Journal of Computational and Applied Mechanics, Vol. 1., No. 2., (2000) pp. 167–178

VIBRATIONS OF CIRCULAR ARCHES SUBJECTED TO HYDROSTATIC FOLLOWER LOADS – COMPUTATIONS BY THE USE OF GREEN FUNCTIONS

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[Received: April 30, 2000]

Abstract. Using the Green function matrix, self adjoint eigenvalue problems governed by degenerate systems of differential equations and homogeneous linear boundary conditions can be replaced – like the case of scalar equations – by an eigenvalue problem for a system of Fredholm integral equations with the Green function matrix as a kernel. We have determined the Green function matrix for simply supported and fixed circular arches provided that the arch is also subjected to a hydrostatic follower load. In the knowledge of the Green function matrix, the self adjoint eigenvalue problem giving the natural frequencies of the vibrations as a function of the follower load can be replaced by an eigenvalue problem described by a system of Fredholm integral equations. The latter is reduced to an algebraic eigenvalue problem and the first eigenvalues are computed by applying the QZ algorithm. The results computed show how the load affects the first natural frequencies of the arches.

Keywords: Circular arch, natural frequencies, stability, hydrostatic follower load, Green function matrix, eigenvalue problem

1. Introduction

There is a classical definition for the Green function of ordinary linear inhomogeneous differential equations associated with homogeneous boundary conditions [1]. The definition has been generalized – see paper [2] for details – for a degenerate system of linear differential equations by keeping up the structure of the definition given in [1]. It is also well known that in the knowledge of the corresponding Green function eigenvalue problems for the differential equation can be replaced by an eigenvalue problem for a Fredholm integral equation with the Green function as a kernel. The latter can effectively be solved by various algorithms – see [3] for details.

There are a lot of works on eigenvalue problems associated with the free vibration and stability of circular arches. Without trying to achieve completeness we should mention the book by Federhoffer [5], and the papers [6,7]. For further references the reader is referred to the papers mentioned. To the author's knowledge the issue how the load affects the vibration of the arch has not been investigated yet. With regard to this fact the main objectives of the present paper are as follows:

- to determine the Green function matrices for simply supported and fixed circular arches subjected to a constant hydrostatic follower load;
- to reduce the eigenvalue problem giving the natural frequencies of the free

vibrations as a function of the hydrostatic follower load to an eigenvalue problem for a system of Fredholm integral equations with the Green function matrix as a kernel and
– to compute and analyse the first natural frequencies as functions of the follower load.

The paper is organized into four sections. Section 2 is devoted to some preliminaries with an emphasis on the definition of the Green function matrix. Section 3 presents the governing equations for the two problems and gives the corresponding Green functions. Section 4 is a brief summary of the solution algorithm and some solutions can also be found there. The last section is a summary of the results.

2. The Green function matrix

Consider the degenerate system of differential equations

$$\begin{gathered}\mathbf{K}(\mathbf{y})=\sum_{\nu=0}^{n}\overset{\nu}{\mathbf{P}}(x)\mathbf{y}^{(\nu)}(x)=\\ =\begin{bmatrix}0 & 0\\ 0 & \overset{n}{\mathbf{P}}_{22}\end{bmatrix}\begin{bmatrix}\mathbf{y}_1\\ \mathbf{y}_2\end{bmatrix}^{(n)}+\cdots+\begin{bmatrix}0 & 0\\ 0 & \overset{k+1}{\mathbf{P}}_{22}\end{bmatrix}\begin{bmatrix}\mathbf{y}_1\\ \mathbf{y}_2\end{bmatrix}^{(k+1)}+\\ +\begin{bmatrix}\overset{k}{\mathbf{P}}_{11} & \overset{k}{\mathbf{P}}_{12}\\ 0 & \overset{k}{\mathbf{P}}_{22}\end{bmatrix}\begin{bmatrix}\mathbf{y}_1\\ \mathbf{y}_2\end{bmatrix}^{(k)}+\cdots+\begin{bmatrix}\overset{s}{\mathbf{P}}_{11} & \overset{s}{\mathbf{P}}_{12}\\ \overset{s}{\mathbf{P}}_{21} & \overset{s}{\mathbf{P}}_{22}\end{bmatrix}\begin{bmatrix}\mathbf{y}_1\\ \mathbf{y}_2\end{bmatrix}^{(s)}+\\ +\cdots+\begin{bmatrix}\overset{0}{\mathbf{P}}_{11} & \overset{0}{\mathbf{P}}_{12}\\ \overset{0}{\mathbf{P}}_{21} & \overset{0}{\mathbf{P}}_{22}\end{bmatrix}\begin{bmatrix}\mathbf{y}_1\\ \mathbf{y}_2\end{bmatrix}=\begin{bmatrix}\mathbf{r}_1\\ \mathbf{r}_2\end{bmatrix}\end{gathered}\tag{2.1}$$

