,|\psi\rangle\,, \quad |\psi\rangle = \alpha\left|^3P_1\right\rangle + \beta\left|^1P_1\right\rangle, \quad (\alpha^2+\beta^2=1), \tag{7}$$

where $|\psi\rangle$ is a wave function expanded in terms of $\left|^{2S+1}L_j\right\rangle$, and α and β are constant coefficients. The mass Hamiltonian is defined by 2×2 matrix of the r.h.s. of Eq. (2), whose matrix elements are expectation values of $M_0 + H^q_{SO}\vec{L}\cdot\vec{S}_q$ between $\left|^3P_1\right\rangle$ and $\left|^1P_1\right\rangle$. Though it might be a rather redundant explanation shown below to solve Eq. (7), we believe that it clarifies the reason why two mixing angles appear.

Now we can reexpress Eq. (7) in the following eigenvalue equation in which all quantities are constant.

$$MP = \lambda P \quad \text{or} \quad M_D P' = \lambda P', \tag{8}$$

where

$$\begin{aligned} &M_D \equiv UMU^T, \quad P' = UP, \quad M = \begin{pmatrix} M_0 - \langle H^q_{SO}\rangle & -\sqrt{2}\,\langle H^q_{SO}\rangle \\ -\sqrt{2}\,\langle H^q_{SO}\rangle & M_0 \end{pmatrix}, \\ &U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \; P = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \; P' = UP = \begin{pmatrix} \alpha' \\ \beta' \end{pmatrix}, \quad (\alpha'^2 + \beta'^2 = 1) \end{aligned} \tag{9}$$

When one solves an eigenvalue equation $MP = \lambda P$, we obtain,

$$\begin{aligned} &\lambda = M_0 - 2\,\langle H^q_{SO}\rangle \quad \text{or} \quad M_0 + \langle H^q_{SO}\rangle\,; \; P' = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{or} \quad P' = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\ &P = \frac{1}{\sqrt{3}}\begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix} \quad \text{or} \quad P = \frac{1}{\sqrt{3}}\begin{pmatrix} -1 \\ \sqrt{2} \end{pmatrix}, \end{aligned} \tag{10}$$

[1] Sign change of $\sin\theta$ in this equation can be absorbed into state redefinitions so that the form of an orthogonal matrix becomes the same as that of Eq. (3).

respectively, and we have only one mixing angle,

$$\theta = \arctan\left(1/\sqrt{2}\right) = 35.3^\circ. \tag{11}$$

Inserting Eq. (10) into Eq. (7), we have eigenfunctions $|\psi\rangle = \left|^{2S+1}P_1\right\rangle'$ as,

$$\left|^3P_1\right\rangle' \equiv \frac{1}{\sqrt{3}}\left(\sqrt{2}\left|^3P_1\right\rangle + \left|^1P_1\right\rangle\right), \quad \left|^1P_1\right\rangle' \equiv \frac{1}{\sqrt{3}}\left(-\left|^3P_1\right\rangle + \sqrt{2}\left|^1P_1\right\rangle\right), \tag{12}$$

where we have named eigenstates $\left|^{2S+1}P_1\right\rangle'$ on the l.h.s. of Eq. (12) according to which coefficient of eigenstates $\left|^{2S+1}P_1\right\rangle$ on the r.h.s. is larger, e.g., in the r.h.s. of the first equation a coefficient of $\left|^3P_1\right\rangle$ $\left(\sqrt{2/3}\right)$ is lager than that of $\left|^1P_1\right\rangle$ $\left(\sqrt{1/3}\right)$, thus we call this $\left|^3P_1\right\rangle'$.

Another way to solve Eq. (8) $M_D P' = \lambda P'$ is to require vanishing off-diagonal elements of M_D, which gives the following two mixing angles θ as in [1, 2],

$$\theta_1 = \arctan\left(\frac{1}{\sqrt{2}}\right) = 35.3^\circ \quad \text{or} \quad \theta_2 = \arctan\left(-\sqrt{2}\right) = -54.7^\circ. \tag{13}$$

