Journal of Computational and Applied Mechanics, Vol. 1., No. 2., (2000), pp. 123–128

A CONSEQUENCE OF THE GENERALIZED CLAPEYRON'S THEOREM

Gyula Béda
Department of Applied Mechanics, Budapest University of Technology and Economics
Műegyetem rkp. 5, 1111 BUDAPEST, Hungary
beda@mm.bme.hu

[Received: June 1, 2000]

Abstract. The investigation of the third order wave necessitates the knowledge of the dynamic compatibility equation. This equation rises from the first equation of motion in case of the acceleration wave. Now it requires the time derivative of the first equation of motion. The material time derivative is not simple in the current configuration. Using the generalized Clapeyron's theorem we obtain an equation of motion of stress-rate. The dynamical compatibility equation can be calculated from it. Many authors have dealt with this question when the body is in equilibrium [8, 9, 10]. The third order wave can be investigated by using the compatibility equations (dynamic, kinematic and constitutive). The generalized acoustic tensor is another important result of these investigations.

Keywords: generalized Clapeyron theorem, equation of motion of stress-rate, acoustic tensor of third order wave

1. Introduction

In its original form, Clapeyron's theorem concerns the transition between two states of equilibrium. If the displacement field u_i takes us from one equilibrium state into the other then the work of the internal forces $t^{ij}u_{i;j}$ is equal to the work done by the body forces q^i and the trections $t^{ij}n_j\big|_A$ acting on the boundary surface A, that is,

$$\int_V t^{ij}u_{i;j}\ \mathrm{d}V = \int_V q^i u_i \mathrm{d}V + \int_A u_i t^{ij} n_j \mathrm{d}A, \tag{1.1}$$

where t^{ij} is the stress tensor, $u_{i;j}$ is the covariant derivative of the displacement vector u_i with respect to the j-th coordinate, n_j is the outward unit normal and V is the part of the geometric space which contains the body B and which is bounded by a closed surface A. Here and in the sequel indicial notations are employed. Accordingly, a Latin index has the range 1,2 and 3; summation over repeated indices is implied and the covariant derivative is denoted by an index preceded by a semicolon. The strain tensor is denoted by e_{ij}.

For moving continua the theory should be modified as follows: the velocity field v_i should be used instead of u_i and, with regard to D'Alembert's principle, q^i is to be replaced by the generalized body force $b^i \equiv q^i - \rho\dot{v}^i$. Let the mass density and the acceleration be denoted by ρ and $\dot{v}^i$, respectively. Obviously, to keep the original meaning of Clapeyron's theorem, the volume and surface integrals should be integrated with respect to time t. This generalization of Clapeyron's theorem remains valid even for finite displacements, if the proper changes are carried out [1]. The generalized Clapeyron's theorem can be formulated as

$$\int_{t_0}^{t_1}\int_V \left(\dot{t}^{ij} - t^{iq}v^{j}_{;q} + t^{ij}v^{p}_{;p}\right) u_{i;j}\mathrm{d}V\mathrm{d}t = \int_{t_0}^{t_1}\int_V \left(\dot{b}^{i} + b^{i}v^{p}_{;p}\right) u_i \mathrm{d}V dt + $$
$$+ \int_{t_0}^{t_1}\int_V \left(\dot{t}^{ij} - t^{iq}v^{j}_{;q} + t^{ij}v^{p}_{;p}\right) u_i n_j \mathrm{d}A_j \mathrm{d}t \ , \tag{1.2}$$

where overdot denotes the material time derivative and $\mathrm{d}A_j \equiv n_j \mathrm{d}A$. If $t^{ij}{}_{;j} + b^i = 0$ then the equation of motion is satisfied [1].

