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Unitarity Boomerang

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For the three family quark flavor mixing, the best parametrization is the original Kobayashi-Maskawa matrix, V_{KM}, with four real parameters: three rotation angles $\theta_{1,2,3}$ and one phase δ. A popular way of presentation is by the unitarity triangle which, however, explicitly displays only three, not four, independent parameters. Here we propose an alternative presentation which displays simultaneously all four parameters: the unitarity boomerang.

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Introduction

As is well known, there are different ways of parameterizing the Kobayashi-Maskawa[1] quark mixing matrix, V_{KM}. For three generations of quarks, V_{KM} is a 3×3 unitary mixing matrix with three rotation angles ($\theta_1,\ \theta_2,\ \theta_3$) and one CP violating phase δ. The magnitudes of the elements V_{ij} of V_{KM} are physical quantities which do not depend on parametrization. However, the value of δ does. For example, in the Particle Data Group (PDG) parametrization [2], adopted from Ref.[3], $\delta \sim 70^\circ$, whereas the phase in the original KM parametrization has a different value, $\delta \sim 90^\circ$. Care must be exercised in quoting a value of δ, as it depends on how the matrix is parameterized. For example, the statement made after Eq. (11.3) in the current edition of PDG is misleading, because it identifies, incorrectly, the phase δ of Ref.[1].

It can therefore be more useful to employ only physically-measurable quantities. To this end, it has long ago been suggested that a unitarity triangle (UT) be used[4] as a useful presentation for the quark flavor mixing, especially of CP violation[5]. Because of the unitary nature of the KM matrix, one has $\sum_i V_{ij}V^*_{ik} = \delta_{jk}$ and $\sum_i V_{ji}V^*_{ki} = \delta_{jk}$, where the first and second indices of V_{ij} take the values u, c, t, ... and d, s, b, ..., respectively. For three generations of quarks, when $j \neq k$, these equations form closed triangles in a plane, the UTs. Six UTs can be formed with all of them having the same area. $A(UT)$, which is equal to half of the value of the Jarlskog determinant [6] J, so that $A(UT) = \frac{1}{2}J$. The inner angles of a given UT are therefore closely related to the CP violating measure J. When the inner angles are measured independently, their sum, whether it turns out to be consistent with precisely 180°, provides a test for the unitarity of the KM matrix. The unitarity triangle is also a popular way, to present CP violation, with three generations of quarks.

A UT, however, does not contain all the information encoded in the KM matrix, V_{KM}. Although a UT has three inner angles and three sides, it contains only three independent parameters. The three parameters can be chosen to be two of the three inner angles and the area, or the three sides, or some combination thereof. One needs an additional parameter fully to represent the physics: this is hardly surprising, as the original UT idea of [4] involved only two, of the three, rows or columns of the 3×3 matrix, V_{KM},

An improved presentation is thus rendered desirable, in order better to present the KM matrix, V_{KM}, diagrammatically. In this Letter, we propose such a new diagram, the unitarity boomerang. The boomerang diagram will contain information from not just one UT, but two UTs, from among [7] the six possible different UTs.

Unitarity Boomerang

We indicate the KM matrix and its elements by $V_{KM} = (V_{KM})_{ij}$, with $i = u, c, t$ and $j = b, s, d$. The unitarity of this matrix implies $\Sigma_i V_{ij}V^*_{ik} = \delta_{jk}$ and $\Sigma_j V_{ij}V^*_{kj} = \delta_{ik}$. The $j \neq k$ and $i \neq k$ cases form, respectively, the six possible different UT presentations for V_{KM} in a convenient two-dimensional plane. There are, thus, a total of 18 inner angles in the six UTs. However, only 9 are different because, by Euclidean geometry, each angle, in any particular UT, must have its equal counterpart in another, different, UT. To understand this simple but crucial discussion consider the two UTs defined by

$$
\begin{aligned}
&UT(a) \qquad (V_{KM})_{ud}(V_{KM})^*_{ub} + (V_{KM})_{cd}(V_{KM})^*_{cb} + (V_{KM})_{td}(V_{KM})^*_{tb} = 0 \\
&UT(b) \qquad (V_{KM})_{ud}(V_{KM})^*_{td} + (V_{KM})_{us}(V_{KM})^*_{ts} + (V_{KM})_{ub}(V_{KM})^*_{tb} = 0
\end{aligned} \tag{1}
$$

