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THE INVERSE OF DIFFERENTIAL OPERATORS AND AN EXTENSION OF TREFFTZ'S METHOD

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Abstract. The purpose of the paper is to present an extension of Trefftz's idea. The essential of this method is to obtain lower bounds for eigenfrequencies of the elastic continuous structures. Let us suppose that we have good upper bounds that are sufficient in number and accuracy as well. This paper also contains a discussion of differential operators and their inverses or generalized inverses. A new example presents a technique for making generalized Green matrix.

Keywords: Trefftz's method, inverse of differential operator, Moore Penrose generalized inverse of differential operator

1. Introduction

The estimation of natural frequencies for elastic continuous rod structures is possible in several ways [1, 2]. For certain procedures - we think of the method of Trefftz or the method of orthogonal invariants - the explicit knowledge of the inverse to the ordinary differential operators is needed [3]. These procedures essentially lead to the construction of Green matrices. The present paper has two aims, one is an extension of Trefftz's idea [4] to obtain lower bounds for the eigenfreqencies of elastic continuous structures if we have good upper bounds, the other is to examine some problems related to the inverses of differential operators. Both will be illustrated by examples.

2. Differential operator and the Green matrix

By the interval $[a, b]$ we mean the set of all real numbers t such that $a \leq t \leq b$. Thus $[a, b] = \{t \in \mathbb{R} : a \leq t \leq b \quad a, b \in \mathbb{R}\}$ is a finite closed set, and $\mathbb{R}$ denotes the set of all real numbers. Let $L^2(a, b)$ denote the set of all Lebesgue integrable n-vector functions with real-valued coordinates. If $\mathbf{x}, \mathbf{y} \in L^2(a, b)$ are n-vector functions with the property that $\mathbf{y}^T\mathbf{x}$ is integrable, then

$$\langle \mathbf{x}, \mathbf{y} \rangle = \int_a^b \mathbf{y}(t)^T \mathbf{x}(t)\,\mathrm{d}t \tag{2.1}$$

is the inner product, where $\mathbf{y}(t)^T$ as a row vector is the transpose of the column vector $\mathbf{y}(t)$. Thus $L^2(a, b)$ with the inner product (2.1) is a Hilbert space [5]. (A complete inner product space is called Hilbert space.) Let $\mathbf{A}_1$ and $\mathbf{A}_0$ denote $n \times n$ matrix functions, $\mathbf{A}_1$ being absolutely continuous and nonsingular, and $\mathbf{A}_0$ being integrable on the interval $[a, b]$. Then

$$\mathcal{A}\mathbf{x} = \mathbf{A}_0(t)\mathbf{x}' + \mathbf{A}_1(t)\mathbf{x} \,. \tag{2.2}$$

is a linear differential operator with the domain $\mathcal{D}_{\mathcal{A}}$ which is the collection of all absolutely continuous n-dimensional vector functions $\mathbf{x}$. In order to formulate the boundary conditions as well, let $\boldsymbol{\xi}$ be a $2n$-vector made up of the n components of $\mathbf{x}(a)$ followed by the n components of $\mathbf{x}(b)$. Thus

$$\mathcal{D}_{\mathcal{A}} \;= \{\mathbf{x} \in L^2(a, b) : \mathcal{A}\mathbf{x} = L^2(a, b) \;\wedge\; \mathbf{M}\boldsymbol{\xi} = \mathbf{0}\}, \tag{2.3}$$

where the boundary conditions are given by the relation $\mathbf{M}\boldsymbol{\xi} = \mathbf{0}$, $\mathbf{M}$ denotes an $r \times 2n$ matrix ($r \leq 2n$) and $\boldsymbol{\xi} \in \mathbb{R}^{2n}$ is composed of the elements of the vectors $\mathbf{x}(a) \in \mathbb{R}^n$, and $\mathbf{x}(b) \in \mathbb{R}^n$. Since the set of solutions of the differential equation $\mathcal{A}\mathbf{x} = \mathbf{0}$ is n-dimensional, Ker$\mathcal{A}$ is a finite-dimensional vector space. If dimKer$\mathcal{A} = k$, it is known [3, 6], that

