Tr. J. of Mathematics
20 (1996) , 203 – 207.

THE GROUP STRUCTURE OF HECKE GROUPS $H(\lambda_q)$ *

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Abstract

In this work a new proof of the fact that the group structure of the Hecke groups is isomorphic to the free product of two cyclic groups of orders 2 and q is given The method used in the proof requires a result of A.M. Macbeath together with the notion of fundamental region. Also given are some new results on the parabolic points of Hecke groups, which still is an open problem.

1. Introduction

Hecke groups $H(\lambda)$ are the subgroups of $PSL(2,\mathbb{R})$ generated by two linear fractional transformations

$$R(z) = -1/z \quad \text{and} \quad T(z) = z + \lambda.$$

They have been introduced by E. Hecke in [2]. He asked the question that for what values of $\lambda, H(\lambda)$ is discrete. In answering this question he proved that $H(\lambda)$ has a fundamental region iff $\lambda \geq 2$ and real or $1 \leq \lambda < 2$ and $\lambda = \lambda_q = 2\cos(\pi/q), q \in \mathbb{N}$. Therefore $H(\lambda)$ is discrete only for these values of λ. We are particularly interested in the case $\lambda = \lambda_q$ and denote the corresponding Hecke group by $H(\lambda_q)$.

The most interesting and worked Hecke group is the modular group $H(\lambda_3)$ obtained for $q = 3$. It is usually denoted by Γ. In this case all coefficients of the elements of $H(\lambda_3)$ are integers and therefore $\Gamma = PSL(2,\mathbb{Z})$. It is isomorphic to the free product of two cyclic groups C_2 and C_3.

The next two important Hecke groups are denoted by $H(\sqrt{2})$ and $H(\sqrt{3})$ obtained for $q = 4$ and 6, respectively. For these groups, the underlying fields are quadratic extensions of $\mathbb{Q}$ by the algebraic numbers $\sqrt{2}$ and $\sqrt{3}$.

In this paper we give a new and elemantary proof of the fact that $H(\lambda_q)$ is isomorphic to the free product of two finite cyclic groups of orders 2 and q, i.e.

$$H(\lambda_q) \cong C_2 * C_q.$$

* This work is originated from the author's PhD thesis.

This is well-known in the case of the modular group where $q = 3$. In proving this result, we also deal with an open problem on the Hecke groups $H(\lambda_q)$, namely the determination of the parabolic points of $H(\lambda_q)$. We find an infinite class of these points for all q by means of some polynomials.

2. Fundamental Regions and Parabolic Points

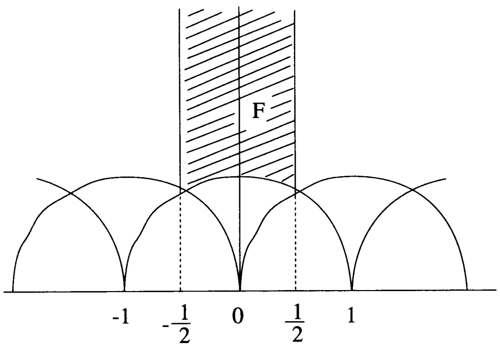
A fundamental region for $\Gamma = H(\lambda_3)$ is given by

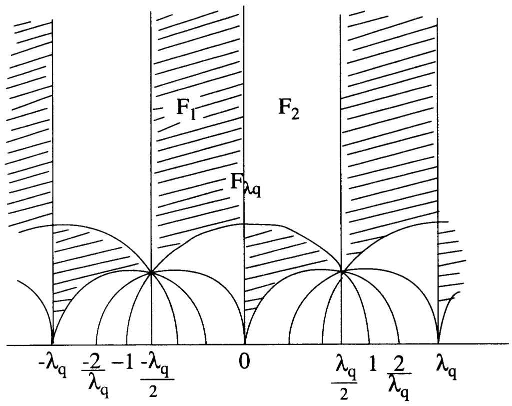
$$F = \left\{ z \in \mathcal{U} : |z| > 1, |Rez| < \frac{1}{2} \right\}.$$

E. Hecke gave a generalisation of F to all Hecke groups $H(\lambda_q)$, [2]:

$$F'_{\lambda_q} = \left\{ z \in \mathcal{U} : |z + \frac{1}{\lambda_q}| > \frac{1}{\lambda_q}, \frac{-\lambda_q}{2} < Rez < 0 \right\}$$

which is $F_1 \cup RF_2$ in Figure 2, as a fundamental region for $H(\lambda_q)$.





We now want to determine the group structure of $H(\lambda_q)$ using some results of Macbeath, [3]. First we have

Definition 1: *Let [G, X] be a topological transformation group and let* $P \subseteq X$. *If for* $g_1, g_2 \in G$, $g_1 \neq g_2$, $g_1 \cap g_2 P = \emptyset$, *then* P *is called a* G*-packing.*

Note that if P *is a* G*-packing, then it contains at most one element from each orbit.*

Theorem 1: *The Hecke group* $H(\lambda_q)$, $q \in \mathbb{N}$, $q \geq 3$, *is isomorphic to the free product of two finite cyclic groups of orders 2 and* q.

The proof depends on the following lemma:

Lemma 1: *Let H and K be two subgroups of a transformation group [G, X]. If P is an H-packing, Q is a K-packing, $A = < H, K >$ – the group generated by the generators of H and K– and $P \cup Q = X$, $P \cap Q \neq \emptyset$, then*

$$A \cong H * K.$$

Also $P \cap Q$ is an A packing.