In order to check whether they are independent or not, we may rewrite the eigenvalue euqation $M_D P' = \lambda P'$ for each angle, which are given as follows. In the case of $\tan\theta = \tan\theta_1 = \frac{1}{\sqrt{2}}$, the diagonalized mass matrix and eigenvectors are given by,

$$\begin{aligned} M_{D1} &= U_1 M U_1^T = \begin{pmatrix} M_0 - 2\left\langle H^q_{SO}\right\rangle & 0 \\ 0 & M_0 + \left\langle H^q_{SO}\right\rangle \end{pmatrix}, \quad P_1' = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\ P_1 &= U_1^T P_1' = \frac{1}{\sqrt{3}}\begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix} \text{ or } \frac{1}{\sqrt{3}}\begin{pmatrix} -1 \\ \sqrt{2} \end{pmatrix}, \end{aligned} \tag{14}$$

respectively, with $U_1 = U(\theta = \theta_1)$ in Eq. (9). In the case of $\tan\theta = \tan\theta_2 = -\sqrt{2}$, those are given by,

$$\begin{aligned} M_{D2} &= U_2 M U_2^T = \begin{pmatrix} M_0 + \left\langle H^q_{SO}\right\rangle & 0 \\ 0 & M_0 - 2\left\langle H^q_{SO}\right\rangle \end{pmatrix}, \quad P_2' = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\ P_2 &= U_2^T P_2' = \frac{1}{\sqrt{3}}\begin{pmatrix} -1 \\ \sqrt{2} \end{pmatrix} \text{ or } \frac{1}{\sqrt{3}}\begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}, \end{aligned} \tag{15}$$

respectively, with $U_2 = U(\theta = \theta_2)$ in Eq. (9). Multiplying the following matrix U_0 on M_{D2}, and P_2' in Eq. (15) as,

$$U_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} \cos 90^\circ & \sin 90^\circ \\ -\sin 90^\circ & \cos 90^\circ \end{pmatrix}, \quad M_{D1} = U_0 M_{D2} U_0^T, \quad P_1' = U_0 P_2', \quad U_1 = U_0 U_2, \tag{16}$$

we can reproduce Eq. (14). Hence Eqs. (14) and (15) are equivalent to each other, which means that two mixing angles are also equivalent. Actually $\theta_2 = \theta_1 - 90^\circ$ as easily seen from Eq. (16). This is consistent with the solution given by Eqs. (10) $\sim$ (12) with the mixing angle $\tan\theta = 1/\sqrt{2}$ when solving the eigenvalue equation $MP = \lambda P$.

When one tries to identify which eigenstate corresponds to a lower-mass or higher-mass state as in Refs. [1, 2], it does not matter which angle one adopts. It depends only on sign of $\langle H^q_{SO}\rangle$. By looking at Eqs. (14, 15), one finds that if $\langle H^q_{SO}\rangle > 0$, then the lower-mass state is identified as $\left|^3P_1\right\rangle'$ and the higher-mass as $\left|^1P_1\right\rangle'$. On the other hand, if $\langle H^q_{SO}\rangle < 0$, the lower-mass state is identified as $\left|^1P_1\right\rangle'$ and the higher as $\left|^3P_1\right\rangle'$ irrespective of a mixing angle.

There is a way to determine which state ($\left|^3P_1\right\rangle'$ or $\left|^1P_1\right\rangle'$) corresponds to which heavy quark symmetric state ($j_\ell^P = (1/2)^+$ or $(3/2)^+$). This is done by expanding heavy quark symmetric states $\left|j^P, j_\ell, j_z\right\rangle$ in terms of states $|j, S, j_z\rangle$ with $\vec{S} = \vec{S}_q + \vec{S}_Q$, i.e., by calculating the Clebsh-Gordan coefficients, which is given in Appendix of [9] as,

$$\begin{pmatrix} |j = L, j_\ell = L - 1/2, m\rangle \\ |j = L, j_\ell = L + 1/2, m\rangle \end{pmatrix} = \frac{1}{\sqrt{2j+1}}\begin{pmatrix} \sqrt{j+1} & \sqrt{j} \\ -\sqrt{j} & \sqrt{j+1} \end{pmatrix}\begin{pmatrix} |j = L, S = 0, m\rangle \\ |j = L, S = 1, m\rangle \end{pmatrix}. \tag{17}$$

By substituting $L = 1$, we obtain Eq. (3). Therefore even discussions given by [1, 2] are enough to uniquely determine the relation between heavy quark symmetric states $\left|j^P, j_\ell\right\rangle$ and non-relativistic states $\left|^{2S+1}L_j\right\rangle$ in heavy quark symmetric limit, which is given by Eq. (3) with only one mixing angle Eq. (6).[11]