$$\max(0, n - r) \leq k \leq \min(n, 2n - r).$$

Now suppose that the operator $\mathcal{A}$ is invertible and its inverse is $\mathcal{A}^{-1}$. Then $\mathcal{A}$ is an injective operator ($k = 0$) and its range is the set $\mathcal{R}_{\mathcal{A}} = L^2(a, b)$. If $\mathcal{A}$ is invertible then $\mathcal{A}^{-1}$ is an integral operator with the kernel $\mathbf{G}(t, s)$. $\mathbf{G}(t, s)$ is an $n \times n$ matrix and is called Green's matrix. The inversion of $\mathcal{A}$ is given by the formula

$$\mathbf{x}(t) = (\mathcal{A}^{-1}\mathbf{y})(t) = \int_a^b \mathbf{G}(t, s)\mathbf{y}(s)\,\mathrm{d}s \tag{2.4}$$

$\forall \mathbf{y} \in L^2(a, b)$, and (2.4) is equivalent to $\mathcal{A}\mathbf{x} = \mathbf{y}$. Existence, uniqueness and properties of the Green matrix have been proved by several authors [5, 6]. The existence theorem for the Green matrix reads as

Theorem 1 *If the boundary value problem $\mathcal{A}\mathbf{x} = \mathbf{0}$, $\mathbf{M}\boldsymbol{\xi} = \mathbf{0}$ has only a trivial solution, then there is one and only one Green matrix $\mathbf{G}(t, s)$ of the differential operator $\mathcal{A}$ generated by the differential expression (2.2). The matrix $\mathbf{G}(t, s)$ has the following properties:*

- *the elements of $\mathbf{G}(t, s)$ are continuous and have continuous first derivatives in t except at $t = s$; $t, s \in [a, b]$;*
- *as t increases through s, $\mathbf{G}(t, s)$ has a jump discontinuity equal to $\mathbf{A}_0(s)^{-1}$, namely*

$$\mathbf{G}(s + 0, s) - \mathbf{G}(s - 0, s) = \mathbf{A}_0(s)^{-1}; \tag{2.5}$$

- *as a function of t, $\mathbf{G}(t, s)$ satisfies the boundary value problem $\mathcal{A}\mathbf{G} = \mathbf{0}$, $\mathbf{M}\boldsymbol{\xi} = \mathbf{0}$ for $a \leq t \leq s \leq b$ and $a \leq s \leq t \leq b$.*

If $\mathcal{A}$ is an invertible differential operator and its kernel is the matrix $\mathbf{G}(t, s)$, we find by using (2.4) that

$$(\mathcal{A}\mathbf{x})(t) = \int_a^b (\mathcal{A}\mathbf{G})(t, s)\mathbf{y}(s)\,\mathrm{d}s = \mathbf{y}(t) \,. \tag{2.6}$$

According to [3, 6] and (2.6) we have the following equations:

$$\mathcal{A}\mathbf{G} = \boldsymbol{\delta}_{\mathcal{A}}, \qquad \mathbf{M}\mathbf{G} = \mathbf{0}, \tag{2.7}$$

where $\boldsymbol{\delta}_{\mathcal{A}} = \delta\mathbf{E}$, $\mathbf{E}$ is a unit matrix and δ is Dirac's distribution [2]. The elements of the Green matrix can be obtained by integrating the equations (2.7) and the arbitrary functions consequent upon the integration can be determined by using the properties in Theorem 1.