where $n > k > s > 0$, l is the number of unknown functions (the size of $\mathbf{y}$), j is the size of $\mathbf{y}_2$ and the matrices $\overset{\nu}{\mathbf{P}}$ and $\mathbf{r}^T = [\,\mathbf{r}_1^T\,|\,\mathbf{r}_2^T\,]$ are continuous for $x \in [a, b]$; $a < b$. The matrices $\overset{n}{\mathbf{P}}_{22}$ and $\overset{k}{\mathbf{P}}_{11}$ are assumed to be invertible if $x \in [a, b]$.

The system of ODEs (2.1) is associated with linear homogeneous boundary conditions

$$\begin{gathered}\mathbf{U}_\mu(\mathbf{y})=\sum_{\nu=0}^{n-1}\Big[\mathbf{A}_{\nu\mu}\mathbf{y}^{(\nu)}(a)+\mathbf{B}_{\nu\mu}\mathbf{y}^{(\nu)}(b)\Big]=\\ =\sum_{\nu=0}^{n-1}\left\{\begin{bmatrix}\overset{11}{\mathbf{A}}_{\nu\mu} & \overset{12}{\mathbf{A}}_{\nu\mu}\\ \overset{21}{\mathbf{A}}_{\nu\mu} & \overset{22}{\mathbf{A}}_{\nu\mu}\end{bmatrix}\begin{bmatrix}\mathbf{y}_1(a)\\ \mathbf{y}_2(a)\end{bmatrix}^{(\nu)}++\begin{bmatrix}\overset{11}{\mathbf{B}}_{\nu\mu} & \overset{12}{\mathbf{B}}_{\nu\mu}\\ \overset{21}{\mathbf{B}}_{\nu\mu} & \overset{22}{\mathbf{B}}_{\nu\mu}\end{bmatrix}\begin{bmatrix}\mathbf{y}_1(b)\\ \mathbf{y}_2(b)\end{bmatrix}^{(\nu)}\right\}=\begin{bmatrix}0\\ 0\end{bmatrix}\end{gathered}\tag{2.2}$$

where $\mu = 1, \ldots, n$; and for $\nu \le k$ the constant matrices $\mathbf{A}_{\nu\mu}$ and $\mathbf{B}_{\nu\mu}$ fulfill the conditions

$$\overset{11}{\mathbf{A}}_{\nu\mu}=\overset{21}{\mathbf{A}}_{\nu\mu}=\overset{11}{\mathbf{B}}_{\nu\mu}=\overset{21}{\mathbf{B}}_{\nu\mu}=0.$$

Solution to the boundary value problem (2.1),(2.2) is sought in the form

$$\mathbf{y}(x) = \int_a^b \mathbf{G}(x,\xi)\mathbf{r}(\xi)d\xi \tag{2.3}$$

in which $\mathbf{G}(x,\xi)$ is the Green function matrix [3,4].

If there exists the Green function matrix for the BVP (2.1), (2.2) then the vector (2.3) satisfies the differential equation (2.1) and the boundary conditions (2.2) [3,4].

As regards a proof of existence for the Green function matrix we refer to [4].

Let the system of differential equations read

$$\mathbf{K}[\mathbf{y}] = \lambda\mathbf{y} \tag{2.4}$$

in which $\mathbf{K}[\mathbf{y}]$ is given by (2.1) and λ is the eigenvalue sought. The ODEs (2.4) are associated with the linear homogeneous boundary conditions (2.2). These are assumed to be independent of λ.

Recalling (2.3) the eigenvalue problem (2.4), (2.2) can be replaced by an eigenvalue problem for the system of integral equations

$$\mathbf{y}(x) = \lambda\int_a^b \mathbf{G}(x,\xi)\mathbf{y}(\xi)d\xi. \tag{2.5}$$

On the basis of [3] a procedure for the numerical solution of the above problem (2.5) has been presented in [4].

3. Vibration of circular arches subjected to a hydrostatic follower load

The arch with radius R is symmetric with respect to the plane of its center line. The cross sectional area and the second moment of inertia with respect to the centroidal axis perpendicular to the plane of the arch are denoted by A and I, respectively. The angle coordinate φ changes in the interval $[-\vartheta,\vartheta]$, the central angle $\bar{\vartheta}$ subtended by the arch is equal to 2ϑ. The Young modulus of elasticity is denoted by E.

