

# BOUNDARY CONTOUR METHOD FOR PLANE PROBLEMS IN A DUAL FORMULATION WITH LINEAR ELEMENTS

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**Abstract.** The present paper is devoted to the boundary contour method for plane problems in the dual system of elasticity. It has been shown that the integrals on the right side of the corresponding boundary integral equations are divergence free in the dual system provided that the unknown functions satisfy the field equations. Consequently these integrals can be given in closed form if appropriate shape functions have been chosen to approximate the unknown functions on the contour. Numerical examples prove the efficiency of this technique.

## 1. Introduction

It has been proved in the article [1] by A. Nagarajan, E. Lutz and S. Mukherjee that the integrand of the direct boundary element method is divergence free in the primal system of the two and three-dimensional elasticity theory. The authors of [1] have come to the conclusion that the numerical solution of three-dimensional problems require the calculation of line integrals instead of surface integrals, while for planar problems evaluation of functions should be performed instead of calculating line integrals. Article [1] supposes linear approximation. The accuracy is greatly increased if one uses quadratic elements [2]. This method can also be employed for rewriting hypersingular integral equations into boundary contour equations. With this technique one can compute stresses and can solve shape optimization problems in two dimensions [3].

The boundary integral equations of the direct method in the dual system of elasticity and for plane problems can be found in a thesis [4]. In view of the formulation presented in [4] there arises the question if it is possible to repeat the line of thought leading to the boundary contour method in a dual formulation as well. The reply to this question is yes and the main result of the present paper is a dual formulation similar, as regard its main features, to that given in paper [1].

The paper is organized into seven sections. Section 2 is devoted to some preliminaries. It is proved in Section 3 that the integrand of the direct boundary element method is divergence free in the dual system of elasticity. In addition we have determined the corresponding shape functions provided that the approximation is linear. Discretized equations are set up in Section 4. The aim is to prepare an algorithm for

our computations. Section 5 is devoted to the question of how to compute stresses at internal points. Section 6 presents some simple numerical examples. The last Section is a summary of the conclusions. The paper is also supplemented with an Appendix in which the shape functions and some manipulations are presented.

## 2. Fundamental solutions and integral equations of the direct method

Cartesian coordinates and indicial notations are used throughout this paper. {Greek} [Latin] subscripts have the range  $\{1, 2\}$ ,  $[1, 2, 3]$ , summation over repeated indices is implied. In accordance with the notations introduced  $\delta_{\kappa\lambda}$  is the Kronecker symbol,  $\partial_{\kappa}$  stands for the derivatives with respect to  $x_{\kappa}$  and  $\varepsilon_{3\kappa\lambda}$  is the permutation tensor.  $\mathcal{F}_{\lambda}$  stands for the stress functions of order one. In plane components of stresses and strains are denoted by  $t_{\kappa\lambda}$  and  $e_{\kappa\lambda}$ . If there are body forces, the particular solution of the equilibrium equations is  $\overset{\circ}{t}_{\kappa\lambda}$ . The shear modulus of elasticity and the Poisson number are denoted by  $\mu$  and  $\nu$ , respectively. The rigid body rotation is denoted by  $\varphi_3$ .

In the dual system of elasticity plane strain problems are governed by the dual kinematic equations

$$t_{\kappa\lambda} = \varepsilon_{\kappa\rho 3} \mathcal{F}_{\lambda} \partial_{\rho} + \overset{\circ}{t}_{\kappa\lambda}, \quad (2.1)$$

the inverse of Hook's law

$$e_{\kappa\lambda} = \frac{1}{2\mu} (t_{\kappa\lambda} - \nu t_{\psi\psi} \delta_{\kappa\lambda}), \quad (2.2)$$

the dual balance equations

$$\varepsilon_{\kappa\rho 3} e_{\lambda\kappa} \partial_{\rho} + \varphi_3 \partial_{\lambda} = 0 \quad (2.3)$$

(equations of compatibility for a simply connected region) and the symmetry condition

$$\varepsilon_{3\lambda\kappa} t_{\lambda\kappa} = 0. \quad (2.4)$$

For simplicity first we shall consider a simply connected inner region  $A_i$ . The contour  $\mathcal{L}_o$  of the region  $A_i$  can be divided into two parts denoted by  $\mathcal{L}_t$  and  $\mathcal{L}_u$ . We shall assume that  $[\mathcal{L}_t]$   $\{\mathcal{L}_u\}$  is the union of those arcs on which [stress functions (loaded arcs)] {displacements} are imposed. These arcs are denoted by  $\mathcal{L}_{t2}$ ,  $\mathcal{L}_{t4}$  and  $\mathcal{L}_{u1}$ ,  $\mathcal{L}_{u3}$ , respectively. The corresponding boundary conditions are of the form

$$F_{\lambda}(s) = \hat{F}_{\lambda} + C_{\lambda}(P_{ti}) \quad s \in \mathcal{L}_{ti} \quad i = 2, 4 \quad (2.5)$$

and

$$\frac{d\hat{u}_{\lambda}}{ds} = n_{\rho} [\varepsilon_{\rho\kappa 3} e_{\kappa\lambda} - \varphi_3 \delta_{\rho\lambda}] \quad s \in \mathcal{L}_u, \quad (2.6)$$

where  $\hat{F}_{\lambda}(s)$  and  $\hat{u}_{\lambda}(s)$  are known functions, while  $C_{\lambda}$  is an integration constant, the number of which equals the number of loaded arcs. Here and in the sequel we shall

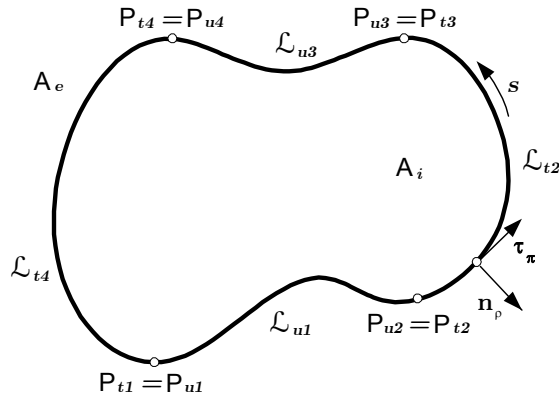


Figure 1.

assume that there are no body forces. In the absence of body forces

$$\hat{F}_\lambda(s) = \int \hat{t}_\lambda(s) ds, \quad (2.7)$$

where  $\hat{t}_\lambda(s)$  is the stress vector on  $\mathcal{L}_t$ .

Substituting the dual kinematic equation (2.1) into Hook's law (2.2) and the result into the compatibility equations (2.3) we get two scalar equations. These equations are associated with the symmetry condition, i.e., we have three equations for the unknowns  $\mathcal{F}_1, \mathcal{F}_2$  and  $\varphi_3$

$$\begin{bmatrix} a\Delta - b\partial_1\partial_1 & -b\partial_2\partial_1 & -\partial_1 \\ -b\partial_2\partial_1 & a\Delta - b\partial_2\partial_2 & -\partial_2 \\ -\partial_1 & -\partial_2 & 0 \end{bmatrix} \begin{bmatrix} \mathcal{F}_1 \\ \mathcal{F}_2 \\ -\varphi_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (2.8)$$

where

$$a = \frac{1}{2\mu}(1 - \nu) \quad b = \frac{1}{2\mu}\left(\frac{1}{2} - \nu\right). \quad (2.9)$$