Proof. See [3].

Let us now prove Theorem 1. Let $H = < R > \cong C_2$ and $K = < S > \cong C_q$. Then H and K are subgroups of $H(\lambda_q)$. Let us now find packings P and Q for H and K, respectively, such that the conditions of Lemma 1 are satisfied:

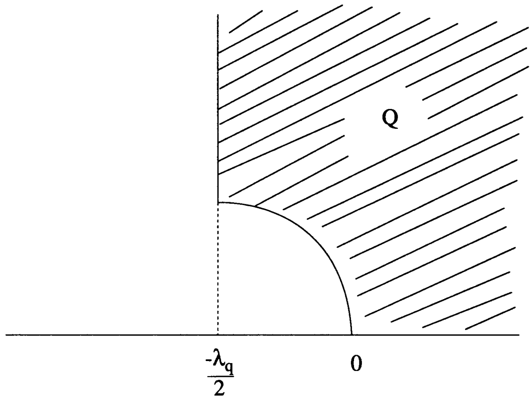
As $R(z) = -1/z = -\bar{z}/|z|^2$, $sign(ReR(z)) = -sign(Rez)$, and the set

$$P = \{z \in \mathcal{U} : Rez < 0\}$$

is an H-packing. Now consider the set

$$Q = \{z \in \mathcal{U} : |z + 1/\lambda_q| > 1/\lambda_q, Rez > -\lambda_q/2\}.$$

□



The elliptic generator $S(z) = -1/(z + \lambda_q)$ can be expressed as a composition of simpler mappings as follows:

i) $T_1(z) = z/|z|^2 = 1/\bar{z}$, reflection in the unit circle,

ii) $T_2(z) = -\bar{z}$, reflection in the line $Rez = 0$,

iii) $T(z) = z + \lambda_q$, translation through λ_q.

Then obviously $S(z) = T_1T_2T(z)$.

Q has the vertices $\zeta, 0$ and ∞. Applying T to Q gives a translation of Q with vertices $\zeta + \lambda_q, \lambda_q$ and ∞. Applying T_2 to TQ we obtain a reflection of TQ with vertices $\zeta, -\lambda_q$ and ∞. Finally applying T_1 to T_2TQ we obtain a reflection of T_2TQ which is SQ with vertices $\zeta, -1/\lambda_q$ and 0.

Similarly, applying T, T_2 and T_1 respectively, this time to SQ, the final region we obtain will be S^2Q with the vertices $\zeta, \lambda_q/(1-\lambda_q^2)$ and $-1/\lambda_q$. Repeating this process another $q-3$ times, we obtain the regions $S^3Q, S^4Q, \cdots, S^{q-1}Q$.

Note that being the fixed point of S, ζ is a vertex of every $S^nQ, 0 \leq n \leq q-1$. Let us now find the other two vertices of S^nQ. An easy calculation shows that

$$S^n(\infty) = -\frac{\alpha_{n-1}(\lambda_q)}{\alpha_n(\lambda_q)},$$

where for $0 \leq n \leq q-1$, α_n's are the polynomials given by the reduction formulae

$$\begin{aligned} &\alpha_{-1}(\lambda_q) = \alpha_o(\lambda_q) = 0 \\ &\alpha_1(\lambda_q) = 1 \\ &\alpha_n(\lambda_q) = \lambda_q \cdot \alpha_{n-1}(\lambda_q) - \alpha_{n-2}(\lambda_q); n \geq 2. \end{aligned}$$

Then we have

Lemma 2: *$S^nQ, 0 \leq n \leq q-1$, has the vertices $\zeta, S^n(\infty)$ and $S^{n+1}(\infty)$.*

Now by Lemma 2 and by the fact that $S^{n+1}(\infty) = S^n(0)$, the images of S^nQ do not overlap.

As we now have an H-packing and a K-packing we can apply Lemma 1. Then the group $H(\lambda_q) = < H, K >$ is isomorphic to the free product of its subgroups H and K, i.e. $H(\lambda_q) \cong C_2 * C_q$. Also

$$\begin{aligned} P \cap Q &= \{z \in \mathcal{U} : |z + 1/\lambda_q| > 1/\lambda_q, \frac{-\lambda_q}{2} < Rez < 0\} \\ &= F_{\lambda_q} \end{aligned}$$

is an $H(\lambda_q)$-packing.

References

[1] İ.N. Cangül: Normal Subgroups of the Hecke groups, PhD Thesis, Southampton (1994)

[2] E. Hecke: Uber die Bestimmung Dirichletscher Reichen durch ihre Funktionalgleichungen, Math. Ann., 112(1936), 664-699.

[3] A.M. Macbeath: Packings, Free Products and Residually Finite Groups, Proc. Camb. Phil. Soc., 59(1963), 555-558.

$H(\lambda_q)$ HECKE GRUPLARININ GRUP YAPISI

Özet

Bu çalışmada Hecke gruplarının grup yapısının mertebesi 2 ve q olan iki devirli grubun serbest çarpımına izomorfik olduğu gerçeğinin yeni bir ispatı verilmiştir. İspatta A.M. Macbeath'ın bir sonucu ile temel bölge kavramı kullanılmıştır. Konuyla bağlantılı olarak hala açık bir problem olan Hecke gruplarının parabolik noktaları ile ilgili bazı sonuçlar da verilmiştir.

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Received 31.5.1995