Vibrations of circular arches subjected to a constant follower load are governed by the differential equation

$$\begin{aligned}\mathbf{K}[\mathbf{y},\varepsilon_0] = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} U \\ W \end{bmatrix}^{(4)} + \begin{bmatrix} -m & 0 \\ 0 & 2-m\varepsilon_0 \end{bmatrix}\begin{bmatrix} U \\ W \end{bmatrix}^{(2)} + \\ + \begin{bmatrix} 0 & -m \\ m & 0 \end{bmatrix}\begin{bmatrix} U \\ W \end{bmatrix}^{(1)} + \begin{bmatrix} 0 & 0 \\ 0 & M-m\varepsilon_0 \end{bmatrix}\begin{bmatrix} U \\ W \end{bmatrix} = \lambda\begin{bmatrix} U \\ W \end{bmatrix}.\end{aligned} \tag{3.1}$$

Here $(\dots)^{(n)} = d^{(n)}/d\varphi^{(n)}$, U and V are the amplitudes of the tangential and normal displacements,

$$M = \frac{AR^2}{I}, \quad m = M-1. \tag{3.2}$$

ε_0 is the axial strain on the center line of the circular arch (this value is constant and uniquely determined constant by the current volume of the load). Neglecting the vibrations, i.e., setting λ to 0 we obtain an ODE

$$U^{(6)} + 2U^{(4)} + U^{(2)} - m\varepsilon_0 \left(U^{(4)} + U^{(2)}\right) = 0 \tag{3.3}$$

which – provided that it is associated with appropriate boundary conditions – gives the critical load. Setting ε_0 to 0 we get a system of ODEs for the free vibrations of the arch [4].

It can be proved that the first critical axial strain (for the first buckling mode) [2] is of the form:

$$\varepsilon_{0crit} = -\frac{1}{m}\left[\left(s_i \frac{\pi}{\vartheta}\right)^2 - 1\right] < 0 \tag{3.4}$$

where

$$s_1 = 1 \tag{3.5}$$

for a simply supported arch and

$$s_2 = s_2(\vartheta) \approx$$
$$\approx \begin{cases} 0.03436558207\vartheta^2 - 0.01102140558\vartheta + 1.431758411 & \text{if} \quad \vartheta \in (0; 1.7] \\ 0.1760886555\vartheta^2 - 0.5259986022\vartheta + 1.899253872 & \text{if} \quad \vartheta \in (1.7; 3.14] \end{cases} \tag{3.6}$$

for a fixed arch.

The corresponding hydrostatic follower load can be obtained from the equations

$$p_{crit} = \frac{1}{k_i(m, \vartheta)} \frac{IE}{R^3}\left[\left(s_i \frac{\pi}{\vartheta}\right)^2 - 1\right] \tag{3.7}$$

where

$$k_i = \frac{g_i(\vartheta)}{\vartheta/m + g_i(\vartheta)}$$

and

$$g_1 = \frac{3}{2}\vartheta + \tan\vartheta\left[\frac{\vartheta}{2}\tan\vartheta - \frac{3}{2}\right], \qquad g_2 = \vartheta - \frac{2\sin^2\vartheta}{\vartheta + \sin\vartheta\cos\vartheta}. \tag{3.8}$$

There are no closed form solutions giving the natural frequencies of the free vibration of the arches.

Depending on the supports applied, the system of ODEs (3.1) is associated with the following boundary conditions:

Simple supported arch $[i = 1]$:

$$\begin{aligned} U(-\vartheta) &= 0 & \qquad U(\vartheta) &= 0 \\ W(-\vartheta) &= 0 & \qquad W(\vartheta) &= 0 \\ W^{(2)}(-\vartheta) &= 0 & \qquad W^{(2)}(\vartheta) &= 0 \end{aligned} \tag{3.9}$$

Fixed arch $[i = 2]$:

$$\begin{aligned} U(-\vartheta) &= 0 & \qquad U(\vartheta) &= 0 \\ W(-\vartheta) &= 0 & \qquad W(\vartheta) &= 0 \\ W^{(1)}(-\vartheta) &= 0 & \qquad W^{(1)}(\vartheta) &= 0 \end{aligned} \tag{3.10}$$