Let  $D_{ik}$  ( $i, k = 1, 2, 3$ ) be the differential operator in equation (2.8). Further let  $\mathbf{u}_k = (\mathcal{F}_1 \mid \mathcal{F}_2 \mid -\varphi_3)$  be the vector of unknowns (or state vector). With these notations equation (2.8) can be rewritten as

$$D_{ik}\mathbf{u}_k = 0. \quad (2.10)$$

Let  $Q(\eta_1, \eta_2)$  and  $M(x_1, x_2)$  be the source point and the point of effect. The position vector of  $M(x_1, x_2)$  relative to  $Q(\eta_1, \eta_2)$  is  $r_\lambda = x_\lambda - \eta_\lambda$ . The distance between  $Q$  and  $M$  is  $R = R(M, Q) = |\mathbf{r}|$ . For a two point function  $f(R)$  it holds that

$$\frac{\partial}{\partial x_\lambda} f(R) = -\frac{\partial}{\partial \eta_\lambda} f(R). \quad (2.11)$$

Let  $e_i(Q)$  be a unit vector at  $Q$ . Solution to the differential equation

$${}^M D_{ik} u_k + \delta(M - Q) e_i(Q) = 0 \quad (2.12)$$

is referred to as fundamental solution. Here the letter  $M$  over  $D_{ik}$  denotes that the derivation is taken with respect to the point  $M$ . It can be shown that

$$u_k = \mathfrak{U}_{kl}(M, Q) e_l(Q), \quad (2.13)$$

where

$$\begin{aligned} [\mathfrak{U}_{kl}(M, Q)] &= \frac{\mu}{4\pi(1-\nu)} \times \\ &\times \begin{bmatrix} -2 \ln R - 3 - 2 \frac{r_2 r_2}{R^2} & 2 \frac{r_1 r_2}{R^2} & \frac{2}{\mu} (1-\nu) \frac{r_1}{R^2} \\ 2 \frac{r_2 r_1}{R^2} & -2 \ln R - 3 - 2 \frac{r_1 r_1}{R^2} & \frac{2}{\mu} (1-\nu) \frac{r_2}{R^2} \\ \frac{2}{\mu} (1-\nu) \frac{r_1}{R^2} & \frac{2}{\mu} (1-\nu) \frac{r_2}{R^2} & 0 \end{bmatrix} \end{aligned} \quad (2.14)$$

For our later considerations we shall introduce the notation

$$t_\lambda = -\frac{du_\lambda}{ds}. \quad (2.15)$$

The vector  $t_\lambda$  is opposite to the derivative of the displacement vector with respect to the arc coordinate  $s$  measured on the contour  $\mathcal{L}_o$ . Omitting the long formal transformations for the vector  $t_\lambda$  calculated from the fundamental solution we get [4]

$$t_\lambda(\overset{\circ}{M}) = e_l(Q) \mathfrak{T}_{l\lambda}(\overset{\circ}{M}, Q) \quad (2.16)$$

where

$$\begin{aligned} \mathfrak{T}_{l\lambda}(\overset{\circ}{M}, Q) &= \frac{1}{8\pi(1-\nu)R^2} \times \\ &\times \begin{bmatrix} n_1 r_1 \left( 4 \frac{r_2^2}{R^2} - 2(3-2\nu) \right) & -n_2 r_1 \left( 4 \frac{r_2^2}{R^2} + 2(1-2\nu) \right) \\ +n_2 r_2 \left( 4 \frac{r_2^2}{R^2} - 2(3-2\nu) \right) & -n_1 r_2 \left( 4 \frac{r_1^2}{R^2} - 2(1-2\nu) \right) \\ -n_1 r_2 \left( 4 \frac{r_1^2}{R^2} + 2(1-2\nu) \right) & n_2 r_2 \left( 4 \frac{r_1^2}{R^2} - 2(3-2\nu) \right) \\ -n_2 r_1 \left( 4 \frac{r_2^2}{R^2} - 2(1-2\nu) \right) & +n_1 r_1 \left( 4 \frac{r_1^2}{R^2} - 2(3-2\nu) \right) \\ -n_1 \frac{2}{\mu} (1-\nu) \frac{r_1^2 - r_2^2}{R^2} & -n_1 \frac{4}{\mu} (1-\nu) \frac{r_1 r_2}{R^2} \\ -n_2 \frac{4}{\mu} (1-\nu) \frac{r_1 r_2}{R^2} & +n_2 \frac{2}{\mu} (1-\nu) \frac{r_1^2 - r_2^2}{R^2} \end{bmatrix} \end{aligned} \quad (2.17)$$

Here and in the sequel the small circle over the letters  $Q$  and  $M$  shows that the corresponding point is located on the contour. The normal  $n_\lambda$  is taken at the point  $\overset{\circ}{M}$ .

If we take two elastic states of the region  $A_i$  – the second state is denoted by asterisk which is placed over the corresponding letters – then the so-called dual Somigliana identity can be written as

$$\int_{A_i} \left[ \mathbf{u}_k(D_{kl}\overset{*}{\mathbf{u}}_l) - \overset{*}{\mathbf{u}}_k(D_{kl}\mathbf{u}_l) \right] dA = \oint_{\mathcal{L}_o} [\mathbf{u}_\lambda \overset{*}{\mathbf{t}}_\lambda - \overset{*}{\mathbf{u}}_\lambda \mathbf{t}_\lambda] ds. \quad (2.18)$$

If the quantities denoted by asterisks are from the fundamental solution and  $Q \in A_i$ , then exploiting the above, we have the first dual Somigliana formula

$$\mathbf{u}_k(Q) = \oint_{\mathcal{L}_o} \mathfrak{u}_{k\lambda}(\overset{\circ}{M}, Q) \mathbf{t}_\lambda(\overset{\circ}{M}) ds_{\overset{\circ}{M}} - \oint_{\mathcal{L}_o} \mathfrak{T}_{k\lambda}(\overset{\circ}{M}, Q) \mathbf{u}_\lambda(\overset{\circ}{M}) ds_{\overset{\circ}{M}}. \quad (2.19)$$

If  $Q = \overset{\circ}{Q} \in \partial A_i = \mathcal{L}_o$ , then equation (2.18) yields the second dual Somigliana formula

$$c_{\kappa\lambda}(\overset{\circ}{Q}) \mathbf{u}_\lambda(\overset{\circ}{Q}) = \oint_{\mathcal{L}_o} \mathfrak{u}_{k\lambda}(\overset{\circ}{M}, \overset{\circ}{Q}) \mathbf{t}_\lambda(\overset{\circ}{M}) ds_{\overset{\circ}{M}} - \oint_{\mathcal{L}_o} \mathfrak{T}_{k\lambda}(\overset{\circ}{M}, \overset{\circ}{Q}) \mathbf{u}_\lambda(\overset{\circ}{M}) ds_{\overset{\circ}{M}}, \quad (2.20)$$

where  $c_{\kappa\lambda}(\overset{\circ}{Q})$  depends on the angle formed by the tangents to the contour at  $\overset{\circ}{Q}$ . The above integral equation is that of the direct method in the dual system of elasticity. Finally if  $Q \notin (A_i \cup \mathcal{L}_o)$ , then the left side of the identity (2.18) is identically equal to zero and the third dual Somigliana formula can immediately be set up

$$0 = \oint_{\mathcal{L}_o} \mathfrak{u}_{k\lambda}(\overset{\circ}{M}, Q) \mathbf{t}_\lambda(\overset{\circ}{M}) ds_{\overset{\circ}{M}} - \oint_{\mathcal{L}_o} \mathfrak{T}_{k\lambda}(\overset{\circ}{M}, Q) \mathbf{u}_\lambda(\overset{\circ}{M}) ds_{\overset{\circ}{M}}. \quad (2.21)$$

### 3. Fundamental relations for linear approximation

We shall assume that the stress functions  $\mathbf{u}_\lambda$  fulfill the basic equations and  $\mathbf{t}_\lambda$  is calculated from  $\mathbf{u}_\lambda$ . Under this condition the opposite to the derivative of the displacement vector can be obtained from (2.6):

$$\mathbf{t}_\lambda(\overset{\circ}{M}) = -n_\rho(\overset{\circ}{M}) \left( \epsilon_{\rho\pi 3} e_{\pi\lambda}(\overset{\circ}{M}) - \delta_{\rho\lambda} \varphi_3(\overset{\circ}{M}) \right). \quad (3.1)$$

We denote again the quantities derived from the fundamental solution by asterisks.