3. An extension of Trefftz's method

Let $\mathcal{A}^{-1}$ be a real symmetric positive semidefinite completely continuous integral operator defined on the interval $[a, b]$ by the equation

$$(\mathcal{A}^{-1}\mathbf{x})(t) = \int_a^b \mathbf{G}(t,s)\mathbf{x}(s)\,\mathrm{d}s,$$

in which $\mathbf{G}(t,s) = \mathbf{G}(s,t)$ is the kernel. Then $\mathcal{A}^{-1}$ is also self-adjoint [6]. According to the Hilbert - Schmidt theorem [3] for $k \in \mathbb{N}$ ($\mathbb{N}$ is the set of all natural numbers)

$$\begin{aligned}(\mathcal{A}^{-1}\mathbf{x})(t) &= \int_a^b \mathbf{G}(t,s)\mathbf{x}(s)\,\mathrm{d}s = \int_a^b \mathbf{x}(s)^T\mathbf{G}(t,s)\,\mathrm{d}s = \\ &= \sum_k \lambda_k \left(\int_a^b \mathbf{x}(s)^T\mathbf{x}_k(s)\,\mathrm{d}s\right)\mathbf{x}_k(t) = \\ &= \int_a^b \mathbf{x}(s)^T\left(\sum_k \lambda_k\mathbf{x}_k(s)\mathbf{x}_k(t)^T\right)\mathrm{d}s.\end{aligned}$$

Since the function $\mathbf{x}(t) \in L^2(a,b)$ is arbitrary we can write

$$\mathbf{G}(t,s) = \sum_k \lambda_k\mathbf{x}_k(t)\mathbf{x}_k(s)^T, \tag{3.1}$$

where $\{\mathbf{x}_i\}$ is an orthonormal collection of the eigenvectors for $\mathcal{A}^{-1}$ with associated eigenvalues $\{\lambda_i\}$ and $\mathbf{x}_i \in L^2(a,b)$, $i \in \mathbb{N}$. If $\mathcal{A}^{-1}$ is a positive compact self-adjoint operator, then the eigenvalues λ_i of $\mathcal{A}^{-1}$ are all real and nonnegative. Moreover, we have

$$\int_a^b \operatorname{Spur}\mathbf{G}(t,t)\,\mathrm{d}t = \sum_k \lambda_k \int_a^b \mathbf{x}_k(t)^T\mathbf{x}_k(t)\,\mathrm{d}t = \sum_k \lambda_k. \tag{3.2}$$

For the application of the Trefftz's method let's suppose $\mathcal{A}^{-1}$ being a symmetric positive semidefinite completely continuous integral operator and we have the first k of the lower bounds

$$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k \tag{3.3}$$

ordered to the first k of the eigenvalues

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \tag{3.4}$$

of the integral operator. If $j \leq k$ according to Trefftz [4] and (3.2) we get

$$\lambda_j \leq \mu_j + \int_a^b \text{Spur } \mathbf{G}(t,t)\ \mathrm{d}t - \sum_{i=1}^{k} \mu_i := \nu_j, \tag{3.5}$$

which is an estimation for the j^{th} eigenvalue. (3.5) is a new extension of the method of Trefftz. Thus, with the knowledge of the lower bounds (3.3) and the Green matrix $\mathbf{G}(t,s)$ one can use the inequality (3.5) to find upper bounds for the first k eigenvalues from the set of eigenvalues $\{\lambda_i\}$ that belong to the integral operator $\mathcal{A}^{-1}$.

4. Numerical example

Suppose that $\mathcal{A}$ is the differential operator

$$\mathcal{A}\mathbf{x} = \mathbf{A}_0(t)\mathbf{x}' + \mathbf{A}_1(t)\mathbf{x}, \tag{4.1}$$

$$\mathbf{A}_0(t) = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{x}'(t) = \begin{pmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \\ x_4'(t) \end{pmatrix},$$

$$\mathbf{A}_1(t) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -p^{-1} & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \qquad \mathbf{x}(\mathbf{t}) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix},$$

on $L^2(0,1)$, with the domain $\mathcal{D}_\mathcal{A} \subset L^2(0,1)$; and $p(t) > 0$, $\forall t \in [0,1]$. The boundary conditions are

$$x_1(0) = x_2(0) = x_3(1) = x_4(1) = 0.$$

We note that the operator $\mathcal{A}$ is self-adjoint, namely

$$\langle \mathcal{A}\mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, \mathcal{A}\mathbf{y} \rangle, \qquad \forall \mathbf{x}, \mathbf{y} \in L^2(0,1).$$