Each of the eigenvalue problems (3.1),(3.9) and (3.1), (3.10) is self adjoint and positive definite if $\varepsilon_0 < 0$. The corresponding Green function matrix assumes the form

$$\underbrace{\mathbf{G}(\varphi,\psi)}_{(2\times2)} = \sum_{j=1}^{4} \mathbf{Y}_j(\varphi)\,[\mathbf{A}_j(\psi) \pm \mathbf{B}_j(\psi)] \tag{3.11}$$

where the sign is {positive}[negative] if $\{\varphi \le \psi\}[\varphi \ge \psi]$ and

$$\begin{aligned} \mathbf{Y}_1 &= \begin{bmatrix} \cos\varphi & 0 \\ \sin\varphi & 0 \end{bmatrix} & \qquad \mathbf{Y}_2 &= \begin{bmatrix} -\sin\varphi & 0 \\ \cos\varphi & 0 \end{bmatrix} \\ \mathbf{Y}_3 &= \begin{bmatrix} \cos(k\varphi) & (M - m\varepsilon_o)\varphi \\ k\sin(k\varphi) & -m \end{bmatrix} & \qquad \mathbf{Y}_4 &= \begin{bmatrix} -\sin(k\varphi) & 1 \\ k\cos(k\varphi) & 0 \end{bmatrix} \end{aligned} \tag{3.12}$$

are solutions of the homogenous $\mathbf{K}[\mathbf{y},\varepsilon_0] = \mathbf{0}$ with

$$k^2 = 1 + \varepsilon_0 - M\varepsilon_0, \tag{3.13}$$

and

$$\mathbf{A}_j = \begin{bmatrix} \overset{j}{A}_{11} & \overset{j}{A}_{12} \\ \overset{j}{A}_{21} & \overset{j}{A}_{22} \end{bmatrix}, \qquad \mathbf{B}_j = \begin{bmatrix} \overset{j}{B}_{11} & \overset{j}{B}_{12} \\ \overset{j}{B}_{21} & \overset{j}{B}_{22} \end{bmatrix} \qquad j = 1,\dots,4 \tag{3.14}$$

are functions of the angle coordinate ψ. The equation systems giving the unknowns

$$a = \overset{1}{B}_{1i},\ b = \overset{2}{B}_{1i},\ c = \overset{3}{B}_{1i},\ d = \overset{3}{B}_{2i},\ e = \overset{4}{B}_{1i},\ f = \overset{4}{B}_{2i} \qquad i = 1,2$$

can be set up from the second property of the Green function matrix – see the definition in [4]. The functions $\overset{1}{B}_{11}(\psi),\dots,\overset{4}{B}_{22}(\psi); \psi \in [-\vartheta,\vartheta]$ are independent of the boundary conditions.

The first system of equations ($i = 1$):

$$\begin{bmatrix} \cos\psi & -\sin\psi & \cos(k\psi) & (M-m\varepsilon_o)\psi & -\sin(k\psi) & 1 \\ \sin\psi & \cos\psi & k\sin(k\psi) & -m & k\cos(k\psi) & 0 \\ -\sin\psi & -\cos\psi & -k\sin(k\psi) & M-m\varepsilon_o & -k\cos(k\psi) & 0 \\ \cos\psi & -\sin\psi & k^2\cos(k\psi) & 0 & -k^2\sin(k\psi) & 0 \\ -\sin\psi & -\cos\psi & -k^3\sin(k\psi) & 0 & -k^3\cos(k\psi) & 0 \\ -\cos\psi & \sin\psi & -k^4\cos(k\psi) & 0 & k^4\sin(k\psi) & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2m} \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{3.15}$$

We have found the following solutions:

$$\left.\begin{aligned} a &= \overset{1}{B}_{11} = -\frac{1}{1-k^2}\frac{\sin\psi}{2} & b &= \overset{2}{B}_{11} = -\frac{1}{1-k^2}\frac{\cos\psi}{2} \\ c &= \overset{3}{B}_{11} = \frac{1}{2}\frac{\sin k\psi}{k^3(1-k^2)} & d &= \overset{3}{B}_{21} = \frac{1}{2mk^2} \\ e &= \overset{4}{B}_{11} = \frac{\cos k\psi}{2k^3(1-k^2)} & f &= \overset{4}{B}_{21} = -\frac{1}{2}\left(k^2+m\right)\frac{\psi}{mk^2} \end{aligned}\right\} \tag{3.16}$$

The second system of equations ($i = 2$):

$$\begin{bmatrix} \cos\psi & -\sin\psi & \cos(k\psi) & (M-m\varepsilon_o)\psi & -\sin(k\psi) & 1 \\ \sin\psi & \cos\psi & k\sin(k\psi) & -m & k\cos(k\psi) & 0 \\ -\sin\psi & -\cos\psi & -k\sin(k\psi) & M-m\varepsilon_o & -k\cos(k\psi) & 0 \\ \cos\psi & -\sin\psi & k^2\cos(k\psi) & 0 & -k^2\sin(k\psi) & 0 \\ -\sin\psi & -\cos\psi & -k^3\sin(k\psi) & 0 & -k^3\cos(k\psi) & 0 \\ -\cos\psi & \sin\psi & -k^4\cos(k\psi) & 0 & k^4\sin(k\psi) & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\frac{1}{2} \end{bmatrix} \tag{3.17}$$