Let  $\overset{*}{e}_{k\pi\lambda}(\overset{\circ}{M}, Q)$  be the strain tensor that follows from the stress function vector  $\overset{*}{\mathbf{u}}_\lambda(\overset{\circ}{M}) = \mathfrak{u}_{k\lambda}(\overset{\circ}{M}, Q)$ . It is also clear that the corresponding rotation  $\overset{*}{\varphi}_3(\overset{\circ}{M})$  is  $\mathfrak{u}_{k3}(\overset{\circ}{M}, Q)$ . Making use of these notations and the relation (3.1) we can write

$$\overset{*}{\mathbf{t}}_\lambda(\overset{\circ}{M}, Q) = \mathfrak{T}_{k\lambda}(\overset{\circ}{M}, Q) = -n_\rho(\overset{\circ}{M}) \left( \epsilon_{\rho\pi 3} \overset{*}{e}_{k\pi\lambda}(\overset{\circ}{M}, Q) - \delta_{\rho\lambda} \mathfrak{u}_{k3}(\overset{\circ}{M}, Q) \right) \quad (3.2)$$

for the derivative  $-du_\lambda^*/ds$ . Upon substitution of  $u_\lambda^*(\overset{\circ}{M})$  for  $u_\lambda$  and  $t_\lambda^*(\overset{\circ}{M}, Q)$  for  $t_\lambda$  in (2.19) we have, after renaming some dummy indices, that

$$I_\kappa(Q) = \oint_{\mathcal{L}_o} n_\rho(\overset{\circ}{M}) \left( -\mathfrak{U}_{\kappa\lambda}(\overset{\circ}{M}, Q) \left( \epsilon_{\rho\pi 3} e_{\pi\lambda}(\overset{\circ}{M}) - \delta_{\rho\lambda} \varphi_3(\overset{\circ}{M}) \right) + \left( \epsilon_{\rho\pi 3} e_{\kappa\pi\lambda}(\overset{\circ}{M}, Q) - \delta_{\rho\lambda} \mathfrak{U}_{\kappa 3}(\overset{\circ}{M}, Q) \right) u_\lambda(\overset{\circ}{M}) \right) ds_{\overset{\circ}{M}}. \quad (3.3)$$

Since those terms for which  $k = 3$  in (2.19) will play no role in the further transformation we have dropped them by writing  $\kappa$  for  $k$ .

Let  $P_{\kappa\rho}$  be the coefficient of  $n_\rho(\overset{\circ}{M})$  in (3.3):

$$P_{\kappa\rho}(\overset{\circ}{M}) = P_{\kappa\rho}(\overset{\circ}{M}, Q) = -\mathfrak{U}_{\kappa\lambda}(\overset{\circ}{M}, Q) \left( \epsilon_{\rho\pi 3} e_{\pi\lambda}(\overset{\circ}{M}) - \delta_{\rho\lambda} \varphi_3(\overset{\circ}{M}) \right) + \left( \epsilon_{\rho\pi 3} e_{\kappa\pi\lambda}(\overset{\circ}{M}, Q) - \delta_{\rho\lambda} \mathfrak{U}_{\kappa 3}(\overset{\circ}{M}, Q) \right) u_\lambda(\overset{\circ}{M}). \quad (3.4)$$

By using Gauss's theorem the line integral  $I_\kappa(Q)$  can be transformed into a surface integral:

$$I_\kappa(Q) = \oint_{\mathcal{L}_o} P_{\kappa\rho}(\overset{\circ}{M}, Q) n_\rho(\overset{\circ}{M}) ds_{\overset{\circ}{M}} = \int_{A_i} P_{\kappa\rho}(M, Q) \overset{M}{\partial}_\rho dA_M, \quad (3.5)$$

where, as can be seen after some hand-made calculations – see the Appendix for details – it holds

$$P_{\kappa\rho}(M, Q) \overset{M}{\partial}_\rho = 0, \quad (3.6)$$

that is, there exists a function  $\phi_\kappa(M, Q)$  such that

$$P_{\kappa 1} = \frac{\partial \phi_\kappa(M, Q)}{\partial x_2} \quad \text{and} \quad P_{\kappa 2} = -\frac{\partial \phi_\kappa(M, Q)}{\partial x_1}. \quad (3.7)$$

This means that the integrand  $P_{\kappa\rho}$  is divergence free.

Taking now the line integral between the contour points  $\overset{\circ}{M}_1$  and  $\overset{\circ}{M}_2$  and using the above results we get the desired solution

$$\int_{\overset{\circ}{M}_1}^{\overset{\circ}{M}_2} P_{\kappa\rho}(\overset{\circ}{M}, Q) n_\rho(\overset{\circ}{M}) ds_{\overset{\circ}{M}} = \int_{\overset{\circ}{M}_1}^{\overset{\circ}{M}_2} \tau_\pi(\overset{\circ}{M}) \phi_\kappa(\overset{\circ}{M}, Q) \overset{M}{\partial}_\pi ds_{\overset{\circ}{M}} = \phi_\kappa(\overset{\circ}{M}_2, Q) - \phi_\kappa(\overset{\circ}{M}_1, Q). \quad (3.8)$$

When deriving the above relations we have not taken the position of  $Q$  relative to the region  $A_i$  into account. In other words the above results remain valid for the second and third dual Somigliana formulae as well.

Assume that the contour is divided into  $n_{be}$  boundary elements. The extremities of the elements are locally denoted by  $M_1$  and  $M_2$ . (Here and in the sequel for simplicity

we have omitted the zero standing over the letter  $M$ .) Then integrating element by element we have

$$I_{\kappa}(Q) = \sum_{e=1}^{n_{be}} [\phi_{\kappa}^e(M_2, Q) - \phi_{\kappa}^e(M_1, Q)] \quad (3.9)$$

where the upper index  $e$  shows that  $\phi_{\kappa}$  is taken on the  $e$ -th element.

Let  $K$  be the middle point of the element  $e$ . Over the element and its neighborhood we shall approximate the unknown vector  $\mathbf{u}_k$  by linear functions for the stress functions, and by a constant for the rigid body rotation, i.e.,

$$\begin{bmatrix} \mathcal{F}_1 \\ \mathcal{F}_2 \\ -\varphi_3 \end{bmatrix}^e = \begin{bmatrix} a_1 + a_2x_1 + a_3x_2 \\ a_4 + a_5x_1 - a_2x_2 \\ a_6 \end{bmatrix}. \quad (3.10)$$

The constants

$$(\mathbf{a}^e)^T = [ a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad a_6 ] \quad (3.11)$$

in (3.10) are related to the six physical quantities

$$(\mathbf{p}^e)^T = [ \mathcal{F}_1^{M_1} \quad \mathcal{F}_2^{M_1} \quad \mathbf{t}_1^K \quad \mathbf{t}_2^K \quad \mathcal{F}_1^{M_2} \quad \mathcal{F}_2^{M_2} ] \quad (3.12)$$

taken on the element  $e$  via the equation

$$\mathbf{T}^e \mathbf{a}^e = \mathbf{p}^e, \quad (3.13)$$

where the transformation matrix  $\mathbf{T}^e$  depends only on the nodal coordinates and the outward unit normal at  $K$ . After some hand-made calculations we have

$$\mathbf{T}^e = \begin{bmatrix} 1 & x_1^{M_1} & x_2^{M_1} & 0 & 0 & 0 \\ 0 & -x_2^{M_1} & 0 & 1 & x_1^{M_1} & 0 \\ 0 & \frac{1}{2\mu}n_1^K & \frac{1}{2\mu}(1-\nu)n_2^K & 0 & \frac{1}{2\mu}\nu n_2^K & n_1^K \\ 0 & -\frac{1}{2\mu}n_2^K & \frac{1}{2\mu}\nu n_1^K & 0 & \frac{1}{2\mu}(1-\nu)n_1^K & n_2^K \\ 1 & x_1^{M_2} & x_2^{M_2} & 0 & 0 & 0 \\ 0 & -x_2^{M_2} & 0 & 1 & x_1^{M_2} & 0 \end{bmatrix}. \quad (3.14)$$

Since  $M_1$ ,  $M_2$  and  $K$  are different, the matrix  $\mathbf{T}^e$  is invertible.