It should be noted that

$$\begin{aligned} \langle \mathcal{A}\mathbf{x}, \mathbf{y} \rangle &= \int_a^b (x_4 y_1' + x_3 y_2' + x_2 y_3' - x_1 y_4' - x_4 y_2 - p^{-1} x_3 y_3 - x_2 y_4)\, \mathrm{d}t + \\ &\quad + [-x_4 y_1 - x_3 y_2 + x_2 y_3 + x_1 y_4]_0^1 . \end{aligned}$$

The integral operator $\mathcal{A}^{-1}$ is self-adjoint as well, and $\mathbf{G}(t,s) = \mathbf{G}(s,t)$. It can be seen easily from [2] and (2.7), that for instance

$$G_{11}(t,s) = H(t-s)\int_s^t \int_s^u \frac{v-s}{p(v)}\,\mathrm{d}s\mathrm{d}u + \int_0^t \int_0^u \frac{s-v}{p(v)}\,\mathrm{d}v\mathrm{d}u \tag{4.2}$$

where H is Heaviside's distribution. Now consider the transverse vibration of a beam [2]. The equation of motion is of the form

$$\frac{\partial^2}{\partial x^2}\left(EI_z\frac{\partial^2 u}{\partial x^2}\right) = -\rho Q\frac{\partial^2 u}{\partial t^2}.$$

where $E = 2.1 \cdot 10^7$ [Pa], $\rho = 7.86 \cdot 10^3$ [kg/m^3], $Q(x) = \frac{\pi}{4}(-10^{-2}x + 4\cdot 10^{-2})^2$ [m^2], $I_z(x) = \frac{\pi}{64}(-10^{-2}x + 4\cdot 10^{-2})^4$ [m^4], and $x \in [0,1]$ for our example. If we assume that $u(x,t) = v(x)\sin(\alpha t)$ we obtain

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(EI_z\frac{\mathrm{d}^2 v}{\mathrm{d}x^2}\right) = \alpha^2\rho Q v\ , \tag{4.3}$$

in which α stands for the angular eigenfrequency. By assumption equation (4.3) is associated with the boundary conditions (beam fixed at the left end and free at the right end):

$$v(0) = v'(0) = v''(1) = v'''(1) = 0.$$

By introducing the functions

$$p(x) = EI_z(x), \quad q(x) = \rho Q(x),$$

we can easily see that the 4^{th}-order differential operator in (4.3) can be reduced by elementary transforms to the differential operator (4.1). Now to obtain lower bounds for eigenfrequencies of the beam we can use the inequality (3.5). By introducing the function

$$K(t,s) = \sqrt{q(t)}\sqrt{q(s)}G_{11}(t,s),$$

the inequality (3.5) becomes in our case (on basis of [2])

$$\frac{1}{\alpha_j^2} \le \frac{1}{(\alpha_j^u)^2} + \int_a^b K(t,s)\,\mathrm{d}t - \sum_{i=1}^{k}\frac{1}{(\alpha_i^u)^2} := \frac{1}{(\alpha_j^l)^2}, \qquad j \le k, \tag{4.4}$$

where α_j is the angular eigenfrequency, α_j^l is the lower bound and α_j^u is the upper bound ($j \in \mathbb{N}$). It is also easy to see from (4.3), if the inequality (3.5) is formulated for the squares of the angular eigenfrequencies as the eigenvalues, that the lower and upper designations exchange their roles. The table shows the numerical results of the calculations:

	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$
α_j^u	2.06152	10.8457	28.8438	55.7227	91.5596
α_j^l	2.03926	8.5541	12.5324	13.4998	13.7565

The first line of the table contains upper bounds calculated by the method of interval-matrices. In the second line lower bounds calculated by the aid of the estimation (4.4) can be found. The numerical investigations show that the estimation is good only for the first eigenfrequency and is not satisfactory in the other cases. A little improvement can be achieved by taking more eigenfrequencies into account. The problem is of theoretical importance and our aim was to show a practical application of the inequality (3.5). Obviously, the estimation becomes a useful tool in the case of complicated structures.