We have found the following solutions:

$$\left.\begin{aligned} a &= \overset{1}{B}_{12} = \frac{\cos\psi}{2(1-k^2)} & b &= \overset{2}{B}_{12} = -\frac{\sin\psi}{2(1-k^2)} \\ c &= \overset{3}{B}_{12} = -\frac{\cos k\psi}{2k^2(1-k^2)} & d &= \overset{3}{B}_{22} = 0 \\ e &= \overset{4}{B}_{12} = \frac{1}{2k^2(1-k^2)}\sin k\psi & f &= \overset{4}{B}_{22} = \frac{1}{2k^2} \end{aligned}\right\} \tag{3.18}$$

Taking into account that the Green function matrix should meet boundary conditions (3.9) (3.10) one can find the functions

$$\overset{1}{A}_{11}(\psi), \dots, \overset{4}{A}_{22}(\psi); \qquad \psi \in [-\vartheta, \vartheta]$$

as well.

Simply supported arch ($i = 1$):

$$
\begin{bmatrix}
\cos\vartheta & \sin\vartheta & \cos(k\vartheta) & -(M - m\varepsilon_o)\vartheta & \sin(k\vartheta) & 1 \\
\cos\vartheta & -\sin\vartheta & \cos(k\vartheta) & (M - m\varepsilon_o)\vartheta & -\sin(k\vartheta) & 1 \\
-\sin\vartheta & \cos\vartheta & -k\sin(k\vartheta) & -m & k\cos(k\vartheta) & 0 \\
\sin\vartheta & \cos\vartheta & k\sin(k\vartheta) & -m & k\cos(k\vartheta) & 0 \\
\sin\vartheta & -\cos\vartheta & k^3\sin(k\vartheta) & 0 & -k^3\cos(k\vartheta) & 0 \\
-\sin\vartheta & -\cos\vartheta & -k^3\sin(k\vartheta) & 0 & -k^3\cos(k\vartheta) & 0
\end{bmatrix}
\begin{bmatrix}
\overset{1}{A}_{1i} \\ \overset{2}{A}_{1i} \\ \overset{3}{A}_{1i} \\ \overset{3}{A}_{2i} \\ \overset{4}{A}_{1i} \\ \overset{4}{A}_{2i}
\end{bmatrix}
=
$$

$$
= \begin{bmatrix}
-a\cos\vartheta - b\sin\vartheta - c\cos(k\vartheta) + d(M - m\varepsilon_o)\vartheta - e\sin(k\vartheta) - f \\
a\cos\vartheta - b\sin\vartheta + c\cos(k\vartheta) + d(M - m\varepsilon_o)\vartheta - e\sin(k\vartheta) + f \\
a\sin\vartheta - b\cos\vartheta + ck\sin(k\vartheta) + dm - ek\cos(k\vartheta) \\
a\sin\vartheta + b\cos\vartheta + ck\sin(k\vartheta) - dm + ek\cos(k\vartheta) \\
-a\sin\vartheta + b\cos\vartheta - ck^3\sin(k\vartheta) + ek^3\cos(k\vartheta) \\
-a\sin\vartheta - b\cos\vartheta - ck^3\sin(k\vartheta) - ek^3\cos(k\vartheta)
\end{bmatrix} \tag{3.19}
$$

Solving the equations (3.19) we have

$$
\left.
\begin{aligned}
\overset{1}{A}_{1i} &= \frac{1}{C}\left[\left(1-k^2\right)b\cos\vartheta + dk^2m\right] \\
\overset{2}{A}_{1i} &= \frac{1}{D}\Big\{k\cos k\vartheta\left[-k^2m\cos\vartheta + \vartheta\left(1-k^2\right)(M - m\varepsilon_o)\sin\vartheta\right] - \\
&\qquad -m\sin k\vartheta\sin\vartheta\Big\}a - \frac{k^3m}{D}\left[c + f\cos k\vartheta\right] \\
\overset{3}{A}_{1i} &= -\frac{1}{k\left(1-k^2\right)\sin k\vartheta}\left[dm - ek\left(1-k^2\right)\cos k\vartheta\right] \\
\overset{3}{A}_{2i} &= \frac{k}{D}\left[a\left(1-k^2\right)\cos k\vartheta + c\left(1-k^2\right)\cos\vartheta + f\left(1-k^2\right)\cos k\vartheta\cos\vartheta\right] \\
\overset{4}{A}_{1i} &= \frac{1}{D}\Big\{am + c\Big[k\vartheta\left(1-k^2\right)(M - m\varepsilon_o)\sin k\vartheta\cos\vartheta + \\
&\qquad +m\left(k^3\sin k\vartheta\sin\vartheta + \cos k\vartheta\cos\vartheta\right)\Big] + fm\cos\vartheta\Big\} \\
\overset{4}{A}_{2i} &= -\frac{1}{k(\sin k\vartheta)C}\left\{bk\left(1-k^2\right)\sin k\vartheta + ek\left(1-k^2\right)\sin\vartheta\right\} + \\
&\qquad + \frac{1}{k(\sin k\vartheta)C}d\Big[k\vartheta\left(1-k^2\right)(M - m\varepsilon_o)(\sin k\vartheta\sin\vartheta) - \\
&\qquad -m\left(k^3\sin k\vartheta\cos\vartheta - \cos k\vartheta\sin\vartheta\right)\Big]
\end{aligned}
\right\} \tag{3.20}
$$