2. Equation of motion for the stress-rate

Equation (1.2) is valid for all kinematically (geometrically) admissible displacement fields, thus it holds also for the virtual field u_i^* [11]. By keeping in mind that remark, the last surface integral of the left-hand side of (1.2) can be transformed into volume integral

$$\int_{t_0}^{t_1}\int_V \left[\left(\dot{t}^{ij} - t^{iq}v^{j}_{;q} + t^{ij}v^{p}_{;p}\right)_{;j} u_i + \left(\dot{t}^{ij} - t^{iq}v^{j}_{;q} + t^{ij}v^{p}_{;p}\right) u_{i;j}\right] \mathrm{d}V\mathrm{d}t\,. \tag{2.1}$$

By substituting (2.1) into (1.2) and performing some rearrangements we have

$$\int_{t_0}^{t_1}\int_V \left[\left(\dot{t}^{ij} - t^{iq}v^{j}_{;q} + t^{ij}v^{p}_{;p}\right)_{;j} + \dot{b}^{i} + b^{i}v^{p}_{;p}\right] u_i^* \mathrm{d}V\mathrm{d}t = 0$$

where the virtual displacement u_i^* is written for u_i. This equation is satisfied for any u_i^* if

$$\left(\dot{t}^{ij} - t^{iq}v^{j}_{;q} + t^{ij}v^{p}_{;p}\right)_{;j} + \dot{b}^{i} + b^{i}v^{p}_{;p} = 0\ . \tag{2.2}$$

Substituting $b^i \equiv q^i - \rho\dot{v}^i$ and introducing the Lie derivative of the stress tensor $L_v\left(t^{ij}\right) = \dot{t}^{ij} - t^{ip}v^{j}_{;p} - t^{pj}v^{i}_{;p}$ as a stress-rate and making use of the continuity equation, the equation of motion (2.2) for the Lie derivative of the stresses assumes the form

$$\left[L_v\left(t^{ij}\right)\right]_{;j} + \left(t^{pj}v^{i}_{;p} + t^{ij}v^{p}_{;p}\right)_{;j} + \dot{q}^{i} + q^{i}v^{p}_{;p} = \rho\ddot{v}^{i}\ . \tag{2.3}$$

This equation is referred to as the equation of motion for the stress-rate [8,9,10,12].

3. The third order wave

When the basic quantities v^k, t^{kl}, e_{kl} and their first derivatives are all continuous, but the second derivatives have a jump when crossing the surface $\varphi\left(x^k, t\right) = 0$, we speak about third order waves [2]. Let us denote the jump of some quantity $v^k_{;p}$ by $\left\langle v^k_{;p}\right\rangle$. When the velocity gradient is $v^k_{;p}$ and we consider a wave of order three then $\left\langle v^k_{;p}\right\rangle = 0$ but $\left\langle v^k_{;qp}\right\rangle \neq 0$. Thus in (2.3) $\left\langle L_v\left(t^{kp}\right)\right\rangle = 0$ but $\left\langle L_v\left(t^{kp}_{\ \ ;p}\right)\right\rangle \neq 0$.

Consequently, the dynamic condition a third order wave should meet is of the form

$$L_v\left(t^{k\ell}_{\ \ ;\ell}\right) + t^{pq}\left\langle v^k_{;qp}\right\rangle + t^{k\ell}\left\langle v^p_{\ ;p\ell}\right\rangle = \rho\left\langle \ddot{v}^k\right\rangle . \tag{3.1}$$

Let the kinematic equation [4, 5] be

$$\left(L_v e_{ij}\right)^{\cdot} = \ddot{e}_{ij} + \left(e_{kj}v^k_{;i} + e_{ik}v^k_{;j}\right) \tag{3.2}$$

where e_{kj} is the Euler strain tensor. If the Lie derivative of the velocity field is $Ł_v$, the expression $L_v \equiv Ł_v + \partial/\partial t$ in (3.2) is a generalization [3]. As is well known $L_v\left(e_{ij}\right) = v_{ij}$, thus the kinematic compatibility condition for the third order wave is

$$\left\langle \ddot{v}_{ij}\right\rangle = \left\langle \ddot{e}_{ij}\right\rangle + e_{kj}\left\langle \dot{v}^k_{;i}\right\rangle + e_{ik}\left\langle \dot{v}^k_{;j}\right\rangle . \tag{3.3}$$

It can easily be shown that $\dot{v}^k_{;i} \neq \left(v^k_{;i}\right)^{\cdot}$, but $\left\langle \dot{v}^k_{;i}\right\rangle = \left\langle \left(v^k_{;i}\right)^{\cdot}\right\rangle$ and this property is the same for the second derivatives of all other functions.