For our later considerations a new local coordinate system  $(\eta_1, \eta_2)$ , centered at the point  $M_1(x_1, x_2)$  is introduced. The axes  $\eta_1$  and  $\eta_2$  are parallel to the axes  $x_1$  and  $x_2$  of the global coordinate system. For the shape functions in the local system we get from (3.10) that

$$\begin{bmatrix} \mathcal{F}_1 \\ \mathcal{F}_2 \\ -\varphi_3 \end{bmatrix}^e = \begin{bmatrix} (a_1 + a_2x_1 + a_3x_2) + a_2\eta_1 + a_3\eta_2 \\ (a_4 + a_5x_1 - a_2x_2) + a_5\eta_1 - a_2\eta_2 \\ a_6 \end{bmatrix} = \begin{bmatrix} \hat{a}_1 + a_2\eta_1 + a_3\eta_2 \\ \hat{a}_4 + a_5\eta_1 - a_2\eta_2 \\ a_6 \end{bmatrix}. \quad (3.15)$$

The vector of constants in the local system is denoted by

$$(\hat{\mathbf{a}}^e)^T = [ \hat{a}_1 \quad a_2 \quad a_3 \quad \hat{a}_4 \quad a_5 \quad a_6 ] . \quad (3.16)$$

It can be shown with ease that the following relation holds

$$\hat{\mathbf{a}}^e = \mathbf{B} \mathbf{a}^e , \quad (3.17)$$

where the transformation matrix  $\mathbf{B}$  depends only on the coordinates  $x_1$  and  $x_2$  of the point  $M_1$ :

$$\mathbf{B} = \begin{bmatrix} 1 & x_1^{M_1} & x_2^{M_1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -x_2^{M_1} & 0 & 1 & x_1^{M_1} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} . \quad (3.18)$$

Relation (3.15) is a linear combination of those linearly independent state vectors which satisfy the fundamental equation:

$$\begin{aligned} \mathbf{u}_1^T &= [ 1 & 0 & 0 ] , & \mathbf{u}_2^T &= [ \eta_1 & -\eta_2 & 0 ] , \\ \mathbf{u}_3^T &= [ \eta_2 & 0 & 0 ] , & \mathbf{u}_4^T &= [ 0 & 1 & 0 ] , \\ \mathbf{u}_5^T &= [ 0 & \eta_1 & 0 ] , & \mathbf{u}_6^T &= [ 0 & 0 & 1 ] . \end{aligned} \quad (3.19)$$

The functions  $\phi_{\kappa i}$  that follow from the vectors  $\mathbf{u}_i$  ( $i = 1, \dots, 6$ ) have been calculated by making use of equation (3.7). These functions are given in the Appendix. In what follows we shall assume that the origin of the local coordinate system is located at the collocation point  $Q_j$ .

#### 4. Discretized equations

The contour  $\mathcal{L}_o$  of the region  $A_i$  is discretized into  $n_{be}$  boundary elements – see Figure 2. The boundary element method equations are enforced only at the end points  $M_1$  and  $M_2$  of the elements. Turning to global numbering we denote these points by  $Q_j$  where  $j = 1, \dots, n_{be}$ . Let  $\mathbf{u}_j^T = [u_1(Q_j) \mid u_2(Q_j)]$  be the matrix of stress functions. The matrix  $\mathbf{C}$  is defined by the equation

$$\mathbf{C}(Q_j) = \begin{bmatrix} c_{11}(Q_j) & c_{12}(Q_j) \\ c_{21}(Q_j) & c_{22}(Q_j) \end{bmatrix} . \quad (4.1)$$

Exploiting equations (3.13) and (3.17) the boundary integral equation (2.19) can be manipulated into the form

$$\mathbf{C}(Q_j) \mathbf{u}_j = \sum_{e=1}^{n_{be}} \Phi^{je} \mathbf{B}^j (\mathbf{T}^e)^{-1} \mathbf{p}^e \quad j = 1, \dots, n_{be} , \quad (4.2)$$



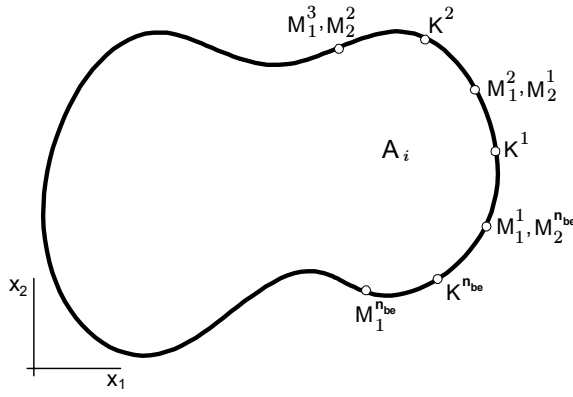


Figure 2.

where

$$\Phi^{je} = \begin{bmatrix} \phi_{11}^{je}(M_2) - \phi_{11}^{je}(M_1) & \phi_{12}^{je}(M_2) - \phi_{12}^{je}(M_1) & \dots & \phi_{16}^{je}(M_2) - \phi_{16}^{je}(M_1) \\ \phi_{21}^{je}(M_2) - \phi_{21}^{je}(M_1) & \phi_{22}^{je}(M_2) - \phi_{22}^{je}(M_1) & \dots & \phi_{26}^{je}(M_2) - \phi_{26}^{je}(M_1) \end{bmatrix}. \quad (4.3)$$

With the notation

$$\mathbf{M}^{je} = \Phi^{je} \mathbf{B}^j (\mathbf{T}^e)^{-1} \quad (4.4)$$

equation system (4.2) can be cast into the form

$$\underbrace{\begin{bmatrix} \mathbf{M}^{11} & \mathbf{M}^{12} & \dots & \mathbf{M}^{1n_{be}} \\ \mathbf{M}^{21} & \mathbf{M}^{22} & \dots & \mathbf{M}^{2n_{be}} \\ \dots & \dots & \dots & \dots \\ \mathbf{M}^{n_{be}1} & \mathbf{M}^{n_{be}2} & \dots & \mathbf{M}^{n_{be}n_{be}} \end{bmatrix}}_{2n_{be} \times 2n_{be}} \underbrace{\begin{bmatrix} \mathbf{p}^1 \\ \mathbf{p}^2 \\ \dots \\ \mathbf{p}^{n_{be}} \end{bmatrix}}_{2n_{be} \times 1} = \underbrace{\begin{bmatrix} \mathbf{C}(Q_1) \mathbf{u}_1 \\ \mathbf{C}(Q_2) \mathbf{u}_2 \\ \dots \\ \mathbf{C}(Q_{n_{be}}) \mathbf{u}_{n_{be}} \end{bmatrix}}_{2n_{be} \times 1}, \quad (4.5)$$

where

$$\begin{aligned} (\mathbf{p}^1)^T &= [ \mathcal{F}_1^{1M_1} & \mathcal{F}_2^{1M_1} & \mathfrak{t}_1^{1K} & \mathfrak{t}_2^{1K} & \mathcal{F}_1^{1M_2} & \mathcal{F}_2^{1M_2} ] \\ (\mathbf{p}^2)^T &= [ \mathcal{F}_1^{2M_1} & \mathcal{F}_2^{2M_1} & \mathfrak{t}_1^{2K} & \mathfrak{t}_2^{2K} & \mathcal{F}_1^{2M_2} & \mathcal{F}_2^{2M_2} ] \end{aligned} \quad (4.6a)$$

and

$$(\mathbf{p}^{n_{be}})^T = [ \mathcal{F}_1^{n_{be}M_1} & \mathcal{F}_2^{n_{be}M_1} & \mathfrak{t}_1^{n_{be}K} & \mathfrak{t}_2^{n_{be}K} & \mathcal{F}_1^{n_{be}M_2} & \mathcal{F}_2^{n_{be}M_2} ]. \quad (4.6b)$$