where

$$
\begin{aligned}
C &= \left(1-k^2\right)\sin\vartheta \\
D &= \vartheta k\left(1-k^2\right)(M - m\varepsilon_o)\cos k\vartheta\cos\vartheta + mk^3\cos k\vartheta\sin\vartheta - m\sin k\vartheta\cos\vartheta.
\end{aligned}
$$

Fixed arch ($i = 2$):

$$\begin{bmatrix} \cos\vartheta & \sin\vartheta & \cos(k\vartheta) & -(M-m\varepsilon_o)\vartheta & \sin(k\vartheta) & 1 \\ \cos\vartheta & -\sin\vartheta & \cos(k\vartheta) & (M-m\varepsilon_o)\vartheta & -\sin(k\vartheta) & 1 \\ -\sin\vartheta & \cos\vartheta & -k\sin(k\vartheta) & -m & k\cos(k\vartheta) & 0 \\ \sin\vartheta & \cos\vartheta & k\sin(k\vartheta) & -m & k\cos(k\vartheta) & 0 \\ \cos\vartheta & \sin\vartheta & k^2\cos(k\vartheta) & 0 & k^2\sin(k\vartheta) & 0 \\ \cos\vartheta & -\sin\vartheta & k^2\cos(k\vartheta) & 0 & -k^2\sin(k\vartheta) & 0 \end{bmatrix} \begin{bmatrix} \overset{1}{A}_{1i} \\ \overset{2}{A}_{1i} \\ \overset{3}{A}_{1i} \\ \overset{3}{A}_{2i} \\ \overset{4}{A}_{1i} \\ \overset{4}{A}_{2i} \end{bmatrix} =$$

$$= \begin{bmatrix} -a\cos\vartheta - b\sin\vartheta - c\cos(k\vartheta) + d(M-m\varepsilon_o)\vartheta - e\sin(k\vartheta) - f \\ a\cos\vartheta - b\sin\vartheta + c\cos(k\vartheta) + d(M-m\varepsilon_o)\vartheta - e\sin(k\vartheta) + f \\ a\sin\vartheta - b\cos\vartheta + ck\sin(k\vartheta) + dm - ek\cos(k\vartheta) \\ a\sin\vartheta + b\cos\vartheta + ck\sin(k\vartheta) - dm + ek\cos(k\vartheta) \\ -a\cos\vartheta - b\sin\vartheta - ck^2\cos(k\vartheta) - ek^2\sin(k\vartheta) \\ a\cos\vartheta - b\sin\vartheta + ck^2\cos(k\vartheta) - ek^2\sin(k\vartheta) \end{bmatrix} \tag{3.21}$$

Solving the equation system (3.21) we have:

$$\left.\begin{aligned} \overset{1}{A}_{1i} &= \frac{1}{D}\left[ek^2 + b\left(k\cos k\vartheta\cos\vartheta + \sin\vartheta\sin k\vartheta\right) - dmk\cos k\vartheta\right] \\ \overset{2}{A}_{1i} &= -\frac{1}{C}a\left[m\left(1-k^2\right)\sin k\vartheta\cos\vartheta - \right. \\ &\quad \left. -\vartheta k\left(M-m\varepsilon_o\right)\left(k\sin\vartheta\sin k\vartheta + \cos k\vartheta\cos\vartheta\right)\right] \\ \overset{3}{A}_{1i} &= -\frac{1}{kD}\left[b + ek\left(k\sin\vartheta\sin k\vartheta + \cos k\vartheta\cos\vartheta\right) - dm\cos\vartheta\right] \\ \overset{3}{A}_{2i} &= -\frac{1}{C}\left[a\left(1-k^2\right)\sin k\vartheta + ck\left(1-k^2\right)\sin\vartheta - \right. \\ &\quad \left. -fk\left(k\sin k\vartheta\cos\vartheta - \cos k\vartheta\sin\vartheta\right)\right] \\ \overset{4}{A}_{1i} &= -\frac{1}{C}c\left[k\vartheta\left(M-m\varepsilon_o\right)\left(k\cos\vartheta\cos k\vartheta + \sin k\vartheta\sin\vartheta\right)\right] - \\ &\quad -\frac{1}{C}\left[a\left(M-m\varepsilon_o\right)\vartheta + cm(1-k^2)\cos k\vartheta\sin\vartheta + mf\sin\vartheta\right] \\ \overset{4}{A}_{2i} &= \frac{1}{D}\left[\tfrac{1}{k}b\left(1-k^2\right)\cos k\vartheta + e\left(1-k^2\right)\cos\vartheta\right] - \\ &\quad -\frac{1}{kD}d\left[k\vartheta\left(M-m\varepsilon_o\right)\left(\sin k\vartheta\cos\vartheta - k\sin\vartheta\cos k\vartheta\right) - \right. \\ &\quad \left. -m\left(1-k^2\right)\cos k\vartheta\cos\vartheta\right] \end{aligned}\right\} \tag{3.22}$$