III. MIXING BETWEEN ^3L_L AND ^1L_L IN HQET

In the relativistic potential model studied by us more than ten years ago, we have derived the relativistic equation for a $Q\bar{q}$ bound state in the heavy quark symmetric limit ($m_Q \to \infty$) treating a light quark as relativistic and a heavy quark as static.[4] In that equation *the angular component is completely solved* and is given by the eigenfunction y_{jm}^k. Because a heavy quark is treated as static in heavy quark limit, a bound state wave function can be separated into (heavy-quark) energy positive and negative components and the lowest non-trivial order wave function is naturally given by a positive energy component which has 2×4 spinor components. In order to classify the states in terms of a non-relativistic $^{2S+1}L_j$ state, only the upper 2×2 component of the wave function is necessary. The relation between y_{jm}^k and angular momentum eigenfunctions is uniquely determiend to be,

$$\begin{pmatrix} y_{j\,m}^{-(j+1)} \\ y_{j\,m}^{j} \end{pmatrix} = U \begin{pmatrix} Y_j^m \\ \vec{\sigma} \cdot \vec{Y}_{j\,m}^{(\mathrm{M})} \end{pmatrix}, \quad \begin{pmatrix} y_{j\,m}^{j+1} \\ y_{j\,m}^{-j} \end{pmatrix} = U \begin{pmatrix} \vec{\sigma} \cdot \vec{Y}_{1\,m}^{(\mathrm{L})} \\ \vec{\sigma} \cdot \vec{Y}_{j\,m}^{(\mathrm{E})} \end{pmatrix}, \quad U = \frac{1}{\sqrt{2j+1}} \begin{pmatrix} \sqrt{j+1} & \sqrt{j} \\ -\sqrt{j} & \sqrt{j+1} \end{pmatrix}. \tag{18}$$

That is, this is the definition of the eigenfunction y_{jm}^k. When $j = 1$, we have the following relation between the eigenstates (l.h.s.) respecting the heavy quark symmetry and the non-relativistic states (r.h.s.) described in terms of $^3P_1$ and $^1P_1$.

$$\begin{pmatrix} y_{1\,m}^{-2} \\ y_{1\,m}^{1} \end{pmatrix} = U \begin{pmatrix} Y_1^m \\ \vec{\sigma} \cdot \vec{Y}_{1\,m}^{(\mathrm{M})} \end{pmatrix} \qquad \text{with } \; U = \frac{1}{\sqrt{3}} \begin{pmatrix} \sqrt{2} & 1 \\ -1 & \sqrt{2} \end{pmatrix}. \tag{19}$$

Here $Y^{(E),\,(M),\,(L)}$ are spinor representations of an intrinsic spin $s = 1$ particle with a total angular momentum j, i.e., photon's wave function with a total angular momentum j. Here $Y^{(E),\,(M),\,(L)}$ have parities, $(-)^{j+1}$, $(-)^j$, and $(-)^{j+1}$, respectively, and Y_j^m has parity, $(-)^j$, i.e., the same as $Y^{(M)}$. That is, $Y^{(M)}$ is a spinor representation of $^3P_1$ and so is $1_{2\times2} \times Y_1^m$ that of $^1P_1$, while wave functions on the l.h.s., $y_{1\,m}^{-2}$ and $y_{1\,m}^{1}$ correspond to $j_\ell = 3/2$ and $1/2$ with $j^P = 1^+$, respectively. Here we have used the relation between j_ℓ and k,[5]

$$j_\ell = |k| - \frac{1}{2}. \tag{20}$$

Equation (19) is our result which is equivalent to Eq. (5). The mixing angle is given by $\theta = \arctan\left(1/\sqrt{2}\right) = 35.3°$ that is not a "magic number" as called in Refs. [6, 7], which is derived from the relation between eigenstates with a k quantum number and $^{2S+1}L_j$ states.

Using the first equation of Eq. (18), we can write down a general relation between heavy quark symmetric states and non-relativistic states ^3L_L and ^1L_L as,

$$\begin{pmatrix} \left|y_{L\,m}^{L}\right\rangle \\ \left|y_{L\,m}^{-(L+1)}\right\rangle \end{pmatrix} = \begin{pmatrix} \left|L^P, j_\ell = L - \frac{1}{2}\right\rangle \\ \left|L^P, j_\ell = L + \frac{1}{2}\right\rangle \end{pmatrix} = \frac{1}{\sqrt{2L+1}} \begin{pmatrix} \sqrt{L+1} & -\sqrt{L} \\ \sqrt{L} & \sqrt{L+1} \end{pmatrix} \begin{pmatrix} \left|^3L_L\right\rangle \\ \left|^1L_L\right\rangle \end{pmatrix}, \quad P = (-1)^{L+1}, \tag{21}$$

which gives Eq. (5) when $j = L = 1$. Here we have used $P = (-1)^{|k|+1}\, k/|k|$ with $k = L$.[5]