The inner angles defined by UT (a), in Eq. (1), are

$$
\begin{aligned}
\phi_1(\beta) &= \arg\left(-\frac{(V_{KM})_{cd}(V_{KM})^*_{cb}}{(V_{KM})_{td}(V_{KM})^*_{tb}}\right) \\
\phi_2(\alpha) &= \arg\left(-\frac{(V_{KM})_{td}(V_{KM})^*_{tb}}{(V_{KM})_{ud}(V_{KM})^*_{ub}}\right) \\
\phi_3(\gamma) &= \arg\left(-\frac{(V_{KM})_{ud}(V_{KM})^*_{ub}}{(V_{KM})_{cd}(V_{KM})^*_{cb}}\right)
\end{aligned} \tag{2}
$$

Correspondingly, the unitarity triangle, $UT(b)$ in Eq. (1), defines another three inner angles

$$
\begin{aligned}
\phi'_1(\beta') &= \arg\left(-\frac{(V_{KM})_{us}(V_{KM})^*_{ts}}{(V_{KM})_{ub}(V_{KM})^*_{tb}}\right) \\
\phi'_2(\alpha') &= \arg\left(-\frac{(V_{KM})_{ub}(V_{KM})^*_{tb}}{(V_{KM})_{ud}(V_{KM})^*_{td}}\right) \\
\phi'_3(\gamma') &= \arg\left(-\frac{(V_{KM})_{ud}(V_{KM})^*_{td}}{(V_{KM})_{us}(V_{KM})^*_{ts}}\right)
\end{aligned} \tag{3}
$$

It is clear that $\phi'_2 = \phi_2$.

Since all the six UTs have the same area $J/2$, not all the different 9 angles are independent. For example $J = |(V_{KM})_{td}(V_{KM})^*_{tb}||(V_{KM})_{ud}(V_{KM})^*_{ub}| \sin\phi_2 = |(V_{KM})_{td}(V_{KM})^*_{tb}||(V_{KM})_{cd}(V_{KM})^*_{cb}| \sin\phi_1 = |(V_{KM})_{us}(V_{KM})^*_{ts}||(V_{KM})_{ub}(V_{KM})^*_{tb}| \sin\phi'_1 = |(V_{KM})_{ud}(V_{KM})^*_{td}||(V_{KM})_{us}(V_{KM})^*_{ts}| \sin\phi'_3$. It can be shown that only 4 independent parameters are needed to parameterize the six UTs, and two different UTs contain the needed 4 parameters.

The values for the angles in $UT(a)$, of Eq.(1), derived from various experiments given by PDG are[2]: $\phi_1 = (21.46 \pm 0.98)^\circ$ (derived from data on $\sin(2\phi_1) = 0.681 \pm 0.025$), and the values for ϕ_2 and ϕ_3 are $(88^{+6}_{-5})^\circ$ and $(77^{+30}_{-32})^\circ$, respectively. These values are consistent with the unitarity of the KM matrix within error bars. $UT(a)$, defined by Eq. (1), is almost a right triangle, by virtue of ϕ_2. Numerically, the angles ϕ'_1 and ϕ'_3 are close to ϕ_1 and ϕ_2, respectively. All the angles in the two UTs are sizable, making experimental determination of them merely challenging, while for the other four choices of UT there is always, at least, one small angle where measurement may be exceptionally difficult. It is therefore easiest to work with the two UTs, $UT(a)$ and $UT(b)$, for practical purposes. We now show that, by combining information from these two UTs, into the boomerang diagram [1] displayed in Fig. 1, all information needed to specify the KM matrix, V_{KM}, can be extracted.

The unitarity boomerang is formed by locating the common angle $\phi_2' = \phi_2$ from the two UTs of $UT(a)$ and $UT(b)$ at the top point A and the shortest sides, $AC = |(V_{KM})_{ud}(V_{KM})^*_{ub}|$ and $AC' = |(V_{KM})_{ub}V^*_{tb}|$, on the opposite sides. The other sides are: $AB = |(V_{KM})_{td}(V_{KM})^*_{tb}|$, $AB' = |(V_{KM})_{ud}(V_{KM})^*_{td}|$, $BC = |(V_{KM})_{cd}(V_{KM})^*_{cb}|$ and $B'C' = |V_{KM})_{us}(V_{KM})^*_{ts}|$.