5. Appendix. The generalized Green matrix

The aim of this section is to give a summary - on the basis of [6,7,8,9] - for the case when the differential operator $\mathcal{A}$ is noninvertible. If $r \neq n$ or $r = n$ but $k \neq 0$, neither the operator $\mathcal{A}$, nor $\mathcal{A}^*$ is invertible. (The adjoint of $\mathcal{A}$ is denoted by $\mathcal{A}^*$.) There are various ways to define a generalized inverse for $\mathcal{A}$ [6]. In this section we shall use an analogue to the Moore Penrose generalized inverse, often called pseudoinverse [6], for differential operators:

Definition 2 *Let $\mathcal{P}$ and $\mathcal{Q}$ denote the projections whose ranges are $\mathcal{R}_\mathcal{P}$ =Ker$\mathcal{A}$ and $\mathcal{R}_\mathcal{Q}$ =Ker$\mathcal{A}^*$. Let $\mathcal{E} = \mathcal{I} - \mathcal{P}$ and $\mathcal{F} = \mathcal{I} - \mathcal{Q}$ be two projections where $\mathcal{I}$ is the unit operator. The generalized inverse of $\mathcal{A}$ denoted by $\mathcal{A}^\dagger$, is given by*

$$\mathcal{A}^\dagger = \mathcal{E}\underline{\mathcal{A}}\mathcal{F}, \tag{5.1}$$

where $\underline{\mathcal{A}}$ is such a generalized inverse of $\mathcal{A}$ that if $\mathbf{z} \in \mathcal{R}_\mathcal{A}$ and $\mathbf{y} = \underline{\mathcal{A}}\mathbf{z}$, then $\mathbf{y} \in \mathcal{D}_\mathcal{A}$ and $\mathcal{A}\mathbf{y} = \mathbf{z}$, and $\mathcal{R}_\mathcal{A} \subset \mathcal{D}_{\underline{\mathcal{A}}}$.

$\mathcal{A}^\dagger$ as defined above is unique. The generalized inverse as defined is an integral operator with a uniquely defined kernel, which will be denoted by $\mathbf{G}^\dagger(t,s)$. $\mathbf{G}^\dagger(t,s)$ is referred to as the generalized Green matrix for the operator $\mathcal{A}$. There are several ways of constructing $\mathbf{G}^\dagger(t,s)$, based on the various properties of the generalized inverse. In this paper we shall give two ways for constructing $\mathbf{G}^\dagger(t,s)$. We shall need explicit expressions for the projections $\mathcal{P}$ and $\mathcal{Q}$.

Let $\mathbf{u}_1(t), \cdots, \mathbf{u}_k(t)$ be linearly independent solutions to the equation $\mathcal{A}\mathbf{x} = \mathbf{0}$. This set is a basis for the vector space Ker$\mathcal{A}$ where dimKer$\mathcal{A} = k$. Let $\mathbf{U}(t)$ denote an $n \times n$ matrix with k columns denoted by $\mathbf{u}_1(t), \cdots, \mathbf{u}_k(t)$. Due to the linear independence of the columns $\mathbf{u}_1(t), \cdots, \mathbf{u}_k(t)$, the $k \times k$ matrix

$$\mathbf{W}_\mathcal{P} = \int_a^b \mathbf{U}(w)^T\mathbf{U}(w)\,\mathrm{d}w \tag{5.2}$$

is a nonsingular, positive definite symmetric one. Let the kernel $\mathbf{G}_\mathcal{P}(t,s)$ be defined by

$$\mathbf{G}_\mathcal{P}(t,s) = \mathbf{U}(w)\mathbf{W}_\mathcal{P}^{-1}\mathbf{U}(w)^T. \tag{5.3}$$

Clearly, $\mathbf{G}_{\mathcal{P}}(t,s)$ is also an $n \times n$ matrix. The projection $\mathcal{P}$ onto Ker$\mathcal{A}$ is then the integral operator

$$(\mathcal{P}\mathbf{x})(t) = \int_a^b \mathbf{G}_{\mathcal{P}}(t,s)\mathbf{x}(s)\,\mathrm{d}s. \tag{5.4}$$