Let the constitutive equation be [14]

$$f_\alpha\left(L_v\left(t^{ij}\right), Q^{ij}, L_v\left(e_{ij}\right), q_{ij}, t^{ij}, e_{ij}\right) = 0, \qquad (\alpha = 1, 2, ..., 6) \tag{3.4}$$

where $Q^{ij} = B^{ijm}_{\ \ \ \ pq}t^{pq}_{\ \ ;m}$ and $q_{ij} = b_{ij}^{\ \ pq\ell}e_{pq;\ell}$, if $B^{ijm}_{\ \ \ \ pq}$ and $b_{ij}^{\ \ pq\ell}$ are appropriate tensors transforming tensors $t^{pq}_{\ \ ;m}$ and $e_{pq;\ell}$ into second order ones.

The constitutive compatibility conditions can be obtained from the material derivative of (3.4)

$$\frac{\partial f_\alpha}{\partial L_v\left(t^{ij}\right)}\left\langle L_v\left(t^{ij}\right)^{\cdot}\right\rangle + \frac{\partial f_\alpha}{\partial Q^{ij}}B^{ijm}_{\ \ \ \ pq}\left\langle \dot{t}^{pq}_{\ \ ;m}\right\rangle + \frac{\partial f_\alpha}{\partial L_v\left(e_{ij}\right)}\left\langle L_v\left(e_{ij}\right)^{\cdot}\right\rangle + \frac{\partial f_\alpha}{\partial q_{ij}}b_{ij}^{\ \ pq\ell}\left\langle \dot{e}_{pq;\ell}\right\rangle = 0, \qquad (\alpha = 1, 2, ..., 6)$$

or introducing the notations $S^{rs}_{\ \ ij}, R^{rs}_{\ \ ij}, T^{rsij}$ and U^{rsij} for the coefficients

$$S^{rs}_{\ \ ij} \equiv \frac{\partial f_\alpha}{\partial L_v\left(t^{ij}\right)}, R^{rs}_{\ \ ij} \equiv \frac{\partial f_\alpha}{\partial Q^{ij}}, T^{rsij} \equiv \frac{\partial f_\alpha}{\partial L_v\left(e_{ij}\right)}, U^{rsij} \equiv \frac{\partial f_\alpha}{\partial q_{ij}}$$

the constitutitve compatibility condition is

$$S^{rs}_{\ \ ij}\left\langle L_v\left(t^{ij}\right)^{\cdot}\right\rangle + R^{rs}_{\ \ ij}B^{ijm}_{\ \ \ \ pq}\left\langle \dot{t}^{pq}_{\ \ ;m}\right\rangle + T^{rsij}\left\langle L_v\left(e_{ij}\right)^{\cdot}\right\rangle + U^{rsij}b_{ij}^{\ \ pq\ell}\left\langle \dot{e}_{pq;\ell}\right\rangle = 0 . \tag{3.5}$$

In the following we use Cartesian coordinates. Let the jumps in the second derivatives of the stress and strain tensors and that of the velocity field, each on the surface $\varphi\left(x^k, t\right) = 0$, are denoted by γ^{ij}, α_{ij} and λ^k. Further denote n_k the unit normal vector of the wavefront and C and c denote the wave propagation velocity with respect to the material and to the reference frame:

$$n_k \equiv \frac{\dfrac{\partial \varphi}{\partial x^k}}{\sqrt{g^{pq}\dfrac{\partial \varphi}{\partial x^p}\dfrac{\partial \varphi}{\partial x^q}}}, \qquad C = c - v^k n_k,$$