where

$$C = m(1-k^2)(\sin\vartheta \sin k\vartheta) + \vartheta k\,(M - m\varepsilon_o)\,(k \sin k\vartheta \cos\vartheta - \sin\vartheta \cos k\vartheta)$$
$$D = k \cos k\vartheta \sin\vartheta - \sin k\vartheta \cos\vartheta .$$

In the knowledge of the above functions we can substitute in the formula (3.11) to get the Green function matrix.

The eigenvalues $\lambda = \lambda\,[\varepsilon_0(p)]$ and the natural frequencies $\alpha = \alpha\,[\varepsilon_0(p)]$ – each as a function of the follower load p – can then be obtained by solving the eigenvalue problem

$$\mathbf{y}(\varphi) = \lambda \int_a^b \mathbf{G}(\varphi, \psi, \varepsilon_0)\mathbf{y}(\psi)d\psi \quad . \tag{3.23}$$

The numerical solution was found by reducing the eigenvalue problem (3.23) to an algebraic eigenvalue problem and solving the latter by the QZ algorithm - see [3,4] for details.

The functions $\lambda_1/\lambda_{1free} = \lambda_1/\lambda_{1free}(\varepsilon_0/\varepsilon_{0crit})$ and $\alpha_1^2/\alpha_{1free}^2 = \alpha_1^2/\alpha_{1free}^2\,(p/p_{crit})$ have proved to be linear for the central angles considered. Here λ_1 and α_{1j} are the first eigenvalues and the natural freqencies computed for a value of ε_0 while λ_{1free} and α_{1free} are also the eigenvalue and the corresponding natural freqency for the same circular arch if it is free of loads ($\varepsilon_0 = 0$).

4. Conclusions

Using the Green function matrix, self adjoint eigenvalue problems, which are governed by a degenerate system of differential equations and homogeneous linear boundary conditions, can be replaced by an eigenvalue problem for a system of Fredholm integral equations with the Green function matrix as kernel.

We have determined the Green function matrix for simply supported and fixed circular arches subjected to hydrostatic and constant follower loads. In the knowledge of the Green function matrix the self adjoint eigenvalue problem giving the natural frequencies of the free vibrations as a function of the hydrostatic follower load has been replaced by an eigenvalue problem described by a system of Fredholm integral equations. The latter is reduced to an algebraic eigenvalue problem and the first eigenvalues as functions of the load are computed by using the QZ algorithm.

The results are shown in Figures 1 and 2.

The variable along the longitudinal axis is the quotient

$$\frac{p}{p_{crit}} = \frac{\varepsilon_0}{\varepsilon_{0crit}} \tag{4.1}$$

where ε_{0crit} and p_{crit} are given by the equation (3.4) and (3.7).

Figures 1 and 2 represent the quotient

$$\frac{\lambda_1}{\lambda_{1free}} = \frac{\alpha_1^2}{\alpha_{1free}^2} \tag{4.2}$$

for simply supported and fixed arches respectively. λ_1 and α_1 are the eigenvalues and the corresponding circular frequencies computed under the assumption that the

Table 1

			Simply supported arch[1]		Fixed arch[2]	
ϑ	m	Symbol	$\varepsilon_0/\varepsilon_{0crit}$	$\lambda_1/\lambda_{1free}$	$\varepsilon_0/\varepsilon_{0crit}$	$\lambda_1/\lambda_{1free}$
0,4	35000	∘	0.1153 0.346 0.5767 0.8074	0.8847 0.654 0.4233 0.1926	0.2229 0.4457 0.6686 0.8914	0.7821 0.562 0.3392 0.1133
0,6	60000	+	0.2271 0.4543 0.6814 0.9086	0.7729 0.5457 0.3185 0.0913	0.1206 0.3619 0.7237 0.965	0.7821 0.562 0.3392 0.1133
0,8	120000	◇	0.0008 0.3328 0.6657 0.9985	0.99 0.667 0.3339 0.009	0.2057 0.4104 0.615 0.8197	0.8022 0.6014 0.3969 0.1881
1	240000[1] and 360000[2]	□	0.1082 0.4329 0.7576 0.8658	0.8918 0.567 0.2423 0.1339	0.1027 0.3068 0.5102 0.9183	0.9056 0.7136 0.5163 0.1001