With regard to the assumed continuity of the stress functions at the points  $Q_j$  we have

$$\mathcal{F}_\lambda^{n_{be}M_2} = \mathcal{F}_\lambda^{1M_1}, \quad \mathcal{F}_\lambda^{eM_2} = \mathcal{F}_\lambda^{(e+1)M_1} \quad e = 1, \dots, n_{be} - 1. \quad (4.7)$$

Introducing the notation

$$\mathbf{M}^e = \begin{bmatrix} \mathbf{M}^{1e} \\ \mathbf{M}^{2e} \\ \dots \\ \mathbf{M}^{n_{be}e} \end{bmatrix} \quad e = 1, \dots, n_{be} \quad (4.8)$$

and taking equation (4.7) into account one can see that the last two columns of the matrix  $\mathbf{M}^e$  and the first two columns of the next matrix  $\mathbf{M}^{e+1}$  are multiplied, due to the continuity, by the same  $\mathcal{F}_\lambda$ . Accordingly, the corresponding columns can be added to each other. It is also obvious that the last two columns of the matrix  $\mathbf{M}^{n_{be}}$  should be added to the first two columns of the matrix  $\mathbf{M}^1$ . After these transformations the size of the equation system (4.5) has been decreased. Finally we have

$$\mathbf{H}\mathbf{f} = \mathbf{q}, \quad (4.9)$$

where  $\mathbf{H}$  is a matrix with size  $(2n_{be} * 4n_{be})$ ,

$$\mathbf{f} = \begin{bmatrix} \mathcal{F}_1^{1M_1} & \mathcal{F}_2^{1M_1} & \mathfrak{t}_1^{1K1} & \mathfrak{t}_2^{1K1} & \mathcal{F}_1^{2M_2} & \mathcal{F}_2^{2M_2} & \dots & \mathcal{F}_1^{n_{be}M_{n_{be}}} & \mathcal{F}_1^{n_{be}M_{n_{be}}} \end{bmatrix} \quad (4.10)$$

is the vector of physical quantities and  $\mathbf{q}$  is the right side of equation system (4.5).

It should be noted that the functions  $\phi_{11}$ ,  $\phi_{21}$ ,  $\phi_{14}$  and  $\phi_{24}$  which have been obtained from the constant shape functions, are singular when the point of effect  $M$  approaches the source point  $Q_j$  because the distance between the two points tends to zero. In order to avoid strongly singular integrals we should take into account that  $\mathfrak{t}_\lambda(\overset{\circ}{M}) = 0$  if  $\mathfrak{u}_\lambda(\overset{\circ}{Q}) = \mathfrak{u}_\lambda(\overset{\circ}{M}) = \text{constant}$ . Under this condition equation (2.20) yields

$$c_{\kappa\lambda}(\overset{\circ}{Q})\mathfrak{u}_\lambda(\overset{\circ}{Q}) = \sum_{e=1}^{n_{be}} \int_{\mathfrak{L}_e} n_\rho(\overset{\circ}{M}) \left( \epsilon_{\rho\pi 3}^* \mathfrak{e}_{\kappa\pi\lambda}(\overset{\circ}{M}, \overset{\circ}{Q}) - \delta_{\rho\lambda} \mathfrak{u}_{\kappa 3}^*(\overset{\circ}{M}, \overset{\circ}{Q}) \right) \mathfrak{u}_\lambda(\overset{\circ}{Q}) \, ds_{\overset{\circ}{M}}. \quad (4.11)$$

Subtracting now equation (4.11) from (2.20) we obtain

$$0 = \sum_{e=1}^{n_{be}} \int_{\mathfrak{L}_e} -n_\rho(\overset{\circ}{M}) \mathfrak{u}_{\kappa\lambda}(\overset{\circ}{M}, \overset{\circ}{Q}) \left( \epsilon_{\rho\pi 3} \mathfrak{e}_{\pi\lambda}(\overset{\circ}{M}) - \delta_{\rho\lambda} \varphi_3(\overset{\circ}{M}) \right) + \quad (4.12)$$

$$+ n_\rho(\overset{\circ}{M}) \left( \epsilon_{\rho\pi 3}^* \mathfrak{e}_{\kappa\pi\lambda}(\overset{\circ}{M}, \overset{\circ}{Q}) - \delta_{\rho\lambda} \mathfrak{u}_{\kappa 3}^*(\overset{\circ}{M}, \overset{\circ}{Q}) \right) \left( \mathfrak{u}_\lambda(\overset{\circ}{M}) - \mathfrak{u}_\lambda(\overset{\circ}{Q}) \right) \, ds_{\overset{\circ}{M}}.$$

After this transformation  $c_{\kappa\lambda}(\overset{\circ}{Q})$  has also been eliminated from the equation. Equation (4.12) differs from equation (2.20) in the extra term

$$n_\rho(\overset{\circ}{M}) \left( \epsilon_{\rho\pi 3}^* \mathfrak{e}_{\kappa\pi\lambda}(\overset{\circ}{M}, \overset{\circ}{Q}) - \delta_{\rho\lambda} \mathfrak{u}_{\kappa 3}^*(\overset{\circ}{M}, \overset{\circ}{Q}) \right) \mathfrak{u}_\lambda(\overset{\circ}{Q}). \quad (4.13)$$

This term is divergence free, therefore the new coefficient of  $n_\rho(\overset{\circ}{M})$  on the right side is also divergence free. Thus, for this purpose,  $\mathfrak{u}_\lambda(\overset{\circ}{Q})$  can be regarded as a constant.

If we consider the  $j$ -th element which involves the source point  $\overset{\circ}{Q}$  at its first nodal point  $M_1$ , then these constants are denoted by  $u_1(\overset{\circ}{Q}) = \hat{a}_1^j$  and  $u_2(\overset{\circ}{Q}) = \hat{a}_4^j$  in the local coordinate system. With these notations for the  $e$ -th element we get

$$\left( u_\lambda(\overset{\circ}{M}) - u_\lambda(\overset{\circ}{Q}) \right)^e = \mathbf{U}^e(\eta_1, \eta_2) \tilde{\mathbf{a}}^e \quad (4.14)$$

where the columns of  $\mathbf{U}^e(\eta_1, \eta_2)$  are formed by the vectors  $u_i$  ( $i = 1, \dots, 6$ ). The vector of constants  $\tilde{\mathbf{a}}^e$  can now be rewritten as

$$(\tilde{\mathbf{a}}^e)^T = \left[ \left( \hat{a}_1^e - \hat{a}_1^j \right) \quad a_2^e \quad a_3^e \quad \left( \hat{a}_4^e - \hat{a}_4^j \right) \quad a_5^e \quad a_6^e \right]. \quad (4.15)$$

$\phi_{11}$ ,  $\phi_{21}$ ,  $\phi_{14}$  and  $\phi_{24}$  are singular when the point of effect approaches the source point, but in this case  $\hat{a}_1^e = \hat{a}_1^j$  and  $\hat{a}_4^e = \hat{a}_4^j$ , therefore we can avoid the evaluation of these potential functions. Consequently the line of thought presented in Section 3 can be repeated word by word and it turns out that the functions  $\phi_{\kappa i}$  will remain unchanged. Finally the discretized equation corresponding to the equation (4.12) assumes the form

$$\mathbf{0} = \sum_{e=1}^{n_{be}} \Phi^{je} \tilde{\mathbf{a}}^e \quad j = 1, \dots, n_{be} \quad (4.16)$$

where the elements of the matrix  $\Phi^{je}$  are those integrals obtained from the shape functions. The structure of the matrix  $\Phi^{je}$  has already been presented – see equation (4.3).