The operator $\mathcal{P}$ as defined by (5.4) is clearly Hermitian and idempotent. Similarly if Ker$\mathcal{A}^*$ (dimKer$\mathcal{A}^* = k'$) is spanned by the linearly independent set of solutions $\mathbf{v}_1(t), \cdots, \mathbf{v}_{k'}(t)$ to the equation $\mathcal{A}^*\mathbf{x} = \mathbf{0}$, we can form the matrix $\mathbf{V}(t)$ with k' columns

$$\mathbf{W}_{\mathcal{Q}} = \int_a^b \mathbf{V}(w)^T\mathbf{V}(w)\mathrm{d}w, \tag{5.5}$$

and the kernel is

$$\mathbf{G}_{\mathcal{Q}}(t,s) = \mathbf{V}(w)\mathbf{W}_{\mathcal{Q}}^{-1}\mathbf{V}(w)^T. \tag{5.6}$$

The projection $\mathcal{Q}$ onto Ker$\mathcal{A}^*$ is given by

$$(\mathcal{Q}\mathbf{x})(t) = \int_a^b \mathbf{G}_{\mathcal{Q}}(t,s)\mathbf{x}(s)\,\mathrm{d}s. \tag{5.7}$$

Note, that if the vectors $\mathbf{u}$ and the vectors $\mathbf{v}$ are chosen to be orthonormal, then $\mathbf{W}_{\mathcal{P}} = \mathbf{W}_{\mathcal{Q}} = \mathbf{E}$. Now we require a kernel for the operator $\underline{\mathcal{A}}$. Let us solve the differential equation

$$\mathcal{A}\mathbf{x} = \mathbf{z} \qquad \mathbf{z} \in \mathcal{R}_{\mathcal{A}} \tag{5.8}$$

by variation of parameters. The solution to this problem can always be made to satisfy the boundary conditions $\mathbf{M}\boldsymbol{\xi} = \mathbf{0}$ of $\mathcal{A}$. In fact, there will always remain k undetermined constants after integration because an arbitrary element of Ker$\mathcal{A}$ can always be added to the solution. The formula for $\mathbf{x}$ in terms of $\mathbf{z}$ is the integral

$$\mathbf{x}(t) = \int_a^b \underline{\mathbf{G}}(t,s)\mathbf{z}(s)\,\mathrm{d}s \tag{5.9}$$

where $\underline{\mathbf{G}}(t,s)$ can be chosen as simply as possible. As a function of t, $\underline{\mathbf{G}}(t,s)$ will satisfy the boundary conditions for $\mathcal{A}$ and will have, as t increases through s, the same continuity properties as the Green matrix for an invertible operator. Therefore the kernels $\mathbf{G}_{\mathcal{P}}(t,s)$, $\underline{\mathbf{G}}(t,s)$ and $\mathbf{G}_{\mathcal{Q}}(t,s)$ are available.

Theorem 3 *The kernel of $\mathcal{A}^\dagger$, i.e., the generalized Green matrix, assumes the form [6]:*

$$\begin{aligned}\mathbf{G}^\dagger(t,s) &= \underline{\mathbf{G}}(t,s) - \int_a^b \mathbf{G}_{\mathcal{P}}(t,u)\underline{\mathbf{G}}(u,s)\,du - \int_a^b \underline{\mathbf{G}}(t,v)\mathbf{G}_{\mathcal{Q}}(v,s)\,dv + \\ &\quad + \int_a^b\int_a^b \mathbf{G}_{\mathcal{P}}(t,u)\underline{\mathbf{G}}(u,v)\mathbf{G}_{\mathcal{Q}}(v,s)\,dv\,du. \end{aligned} \tag{5.10}$$