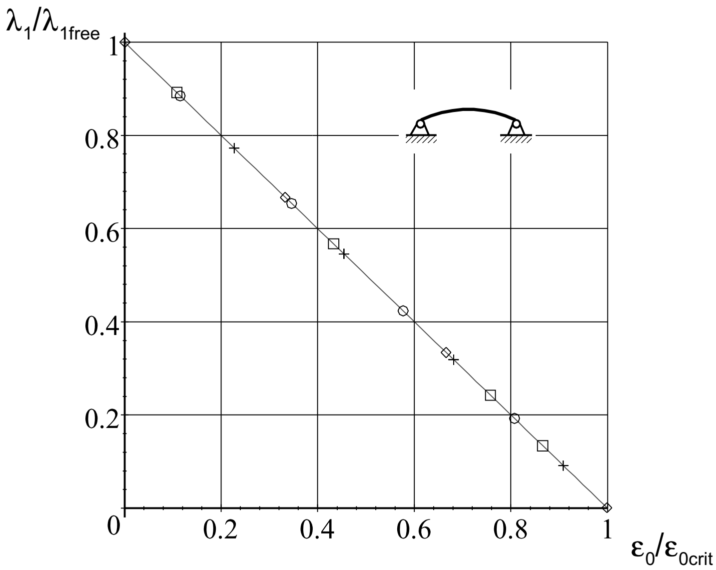

Figure 1.

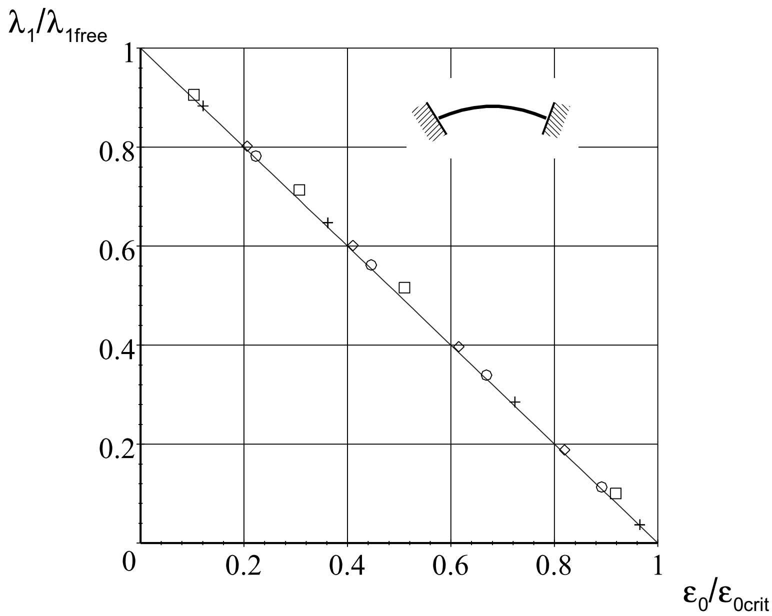

Figure 2.

arch is subjected to a follower load p which produces the axial strain ε_0. λ_{1free} and α_{1free} are also the eigenvalue and corresponding circular frequency for the same but unloaded arches.

Computations were carried out for $m = 20000; 35000; 60000; 120000; 240000; 360000$ provided that $\vartheta \in [0.1; 3]$. Some numerical results are presented in Table 1.

Figures 1 and 2 represent the quotient $\lambda_1/\lambda_{1free}$ as a function of the quotient $\varepsilon_0/\varepsilon_{0crit}$. It is clear from Figures 1 and 2 that

$$\frac{\lambda_1}{\lambda_{1free}} = 1 - \frac{\varepsilon_0}{\varepsilon_{0crit}} \tag{4.3}$$

for both cases, i.e., this result is the same for both support arrangements. We remark that the agreement of (4.3) with the results computed - see Figure 2 - is not as good for the fixed arch as for the simple supported arch. The reason for this is probably the fact that the solution for s_2 (3.8) is also an approximation obtained by the method of least squares [2]. It is also worthy of mention that function (4.3) is independent of m.

Acknowledgement. The support provided by the Hungarian National Research Foundation (project No. T031998) is gratefully acknowledged.

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