The next transformation becomes clearer, if the equation (4.16) is written out in full:

$$\underbrace{\begin{bmatrix} \Phi^{11} & \Phi^{12} & \dots & \Phi^{1n_{be}} \\ \Phi^{21} & \Phi^{22} & \dots & \Phi^{2n_{be}} \\ \dots & \dots & \dots & \dots \\ \Phi^{n_{be}1} & \Phi^{n_{be}2} & \dots & \Phi^{n_{be}n_{be}} \end{bmatrix}}_{\Phi} \begin{bmatrix} \tilde{\mathbf{c}}^1 \\ \tilde{\mathbf{c}}^2 \\ \dots \\ \tilde{\mathbf{c}}^{n_{be}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dots \\ 0 \end{bmatrix}. \quad (4.17)$$

Clearly, if the  $j$ -th element ( $j$  is the equation counter or, which is the same, the block row counter) coincides with the  $e$ -th element ( $e$  is the element counter or which is the same the block column counter), then we are in the main diagonal, i.e., the  $j$ -th element involves the source point, consequently  $\hat{a}_1^e - \hat{a}_1^j = 0$  and  $\hat{a}_4^e - \hat{a}_4^j = 0$ . Thus the singular terms  $\phi_{\kappa 1}^{jj}(M_1)$ ,  $\phi_{\kappa 4}^{jj}(M_1)$  of the matrix  $\phi_{\kappa i}^{jj}(M_1)$  drop out and the other terms in matrix  $\phi_{\kappa i}^{jj}(M_1)$  have already been zero. With regard to the assumed continuity of the stress functions at the nodal points, the previous establishment is true for the matrix  $\phi_{\kappa i}^{jj-1}(M_2)$ .

Decompose  $\tilde{\mathbf{a}}^e$  into two parts. The first part is the vector  $\hat{\mathbf{a}}^e$ , the part left is denoted by

$$(\mathbf{d}^e)^T = \left[ -\hat{a}_1^j \quad 0 \quad 0 \quad -\hat{a}_4^j \quad 0 \quad 0 \right]. \quad (4.18)$$

The point  $M_2$  of the element  $e$  coincides with the point  $M_1$  of the element  $e + 1$ . Assuming continuity it follows that  $\phi_{\kappa i}^e(M_2) = \phi_{\kappa i}^{e+1}(M_1)$ , therefore we can write

$$\mathbf{0} = \sum_{e=1}^{n_{be}} \Phi^{je} \mathbf{d}^e \quad j = 1, \dots, n_{be}. \quad (4.19)$$

With regard to equation (4.19), the following relations hold for the diagonal blocks of the matrix  $\Phi$  :

$$\Phi^{kk} = \begin{bmatrix} \phi_{11}^{kk}(M_2) & \phi_{12}^{kk}(M_2) & \dots & \phi_{16}^{kk}(M_2) \\ \phi_{21}^{kk}(M_2) & \phi_{22}^{kk}(M_2) & \dots & \phi_{26}^{kk}(M_2) \end{bmatrix} \quad k = 1, \dots, n_{be}. \quad (4.20a)$$

In the same way we have

$$\Phi^{k,k-1} = \begin{bmatrix} -\phi_{11}^{k,k-1}(M_1) & -\phi_{12}^{k,k-1}(M_1) & \dots & -\phi_{16}^{k,k-1}(M_1) \\ -\phi_{21}^{k,k-1}(M_1) & -\phi_{22}^{k,k-1}(M_1) & \dots & -\phi_{26}^{k,k-1}(M_1) \end{bmatrix} \quad k = 2, \dots, n_{be} \quad (4.20b)$$

and

$$\Phi^{1,n_{be}-1} = \begin{bmatrix} -\phi_{11}^{1,n_{be}-1}(M_1) & -\phi_{12}^{1,n_{be}-1}(M_1) & \dots & -\phi_{16}^{1,n_{be}-1}(M_1) \\ -\phi_{21}^{1,n_{be}-1}(M_1) & -\phi_{22}^{1,n_{be}-1}(M_1) & \dots & -\phi_{26}^{1,n_{be}-1}(M_1) \end{bmatrix}. \quad (4.20c)$$

Making use of equations (4.20a,b,c), (3.13) and (3.17), equation system (4.17) can be manipulated into the form

$$\mathbf{0} = \sum_{e=1}^{n_{be}} \Phi^{je} \hat{\mathbf{a}}^e = \sum_{e=1}^{n_{be}} \Phi^{je} \mathbf{B}^j (\mathbf{T}^{-1})^e \mathbf{p}^e \quad j = 1, \dots, n_{be}. \quad (4.21)$$

We remark that this equation is originated from the second dual Somigina formula (2.20). Introducing the notation

$$\mathbf{N}^{je} = \Phi^{je} \mathbf{B}^j (\mathbf{T}^e)^{-1} \quad (4.22)$$

the equation system (4.21) can be written as

$$\sum_{e=1}^{n_{be}} \mathbf{N}^{je} \mathbf{p}^e = \mathbf{0}^e \quad j = 1, \dots, n_{be}. \quad (4.23)$$

It has not been taken into account so far that the matrices  $\mathbf{p}^e$  should meet the continuity condition (4.7). Under this condition the line of thought leading to (4.9) can be repeated word by word. Finally we get

$$\mathbf{K} \mathbf{f} = \mathbf{0}, \quad (4.24)$$

where  $\mathbf{K}$  is a matrix with size  $(2n_{be} * 4n_{be})$  and  $\mathbf{f}$  denotes the vector of physical quantities – see (4.10) for details.

### 5. Stresses at internal points

If the physical quantities at the nodal points are known, then for computing the vectors of constants we should apply the equation

$$\mathbf{a}^e = [\mathbf{T}^e]^{-1} \mathbf{p}^e. \quad (5.1)$$

With the knowledge of the vector  $\mathbf{a}^e$  we can compute the stress functions at an arbitrary point. Making use of the equation (2.19) and applying the notations we have introduced, equation (2.19) can be manipulated into the form

$$\mathbf{u}(Q) = \sum_{e=1}^{n_{be}} \Phi^{Qe} \mathbf{B}^Q \mathbf{a}^e \quad (5.2)$$

where  $\mathbf{B}^Q$  is the transformation matrix corresponding to the internal source point  $Q$ . Recalling equation (2.1) for the stress components at  $Q$  we can write

$$\begin{aligned} \sigma_{11} &= \frac{\partial \mathbf{u}_1(Q)}{\partial x_2}, & \tau_{12} &= \frac{\partial \mathbf{u}_2(Q)}{\partial x_2}, \\ \tau_{21} &= -\frac{\partial \mathbf{u}_1(Q)}{\partial x_1}, & \sigma_{22} &= -\frac{\partial \mathbf{u}_2(Q)}{\partial x_1}. \end{aligned} \quad (5.3)$$

Derivatives of the stress functions are obtained from equation (5.2)

$$\mathbf{u}(Q) \partial x_\alpha = \sum_{e=1}^{n_{be}} [(\Phi^{Qe} \partial x_\alpha) \mathbf{B}^Q + \Phi^{Qe} (\mathbf{B}^Q \partial x_\alpha)] \mathbf{a}^e. \quad (5.4)$$