In our model[4], a spin doublet $(0^+, 1^+)$ degenerates and so does another spin double $(1^+, 2^+)$ in heavy quark symmetric limit, which are corresponding to $j_\ell^P = (1/2)^+$ and $(3/2)^+$ multiplets, respectively. Our most recent numerical calculations[8] show that $M\left((1/2)^+\right) < M\left((3/2)^+\right)$ in the cases of $c\bar{q}$ and $b\bar{q}$ which is equivalent to $M\left(\left|^3P_1\right\rangle'\right) < M\left(\left|^1P_1\right\rangle'\right)$. These values of M are degenerate eigenvalues of a first-order differential equation and can not be predicted beforehand by just looking at the equation.

There appear a couple of quantum numbers to distinguish heavy-light mesons, which is summarized in TABLE I. Here j stands for a total angular momentum, P its parity, k a quantum number whose relation with other quantum numbers is given by, e.g., Eq. (20),[5, 10] j_ℓ^P a total angular momentum of a light quark with parity P, and $^{2S+1}L_j$ a non-relativistic quantum number describing a total intrinsic spin S, an internal angular momentum L, and a total angular momentum j.

IV. SUMMARY

We conclude from the previous sections' results that the heavy quark symmetry can uniquely determine the relation between heavy quark symmetric eigenstates and states with $^{2S+1}P_1$ with the mixing angle $\theta = 35.3°$ between $^3P_1$ and

TABLE I: States classified by various quantum numbers.

j^P	0^-	1^-	0^+	1^+	1^+	2^+	1^-	2^-	2^-	3^-
k	-1	-1	1	1	-2	-2	2	2	-3	-3
j_ℓ^P	$\frac{1}{2}^-$	$\frac{1}{2}^-$	$\frac{1}{2}^+$	$\frac{1}{2}^+$	$\frac{3}{2}^+$	$\frac{3}{2}^+$	$\frac{3}{2}^-$	$\frac{3}{2}^-$	$\frac{5}{2}^-$	$\frac{5}{2}^-$
${}^{2S+1}L_j$	${}^1S_0$	${}^3S_1$	${}^3P_0$	$\left({}^3P_1,\ {}^1P_1\right)$	$\left({}^1P_1,\ {}^3P_1\right)$	${}^3P_2$	${}^3D_1$	$\left({}^3D_2,\ {}^1D_2\right)$	$\left({}^1D_2,\ {}^3D_2\right)$	${}^3D_3$

${}^1P_1$ as shown by Eq. (5) as,

$$\begin{pmatrix} \left|1^+, j_\ell = \frac{1}{2}\right\rangle \\ \left|1^+, j_\ell = \frac{3}{2}\right\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \left|{}^3P_1\right\rangle \\ \left|{}^1P_1\right\rangle \end{pmatrix} \quad \text{with} \quad \tan\theta = \frac{1}{\sqrt{2}}.$$

In heavy quark symmetric limit, our relativistic potential model[8] predicts the lower-mass state is $\left|{}^3P_1\right\rangle'$ and the higher-mass state $\left|{}^1P_1\right\rangle'$, while the model with the Breit-Fermi type non-relativistic potential model[1, 2] predicts either $\left|{}^3P_1\right\rangle'$ or $\left|{}^1P_1\right\rangle'$ as the lower mass state depending on sign of $\langle H_{SO}^q\rangle$.

A general mixing angle between ${}^3L_L$ and ${}^1L_L$ in HQET is given by $\tan\theta = \sqrt{L/(L+1)}$ as readily seen from Eq. (18) and when one takes into account breaking of the heavy quark symmetry, it is given by

$$\tan\left(\theta_1 + \delta\theta\right) = \sqrt{\frac{L}{L+1}} + \frac{(2L+1)}{L+1}\delta\theta, \quad \tan\theta_1 = \sqrt{\frac{L}{L+1}}, \quad \delta\theta = O\left(\frac{1}{m_Q}\right). \tag{22}$$

Because $\tan\theta_1 = 1/\sqrt{2}$ is the result of heavy quark symmetry, $\delta\theta$ gives mixing between states with different j_ℓ as

$$\begin{pmatrix} \left|L^P, j_\ell = L - \frac{1}{2}\right\rangle' \\ \left|L^P, j_\ell = L + \frac{1}{2}\right\rangle' \end{pmatrix} = \begin{pmatrix} 1 & -\delta\theta \\ \delta\theta & 1 \end{pmatrix} \begin{pmatrix} \left|L^P, j_\ell = L - \frac{1}{2}\right\rangle \\ \left|L^P, j_\ell = L + \frac{1}{2}\right\rangle \end{pmatrix}, \quad P = (-1)^{L+1}, \tag{23}$$

where $k = j = L$ is assumed. See [12] for discussions on this kind of mixing.