With these notations one can conclude that the equations (3.1), (3.3) and (3.5) imply [2], [4]:

$$\gamma^{k\ell} n_\ell = -\rho C \lambda^k \tag{3.6}$$

$$\alpha_{ij} = \frac{1}{2C}\left[n_i\left(2e_{kj} - g_{kj}\right) + n_j\left(2e_{ik} - g_{ik}\right)\right]\lambda^k \tag{3.7}$$

$$S^{rs}{}_{ij}\left(C\gamma^{ij} + t^{iq}n_q\lambda^j + t^{qj}n_q\lambda^i - t^{ij}n_\ell\lambda^\ell\right) - R^{rs}{}_{ij}B^{ijm}{}_{pq}n_m\gamma^{pq} + \\ + T^{rsij}\left[C\alpha_{ij} - \lambda^k\left(e_{ik}n_j + e_{kj}n_i\right)\right] - U^{rsij}b_{ij}{}^{pq\ell}n_\ell\alpha_{pq} = 0 \tag{3.8}$$

Making use of equations (3.6) and (3.7) we get from (3.8) that

$$\begin{gathered}\left\{2\rho S^{rs}{}_{k\ell}C^3 - 2\rho \bar{R}^{rs}{}_{k\ell}C^2 + \right.\\ \left[T^{rsij}\left(g_{kj}n_i + g_{ik}n_j\right) - 2S^{rs}{}_{ij}\left(t^{iq}g^j{}_k + t^{qj}g^i{}_k - t^{ij}g^q{}_k\right)n_q\right]n_\ell C + \\ \left.\bar{U}^{rsij}\left[n_i\left(2a_{kj} - g_{kj}\right) + n_j\left(2a_{ik} - g_{ik}\right)\right]n_\ell\right\}\gamma^{k\ell} = 0,\end{gathered} \tag{3.9}$$

where $g^j{}_k$ denotes Kronecker's symbol. Since $\gamma^{k\ell}$ is different from zero, the determinant of its coefficient matrix must vanish [13], that is,

$$\det\{\} = 0.$$

This is the equation of propagation for the third order wave. Clearly this equation is an algebraic equation of order 18 for the propagation velocity C. Observe that the notations $\bar{R}^{rs}{}_{k\ell} \equiv R^{rs}{}_{pq}B^{pqm}{}_{k\ell}n_m$ and $\bar{U}^{rsij} \equiv U^{rspq}b_{pq}{}^{ijm}n_m$ are used in the matrix $\{\}$. These notations can also be employed in equation (3.8). The matrix $\{\}$ can be considered as the characteristic matrix of a generalization of the acoustic tensor.

Equation (3.8) can also be written in a form consisting of λ^k when $S^{rs}{}_{ij}$ takes some of the values 0, $S^r_i g^s_j$, $g^r_i g^s_j$ and $\bar{R}^{rs}{}_{ij}$ one of 0, $\bar{R}^r_i g^s_j$, $\bar{R}^{rs}{}_i n_j$ and we multiply the equations by n_s. (In the case of a solid body S and $\bar{R}$ are probably impossible to be zero at the same time).

In the 9 cases under consideration, the equation for λ^k is

$$\left(E^r_k C^3 + F^r_k C^2 + G^r_k C + H^r_k\right)\lambda^k = 0\,. \tag{3.10}$$

In the second case $E^r_k = g^r_k$ in (3.10).

The wave propagation equation is

$$\det\left(E^r_k C^3 + F^r_k C^2 + G^r_k C + H^r_k\right)\lambda^k = 0\ . \tag{3.11}$$

This is a 9-th order equation for the propagation velocity .