Using equation (2.11) the above equation (5.4) can be rewritten as

$$\mathbf{u}(Q) \partial x_\alpha = \sum_{e=1}^{n_{be}} [\Phi^{Qe} (\mathbf{B}^Q \partial x_\alpha) - (\Phi^{Qe} \partial \eta_\alpha) \mathbf{B}^Q] \mathbf{a}^e. \quad (5.5)$$

### 6. Examples

The main step of the numerical computations consists in solving the equation system (4.24). It is worthy of repeating that the matrix  $\mathbf{K}$  is not a square one and the vector  $\mathbf{f}$  involves the physical quantities. The total number of physical quantities is  $4n_{be}$  on the contour  $\mathcal{L}_o$ . Some of them are known from the boundary conditions. (Three physical quantities can be prescribed from the six possible physical quantities on each element. The total number of the physical quantities that can be prescribed is, however, less than  $3 \times 2n_{be}$  since continuity must hold at the extremities.) The columns of the matrix  $\mathbf{K}$  that are multiplied by the prescribed quantities should be grouped on the right side of the equation to get the right side of the equation system to be solved. We should know at least  $2n_{be}$  physical quantities from the boundary conditions to get a solvable linear equation system.

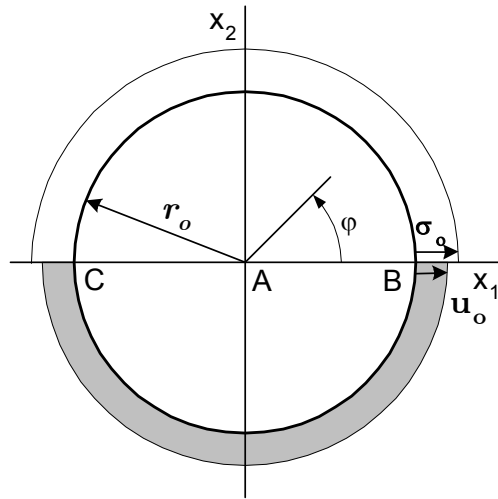


Figure 3.

In case of mixed boundary value problems we have more equations than the number of unknowns. If we regard some of the prescribed quantities as unknowns, then we can obtain a solvable linear equation system.

Two simple numerical examples are presented. The material properties ( $\mu = 8 \cdot 10^4$  MPa,  $\nu = 0.3$ ) are the same for each example.

**Problem 1.** First we shall consider a circular region with radius  $r_0 = 10$  mm. On the arc  $BC$  of the contour for which the polar angle  $\varphi \in [0, \pi]$  the normal stress is  $\sigma_o = 100$  MPa and there is no shear stress. On the arc  $CB$  of the contour  $u_o = (1 - 2\nu)\sigma_o r_o / 2\mu$  is the radial displacement and there is no displacement in the circumferential direction – see Figure 3. In this case

$$\mathcal{F}_1 = \sigma_o x_2 = \sigma_o r \sin \varphi, \quad \mathcal{F}_2 = -\sigma_o x_1 = -\sigma_o r \cos \varphi,$$

$$\sigma_{11} = \sigma_{22} = \sigma_o = 100 \text{ MPa}, \quad \tau_{12} = 0,$$

$$u_1 = \frac{1 - 2\nu}{2\mu} \sigma_o r \sin \varphi, \quad u_2 = \frac{1 - 2\nu}{2\mu} \sigma_o r \cos \varphi.$$

The exact solutions for this problem are given by the equations

$$u_1 = \mathcal{F}_1 = \sigma_o r_o \sin \varphi, \quad u_2 = \mathcal{F}_2 = -\sigma_o r_o \cos \varphi,$$

$$t_1 = \frac{1 - 2\nu}{2\mu} \sigma_o \sin \varphi, \quad t_2 = -\frac{1 - 2\nu}{2\mu} \sigma_o \cos \varphi.$$

where  $r$  and  $\varphi$  are polar coordinates. One can check with ease that these solutions determine a homogenous state of stress. At the internal points the exact solutions for

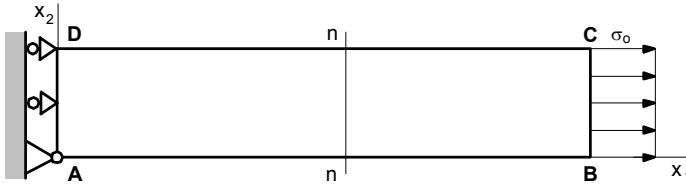


Figure 4.

the stresses are as follows:

$$\sigma_{11} = \sigma_{22} = \sigma_o = 100 \text{ [MPa]}, \quad \tau_{12} = 0.$$

Table 1 below represents numerical results at various internal points.

Table 1: Solutions for stress components

$x_1$ [mm]	$x_2$ [mm]	$\sigma_{11}$ [MPa]	$\tau_{12}$ [MPa]	$\sigma_{22}$ [MPa]
-8.00	0.00	100.00	$1.69 \cdot 10^{-15}$	100.00
-6.00	0.00	100.00	$7.61 \cdot 10^{-15}$	100.00
-4.00	0.00	100.00	$1.07 \cdot 10^{-14}$	100.00
-2.00	0.00	100.00	$1.12 \cdot 10^{-14}$	100.00
0.00	0.00	99.99	0.08	100.00
2.00	0.00	100.00	$1.69 \cdot 10^{-13}$	100.00
4.00	0.00	100.00	0	100.00
5.00	5.00	100.00	$5.73 \cdot 10^{-15}$	100.00
5.00	7.00	100.00	$1.73 \cdot 10^{-14}$	100.00

**Problem 2.** The width and length of the rectangle  $ABCD$  in plane strain are 20 mm and 100 mm, respectively. The rectangle is subjected to a horizontal and uniform load  $\sigma_o = 200$  MPa on the line  $BC$ . The upper and lower sides are, however, unloaded. The right end of the region is supported as shown in Figure 3. In this case the solutions computed are comparable with the solutions  $\sigma_{11} = 200$  MPa,  $\tau_{12} = 0$  and  $\sigma_{22} = 0$  valid for a bar in tension. Table 2 below contains numerical results for the stresses at some internal points located on the line  $n - n$  ( $x_1 = 50$  mm).

Table 2: Solutions for stress components

$x_1$ [mm]	$x_2$ [mm]	$\sigma_{11}$ [MPa]	$\tau_{12}$ [MPa]	$\sigma_{22}$ [MPa]
50.00	2.00	200.00	$7.42 \cdot 10^{-17}$	$-3.46 \cdot 10^{-15}$
50.00	4.00	200.00	$-1.38 \cdot 10^{-17}$	$2.53 \cdot 10^{-15}$
50.00	6.00	200.00	$-7.63 \cdot 10^{-17}$	$-4.73 \cdot 10^{-15}$
50.00	8.00	199.99	$8.60 \cdot 10^{-16}$	$3.88 \cdot 10^{-16}$

## 7. Conclusion

The boundary contour method for plane problems of elasticity in a dual formulation (regarding the stress functions of order one and the rigid body rotation as unknowns) is presented in this paper. After having shown what form the Somigliana formulae have in the dual system of elasticity we proved that the integrand of the direct method is divergence free like the case of primal system [1,2]. Making use of this property an implementation is carried out with linear approximation and the idea that there are no stresses due to constant stress functions has also been taken into account. The corresponding shape functions  $\phi_{ki}$  are also given. It is an advantage of the resulting system of linear equations that there is no need to perform numerical integration when one computes the coefficient matrix and the right side. It is a further advantage that computation of stresses on the boundary elements requires derivations, that is, on the contrary to conventional BEM, one can avoid computation of singular integrals. Two simple examples are given to illustrate the applicability of the method.

It is the aim of our further investigations to apply quadratic approximation and to clarify how to use the method for outer regions if there is a constant stress state at infinity. This work is in progress.