Finally let us discuss the reason why the mass matrix given by Rosner[1] or Godfrey and Kokoski[2] gives the same eigenstates $\left|{}^3P_1\right\rangle'$ and $\left|{}^1P_1\right\rangle'$ as our model. Interactions including only the spin-orbit terms can be diagonalized by y_{jm}^k because these are eigenfunctions of the operator $\vec{L}\cdot\vec{\sigma}_q$ as,

$$\vec{L}\cdot\vec{\sigma}_q \left|y_{jm}^k\right\rangle = -(k+1)\left|y_{jm}^k\right\rangle. \tag{24}$$

Because

$$\left|y_{1m}^{-2}\right\rangle \equiv \left|{}^1P_1\right\rangle', \quad \left|y_{1m}^{1}\right\rangle \equiv \left|{}^3P_1\right\rangle',$$

the operator $\vec{L}\cdot\vec{\sigma}_q$ has the following eigenvalues,

$$\vec{L}\cdot\vec{\sigma}_q \left|{}^3P_1\right\rangle' = -2\left|{}^3P_1\right\rangle', \quad \vec{L}\cdot\vec{\sigma}_q \left|{}^1P_1\right\rangle' = \left|{}^1P_1\right\rangle'.$$

Hence we have

$$\vec{j_\ell}^{\,2} = \left(\vec{L} + \vec{S}_q\right)^2 = \frac{3}{4}, \quad \frac{15}{4}, \quad \text{or} \quad j_\ell = \frac{1}{2}, \quad \frac{3}{2}, \tag{25}$$

respectively. Here $L = 1$ and $\vec{S}_q = \vec{\sigma}_q/2$. This simply is the reason why they have obtained the same eigenstates $\left|{}^3P_1\right\rangle'$ and $\left|{}^1P_1\right\rangle'$ as our model. The functions y_{jm}^k specified by k,[5, 10] j, and m quantum numbers are equivalent to the heavy quark eigenstates specified by j^P and j_ℓ,

$$\left|y_{jm}^k\right\rangle = \left|j^P, j_\ell\right\rangle \quad \left(\text{with} \ \ j = |k| \ \text{ or } \ |k|-1, \quad j_\ell = |k| - \frac{1}{2}, \quad P = \frac{k}{|k|}(-1)^{|k|+1}\right), \tag{26}$$

where $k \neq 0$ and we have omitted a quantum number m on the r.h.s..

Acknowledgments

One of the authors (T. Matsuki) would like to thank Dr. Xiang Liu for discussions.

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[10] M. E. Rose, *Relativistic Electron Theory*, John Wiley & Sons, 1961; J. J. Sakurai, *Advanced Quantum Mechanics*, Addison-Wesley, 1967.
[11] Some papers adopt erroneous relation between heavy quark symmetric eigenstates and $\left|{}^3P_1\right\rangle$ and $\left|{}^1P_1\right\rangle$ as

$$\begin{pmatrix} \left|j^P = 1^+, j_\ell^P = \frac{1}{2}^+\right\rangle \\ \left|j^P = 1^+, j_\ell^P = \frac{3}{2}^+\right\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \left|{}^1P_1\right\rangle \\ \left|{}^3P_1\right\rangle \end{pmatrix},$$

with two mixing angles,

$$\tan\theta = \frac{1}{\sqrt{2}} \quad \text{or} \quad -\sqrt{2}.$$

An incomplete list of papers, as far as we know, which adopt this transformation are given as follows :
B. Chen, D.-X. Wang, and A. Zhang, Phys. Rev. D **80**, 071502 (2009), see Table II.;
Z.-F. Sun and X. Liu, Phys. Rev. D **80**, 074037 (2009);
Z.-G. Luo, X.-L. Chen, and X. Liu, Phys. Rev. D **79**, 074020 (2009);

[12] V. Balagura *et al.*(BELLE collaboration), Phys. Rev. D **77**, 032001 (2007).