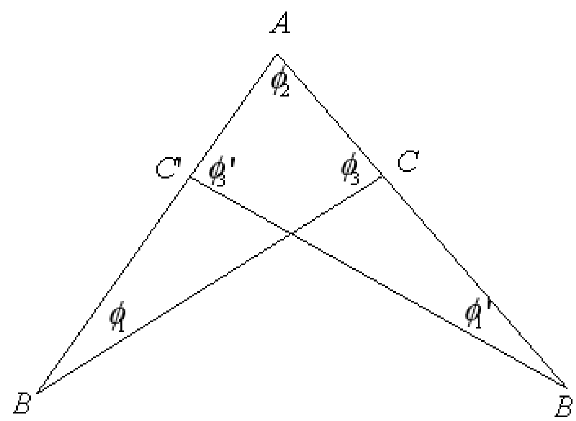

FIG. 1: The unitarity boomerang. The sides are: $AC = |V_{ud}V^*_{ub}|$, $AC' = |V_{ub}V^*_{tb}|$, $AB = |V_{td}V^*_{tb}|$, $AB' = |V_{ud}V^*_{td}|$, $BC = |V_{cd}V^*_{cb}|$ and $B'C' = |V_{us}V^*_{ts}|$.

One can choose the area $(J/2)$ of the triangles, two inner angles from one of the UTs (for example ϕ_1 and ϕ_2), and a third angle from the other UT (for example ϕ_3') as the four independent parameters.

Original KM parametrization and Unitarity Boomerang

To show explicitly how the unitarity boomerang can provide all information needed to specify the quark flavor mixing, we work with a specific parametrization, V_{KM}, originally given by Kobayashi and Maskawa[1]

[1] The name arises from resemblance to the hunting instrument.

$$V_{KM} = \begin{pmatrix} c_1 & -s_1 c_3 & -s_1 s_3 \\ s_1 c_2 & c_1 c_2 c_3 - s_2 s_3 e^{i\delta} & c_1 c_2 s_3 + s_2 c_3 e^{i\delta} \\ s_1 s_2 & c_1 s_2 c_3 + c_2 s_3 e^{i\delta} & c_1 s_2 s_3 - c_2 c_3 e^{i\delta} \end{pmatrix} . \tag{4}$$

One can also work with other parameterizations, such as that adopted by the PDG. But we find an interesting feature of the original KM parametrization which turns out to be very convenient for the discussions of the unitarity boomerang.

Using experimental values[2] for $(V_{KM})_{us} = 0.2257 \pm 0.0010$, $(V_{KM})_{ub} = 0.00359 \pm 0.00016$ and $(V_{KM})_{td} = 0.00874^{+0.00026}_{-0.00037}$, one finds that $s_2 s_3 << 1$. At a few percent level, one has $(V_{KM})_{tb} = (c_1 s_2 s_3 - c_2 c_3 e^{-i\delta}) \approx -c_2 c_3 e^{-i\delta}$.

Then

$$\begin{aligned} \phi_2 &= \arg(-\frac{s_1 s_2 * (c_1 s_2 s_3 - c_2 c_3 e^{-i\delta})}{c_1 * (-s_1 s_3)}) \\ &\approx \arg(\frac{s_1 s_2 * (-c_2 c_3 e^{-i\delta})}{c_1 * s_1 s_3}) = \pi - \delta. \end{aligned} \tag{5}$$

The CP violating phase δ, in this parametrization, is equal to $\pi - \phi_2$, to a good approximation.

The fact that $\phi_2 = (88^{+6}_{-5})^\circ$ implies $\delta \approx 90^\circ$. The almost right triangle UT may indicate that CP, from a deeper perspective, is maximally violated[8, 9]. Kobayashi and Maskawa, with remarkable prescience, made an excellent choice of parametrization. We suggest that the original parametrization of Kobayashi-Maskawa matrix be used as the standard parametrization. A parametrization suggested by Fritzsch and Xing[8], which also has its phase close to ϕ_2, is another alternative interesting parametrization. From the unitarity boomerang, one can easily obtain approximation solutions for the four physical parameters. One first notices that the relation in Eq.(5) allows one to read off the δ from the top angle in the diagram. Taking the ratio, of the two sides AC/AC' or AB/AB', one obtains $|(V_{KM})_{ud}/(V_{KM})^*_{tb}| \approx c_1$ since $|(V_{KM})_{tb}|$ is very close to 1. With c_1 and therefore s_1 known, the length of the sides AB and AC' then provide the values for s_2 and s_3.