In the noninvertible case a similar calculation is possible as in the invertible case. The kernels $\underline{\mathbf{G}}$ and $\mathbf{G}^\dagger$ have the same discontinuity at $t = s$ as the ordinary Green matrix. When the operator $\mathcal{A}$ is applied to

$$\mathbf{g}(t) = \int_a^b \underline{\mathbf{G}}(t, v)\mathbf{f}(v)\,\mathrm{d}v, \tag{5.11}$$

the result is $\mathbf{f}(t) = (\mathcal{A}\mathbf{g})(t)$, whereas it is $\mathcal{A}\underline{\mathbf{G}} = \mathbf{0}$ for $a \le t < s \le b$ and for $a \le s < t \le b$. Since $\mathcal{A}\mathbf{G}_\mathcal{P} = \mathbf{0}$, we find from (5.10) that

$$\mathcal{A}\mathbf{G}^\dagger = -\mathbf{G}_\mathcal{Q} \quad a \le t < s \le b \quad a \le s < t \le b. \tag{5.12}$$

We can recover $\mathbf{G}^\dagger(t, s)$ by determining the solution of the differential equation (5.12), so as to satisfy, as a function of t, the known properties of $\mathbf{G}^\dagger(t, s)$, namely

1. $\mathbf{G}^\dagger(t, s)$ should have the appropriate discontinuity at $t = s$,
2. $\mathbf{G}^\dagger(t, s)$ should satisfy the boundary conditions $\mathbf{M}\boldsymbol{\xi} = \mathbf{0}$ of the operator $\mathcal{A}$ and
3. $\mathbf{G}^\dagger(t, s)$ should be orthogonal to Ker$\mathcal{A}$.

It can also be proved that these properties uniquely determine $\mathbf{G}^\dagger(t, s)$.

6. A new example for the generalized Green matrix

As a brief example, consider the differential equation of the simple harmonic motion

$$x'' + \alpha^2 x = 0. \tag{6.1}$$

We obtain a system of first order equations by the elementary transforms

$$\begin{aligned} x_1 + x_2' &= 0, \\ -x_1' + \alpha^2 x_2 &= 0. \end{aligned} \tag{6.2}$$

Now consider the differential operator from (6.2)

$$\mathcal{A}\mathbf{x} = \begin{pmatrix} x_1 + x_2' \\ -x_1' + \alpha^2 x_2 \end{pmatrix}$$

with the boundary conditions $x_1(0) = x_1(\pi)$ and $x_2(0) = x_2(\pi)$. For simplicity we shall assume that $\alpha = 2m$, $m \in \mathbb{N}$. Under these conditions the fundamental matrix $\Phi(t)$ for the equation $\mathcal{A}\mathbf{x} = \mathbf{0}$ is:

$$\Phi(t) = \begin{pmatrix} \alpha \sin \alpha t & -\alpha \cos \alpha t \\ \cos \alpha t & \sin \alpha t \end{pmatrix}.$$

The orthonormal base vectors are given by

$$\mathbf{U}(t) = \begin{pmatrix} \mathbf{u}_1(t) & \mathbf{u}_2(t) \end{pmatrix} = \sqrt{\frac{2}{\pi(1+\alpha^2)}}\,\Phi(t).$$

On the basis of (5.6) we have

$$\mathbf{G}_{\mathcal{Q}}(t,s) = \frac{2}{\pi(1+\alpha^2)} \begin{pmatrix} \alpha^2 \cos(\alpha(t-s)) & \alpha \sin(\alpha(t-s)) \\ -\alpha \sin(\alpha(t-s)) & \cos(\alpha(t-s)) \end{pmatrix}.$$

For example one of the solutions of the differential equation $\mathcal{A}\mathbf{G}^\dagger = -\mathbf{G}_{\mathcal{Q}}$ is:

$$G_{21}^\dagger(t,s) = \begin{cases} C_{11}(s)\cos\alpha t + C_{21}(s)\sin\alpha t - \dfrac{t}{\pi}\cos(\alpha(t-s)) & t < s \\ C_{12}(s)\cos\alpha t + C_{22}(s)\sin\alpha t - \dfrac{t}{\pi}\cos(\alpha(t-s)) & t > s \end{cases}.$$