By using [6] the matrix of acoustic tensor in case of (3.10) can be obtained. Let us denote the coefficients of C in the form of 3x3 matrices by $\mathbf{E}, \mathbf{F}, \mathbf{G}, \mathbf{H}$. By introducing the inverse and unit matrices $\mathbf{E}^{-1}$, $\mathbf{I}$ the acoustic tensor is

$$\begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \\ -\mathbf{E}^{-1}\cdot\mathbf{H} & -\mathbf{E}^{-1}\cdot\mathbf{G} & -\mathbf{E}^{-1}\cdot\mathbf{F} \end{bmatrix}. \tag{3.12}$$

Comparing the expression (3.1) of [4] and expression (3.9) of this paper we find that these are the same in the case of acceleration wave [4] and in the present one.

The most general acoustic tensor can be obtained from (3.9) when the coefficients have been denoted in the form of 6x6 matrices $\mathbf{E}, \mathbf{F}, \mathbf{G}, \mathbf{H}$. The shape of acoustic tensor is identical to (3.12) [7]. Matrix (3.12) is a 9x9 matrix while this generalized matrix is a 18x18 one.

4. Concluding remarks

Our starting point was the generalized Clapeyron's equation, which, by using continuity equation, resulted in the first Cauchy equation of motion and the equation of motion for stress-rate. In such a way we obtain a more general equation for stress-rate which enables us to write the compatibility equations for the third order wave. These two sets of equations are the main result of this paper. As an application these equations lead to an acoustic tensor, which is identical to the acoustic tensor obtained by acceleration waves, because we used quasilinear integrable second order constitutive equations. This computation justifies that we could similarly get an acoustic tensor in case of a general second order constitutive equation.

REFERENCES

1. Béda, G.: *Generalization of Clapeyron's theorem of solids*, Periodica Polytechnica Ser. Mech. Eng., **44**(1), (2000), 5-7.
2. Eringen, A. C. and Suhubi, E. S.: *Elastodynamics*, Academic Press, New York and London, 1974.
3. Marsden, J. E. and Hughes, T. J. R.: *Mathematical Foundations of Elasticity*, Prentice-Hall, Englewood Cliffs, N.Y., 1983.
4. Béda, G.: *The possible fundamental equations of the constitutive equations of solids*, Newsletter TU Budapest, 10(3), (1992), 5-11.
5. Béda, G.: *The possible fundamental equations of continuum mechanics*, Periodica Polytechnica Ser. Mech. Eng. **35**(1-2), (1991), 15-22.

6. Bishop, R.E.D., Gladwell, G. M. L. and Michaelson, S.: *The Matrix Analysis of Vibration*, Cambridge University Press, 1965.
7. Béda, G.: *Constitutive Equations of Moving Plastic Bodies*, DSc. thesis, Budapest 1982 (in Hungarian).
8. Hill, R.: *Some basic principles in the mechanics of solids without a natural time.* Journal of the Mechanics and Physics of Solids, **7**, (1959), 209-225.
9. Thompson, E.G. and Szu-Wei Yu: *A flow formulation for rate equilibrium equations*, Int. J. for Numerical Methods in Engineering, **30**, (1990), 1619-1632.
10. Dubey R. N.: *Variation method for nonconservative problems,* Trans. ASME Journal of Applied Mechanics, **37(1)**, (1970), 133-136.
11. Béda, G., Kozák I. and Verhás J.: *Continuum mechanics*, Akadémiai Kiadó, Budapest, 1995.
12. Béda, G.: *The principle of virtual work on continuous media*, Newsletter TU of Budapest, 10(1), (1992), 5-9.
13. Béda G. and Béda P.: *A study on constitutive relations of copper using the existence of acceleration waves and dynamical systems*, Proc. of Estonian Academy of Sci. Engin. **5**, (1999), 101-111.
14. Béda, P. and Béda, G.: *Gradient constitutive equation for finite deformation by using acceleration waves and dynamical Systems*, Proceedings of ICES2K, Tech. Science Press, 2000 (in press).