One can obtain more precise solutions by using the following information from four sides, $AC = a$, $BC = b$, $AB = c$ and $AB' = d$ of the unitarity boomerang:

$$\begin{aligned}
&a = |(V_{KM})_{ud}(V_{KM})^*_{ub}| = c_1 s_1 s_3\ , \quad b = |(V_{KM})_{cd}(V_{KM})^*_{cb}| = s_1 c_2 |c_1 c_2 s_3 + s_2 c_3 e^{-i\delta}|\ ,\\
&c = |(V_{KM})_{td} V_{KM})^*_{tb}| = s_1 s_2 |c_1 s_2 s_3 - c_2 c_3 e^{-i\delta}|\ , \quad d = |(V_{KM})_{ud}(V_{KM})^*_{td}| = c_1 s_1 s_2\ . \qquad (6)
\end{aligned}$$

Using the above, one can express $s_{1,2,3}$ and δ as functions of a, b, c and d. The KM parameters can be determined. For example

$$a^2 - c_1^2 + c_1^4 \left(\frac{c^2}{d^2} - \frac{b^2}{c_1^4 - c_1^2 + d^2} \right) = 0\ . \qquad (7)$$

Solving for the roots of the above equations, the c_1^2 is determined up to four possible discrete solutions. Restricting to real positive solutions with magnitude less than 1, one can further limit the choices.

The other angles, and the phase, can be determined from the following relations

$$\begin{aligned}
&s_2 = \frac{d}{c_1 s_1}\ , \quad s_3 = \frac{a}{c_1 s_2}\ ,\\
&\cos\delta = \frac{b^2/s_1^2 c_2^2 - (c_1^2 c_2^2 s_3^2 + s_2^2 c_3^2)}{2 c_1 c_2 s_2 c_3 s_3} = \frac{c_1^2 s_2^2 s_3^2 + c_2^2 c_3^2 - c^2/s_1^2 s_2^2}{2 c_1 c_2 s_2 c_3 s_3}\ . \qquad (8)
\end{aligned}$$

After applying the constraint on $c_{2,3}^2$, that they satisfy $0 \le c_{2,3}^2 \le 1$, the solution is even more restricted. Putting in numerical values, for the sides, and comparing with the approximate solution above, we find that a unique solution survives.

Discussion

The most popular way to present the flavor mixing for three generations of quarks is by a unitarity triangle which, however, explicitly displays only three of four independent parameters. To have a diagrammatical representation for the full four independent parameters, we have proposed improvement to the unitarity boomerang.

By studying the unitarity boomerang, one can obtain all the information enshrined in KM matrix. We find that the original parametrization by Kobayashi and Maskawa is particularly convenient for this purpose. The angle ϕ_2 in the boomerang diagram, to a good approximation, can be identified with the phase δ in the original KM parametrization [1].

The fact that $\phi_2 = (88^{+6}_{-5})^\circ$ implies $\delta \approx 90^\circ$, so that this parametrization may be the right one to study assiduously, in order to probe further the connection to the origin of, possibly maximal, CP violation. We, therefore, humbly submit that the original parametrization of KM matrix be kept as the standard.

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[1] M. Kobayashi and T. Maskawa, Prog. Theor. Phys. **49**, 652 (1973).

[2] Particle Data Group, C. Amsler, et al., Phys. Lett. **B667**, 1 (2008).

[3] L. L. Chau and W. Y. Keung, Phys. Rev. Lett. **53**, 1802 (1984)

[4] J.D. Bjorken, Nucl. Phys. **B**(Proc. Suppl.)**11**, 325(1989).

[5] J.H. Christensen, J.W. Cronin, V.L Fitch and R. Turlay, Phys. Rev. Lett. **13,** 138 (1964).

[6] C. Jarlskog, Phys. Rev. Lett. **55**, 1039(1985); Z. Phys. **C29**, 491(1986).
P.H. Frampton and C. Jarkskog, Phys. Lett. **B152,** 421 (1985).

[7] P.H. Frampton and X.-G. He, work in progress.

[8] H. Fritzsch, Z.Z. Xing, Phys. Lett. **B 413** 396 (1997).

[9] Y. Koide, Phys. Lett. **B607**, 123 (2005).