Since

$$\mathbf{A}_0^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

the elements of the matrix $\mathbf{G}^\dagger$ satisfy the discontinuity conditions:

$$\begin{aligned} G_{11}^\dagger(s+0,s) - G_{11}^\dagger(s-0,s) &= 0, \\ G_{12}^\dagger(s+0,s) - G_{12}^\dagger(s-0,s) &= -1, \\ G_{21}^\dagger(s+0,s) - G_{21}^\dagger(s-0,s) &= 1, \\ G_{22}^\dagger(s+0,s) - G_{22}^\dagger(s-0,s) &= 0. \end{aligned}$$

Hence

$$G_{21}^\dagger(t,s) = \begin{cases} C_{11}(s)\cos\alpha t + C_{21}(s)\sin\alpha t - \dfrac{t}{\pi}\cos(\alpha(t-s)) & t < s \\ C_{11}(s)\cos\alpha t + C_{21}(s)\sin\alpha t - (1-\dfrac{t}{\pi})\cos(\alpha(t-s)) & t > s \end{cases}.$$

It can be seen easily that if

$$\mathbf{g}_1(t) = \begin{pmatrix} G_{11}^\dagger(t,s) \\ G_{21}^\dagger(t,s) \end{pmatrix}, \quad \mathbf{g}_2(t) = \begin{pmatrix} G_{12}^\dagger(t,s) \\ G_{22}^\dagger(t,s) \end{pmatrix},$$

and

$$\int_0^\pi \mathbf{g}_i(t)\mathbf{u}_j(t)\mathrm{d}t = 0 \quad i,j = 1,2,$$

then the elements of the generalized Green's matrix $\mathbf{G}^\dagger(t,s)$ are

$$G_{11}^\dagger(t,s) = \begin{cases} \gamma_1(s)\cos(\alpha(s-t)) + \dfrac{\alpha}{2\pi}(\pi - 2(s-t))\sin(\alpha(s-t)) & t < s \\ \gamma_1(s)\cos(\alpha(t-s)) + \dfrac{\alpha}{2\pi}(\pi - 2(t-s))\sin(\alpha(t-s)) & t > s \end{cases},$$

$$G_{21}^\dagger(t,s) = \begin{cases} \gamma_2(s)\sin(\alpha(s-t)) - \dfrac{1}{2\pi}(\pi - 2(s-t))\cos(\alpha(s-t)) & t < s \\ -\gamma_2(s)\sin(\alpha(t-s)) + \dfrac{1}{2\pi}(\pi - 2(t-s))\cos(\alpha(t-s)) & t > s \end{cases},$$

$$G_{12}^\dagger(t,s) = \begin{cases} -\gamma_2(s)\sin(\alpha(s-t)) + \dfrac{1}{2\pi}(\pi - 2(s-t))\cos(\alpha(s-t)) & t < s \\ \gamma_2(s)\sin(\alpha(t-s)) - \dfrac{1}{2\pi}(\pi - 2(t-s))\cos(\alpha(t-s)) & t > s \end{cases},$$

$$G_{22}^\dagger(t,s) = \begin{cases} \gamma_3(s)\cos(\alpha(s-t)) + \dfrac{1}{2\alpha\pi}(\pi - 2(s-t))\sin(\alpha(s-t)) & t < s \\ \gamma_3(s)\cos(\alpha(t-s)) + \dfrac{1}{2\alpha\pi}(\pi - 2(t-s))\sin(\alpha(t-s)) & t > s \end{cases}$$

where γ_1, γ_2, and γ_3 are constants depending on α^2 [8].

7. Conclusions

The method outlined in Section 3 will produce lower bounds for eigenfrequencies of the elastic continuous structures by using (3.5) if we have good upper bounds calculated by some kind of a simple method. Since the solution of our problem requires the invertation of differential operators we expatiated on the constructions of Green matrices by solving the matrix equation (2.7). In the first part of the Appendix we give a short summary of the results connected to the generalized Green matrix on the basis of [6]. In Section 6 with the help of the example, not published yet, we present a technique for making a generalized Green matrix.

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