## 8. Appendix

The compatibility equation and the symmetry conditions for the elastic state  $\overset{*}{\mathbf{u}}_\lambda$  are of the form

$$\left[ \epsilon_{\rho\pi 3} \overset{*}{e}_{\kappa\pi\lambda}(\overset{\circ}{M}, Q) - \delta_{\rho\lambda} \mathfrak{U}_{\kappa 3}(\overset{\circ}{M}, Q) \right] \overset{M}{\partial}_\rho = 0 \quad (8.1)$$

and

$$\epsilon_{\lambda\pi 3} \overset{*}{\mathfrak{t}}_{\kappa\pi\lambda} = 0 \quad (8.2)$$

Since  $\overset{*}{\mathfrak{t}}_{\kappa\pi\lambda}$  and  $e_{\kappa\lambda}$  are 'elastic states' we can write

$$\overset{*}{\mathfrak{t}}_{\kappa\pi\lambda} e_{\pi\lambda} = \mathfrak{t}_{\pi\lambda} \overset{*}{e}_{\kappa\pi\lambda} \quad (8.3)$$

With regard to (2.3), (2.4), (8.1), (8.2) and (8.3) it follows from (3.6) that

$$\begin{aligned} P_{\kappa\rho}(M, Q) \overset{M}{\partial}_\rho &= - \left( \mathfrak{U}_{\kappa\lambda}(\overset{\circ}{M}, Q) \overset{M}{\partial}_\rho \right) \left( \epsilon_{\rho\pi 3} e_{\pi\lambda}(\overset{\circ}{M}) - \delta_{\rho\lambda} \varphi_3(\overset{\circ}{M}) \right) - \\ &\quad - \mathfrak{U}_{\kappa\lambda}(\overset{\circ}{M}, Q) \left( \left[ \epsilon_{\rho\pi 3} e_{\pi\lambda}(\overset{\circ}{M}) - \delta_{\rho\lambda} \varphi_3(\overset{\circ}{M}) \right] \overset{M}{\partial}_\rho \right) + \\ &\quad + \left( \left[ \epsilon_{\rho\pi 3} \overset{*}{e}_{\kappa\pi\lambda}(\overset{\circ}{M}, Q) - \delta_{\rho\lambda} \mathfrak{U}_{\kappa 3}(\overset{\circ}{M}, Q) \right] \overset{M}{\partial}_\rho \right) \mathfrak{u}_\lambda(\overset{\circ}{M}) + \\ &\quad + \left( \epsilon_{\rho\pi 3} \overset{*}{e}_{\kappa\pi\lambda}(\overset{\circ}{M}, Q) - \delta_{\rho\lambda} \mathfrak{U}_{\kappa 3}(\overset{\circ}{M}, Q) \right) \left( \mathfrak{u}_\lambda(\overset{\circ}{M}) \overset{M}{\partial}_\rho \right) = \end{aligned}$$



$$\begin{aligned}
&= - \left( \mathfrak{u}_{\kappa\lambda}(\overset{\circ}{M}, Q) \overset{M}{\partial}_\rho \right) \epsilon_{\rho\pi 3} e_{\pi\lambda}(\overset{\circ}{M}) + \epsilon_{\rho\pi 3} e_{\kappa\pi\lambda}^*(\overset{\circ}{M}, Q) \left( \mathfrak{u}_\lambda(\overset{\circ}{M}) \overset{M}{\partial}_\rho \right) + \\
&+ \left( \mathfrak{u}_{\kappa\lambda}(\overset{\circ}{M}, Q) \overset{M}{\partial}_\rho \epsilon_{\rho\pi 3} \right) \epsilon_{\lambda\pi 3} \varphi_3(\overset{\circ}{M}) - \left( \mathfrak{u}_\lambda(\overset{\circ}{M}) \overset{M}{\partial}_\rho \epsilon_{\rho\pi 3} \right) \epsilon_{\lambda\pi 3} \mathfrak{u}_{\kappa 3}(\overset{\circ}{M}, Q) = \\
&= - \mathfrak{t}_{\kappa\pi\lambda}^* e_{\pi\lambda} + \mathfrak{t}_{\pi\lambda} e_{\kappa\pi\lambda}^* + \epsilon_{\lambda\pi 3} \mathfrak{t}_{\kappa\pi\lambda}^* \varphi_3(\overset{\circ}{M}) - \epsilon_{\lambda\pi 3} \mathfrak{t}_{\kappa\pi}^* \varphi_3(\overset{\circ}{M}) = 0
\end{aligned}$$

In other words  $P_{\kappa\rho}$  is divergence free.

Without entering into details, below we list the shape functions:

$$\begin{aligned}
\phi_{11} &= \frac{1}{2\pi} \arctan \frac{\eta_2}{\eta_1} + \frac{1}{4\pi(1-\nu)} \frac{\eta_1 \eta_2}{\eta_1^2 + \eta_2^2} \\
\phi_{12} &= \frac{-1}{4\pi(1-\nu)} \eta_2 \left( \ln \sqrt{\eta_1^2 + \eta_2^2} + \frac{4\nu - 3}{2} + \frac{2\eta_2^2}{\eta_1^2 + \eta_2^2} \right) \\
\phi_{13} &= \frac{1}{4\pi(1-\nu)} \eta_1 \left( (1-\nu) \ln \sqrt{\eta_1^2 + \eta_2^2} + \frac{3-\nu}{2} - \frac{\eta_1^2}{\eta_1^2 + \eta_2^2} \right) \\
\phi_{14} &= \frac{-1}{4\pi(1-\nu)} \left( (1-2\nu) \ln \sqrt{\eta_1^2 + \eta_2^2} + \frac{\eta_1^2}{\eta_1^2 + \eta_2^2} \right) \\
\phi_{15} &= \frac{1}{4\pi(1-\nu)} \eta_1 \left( \nu \ln \sqrt{\eta_1^2 + \eta_2^2} + \frac{5\nu}{2} - \frac{\eta_1^2}{\eta_1^2 + \eta_2^2} \right) \\
\phi_{16} &= \frac{-\mu}{4\pi(1-\nu)} \eta_2 \left( 2 \ln \sqrt{\eta_1^2 + \eta_2^2} + 3 \right) \\
\phi_{21} &= \frac{1}{4\pi(1-\nu)} \left( (1-2\nu) \ln \sqrt{\eta_1^2 + \eta_2^2} + \frac{\eta_2^2}{\eta_1^2 + \eta_2^2} \right) \\
\phi_{22} &= \frac{-1}{4\pi(1-\nu)} \eta_1 \left( \ln \sqrt{\eta_1^2 + \eta_2^2} + \frac{4\nu - 3}{2} + \frac{2\eta_1^2}{\eta_1^2 + \eta_2^2} \right) \\
\phi_{23} &= \frac{-1}{4\pi(1-\nu)} \eta_2 \left( \nu \ln \sqrt{\eta_1^2 + \eta_2^2} + \frac{5\nu}{2} - \frac{\eta_2^2}{\eta_1^2 + \eta_2^2} \right) \\
\phi_{24} &= -\frac{1}{2\pi} \arctan \frac{\eta_1}{\eta_2} - \frac{1}{4\pi(1-\nu)} \frac{\eta_1 \eta_2}{\eta_1^2 + \eta_2^2} \\
\phi_{25} &= \frac{-1}{4\pi(1-\nu)} \eta_2 \left( (1-\nu) \ln \sqrt{\eta_1^2 + \eta_2^2} + \frac{3-\nu}{2} - \frac{\eta_2^2}{\eta_1^2 + \eta_2^2} \right) \\
\phi_{26} &= \frac{\mu}{4\pi(1-\nu)} \eta_1 \left( 2 \ln \sqrt{\eta_1^2 + \eta_2^2} + 3 \right)
\end{aligned}$$

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