Tr. J. of Mathematics
20 (1996) , 233 – 236.

(σ,τ)- LIE IDEALS IN PRIME RINGS WITH DERIVATION

M. Soytürk

Abstract

Let R be a prime ring, char $R \neq 2, 3\,\sigma, \tau : R \to R$ two automorphisms, U a nonzero $(\sigma,\tau)-$ Lie ideal of R and $o \neq d : R \to R$ a derivation such that $\sigma d = d\sigma, \tau d = d\tau$. In this paper we have proved the following results. (1) If $d(U) \subset Z$ then $U \subset Z$ (2) If $d(U) \subset U$ and $d^2(U) \subset Z$ then $U \subset Z$.

Introduction

Let R be a ring and U an additive subgroup of R, and $\sigma,\tau : R \to R$ two mappings. We set $[x,y]_{\sigma,\tau} = x\sigma(y) - \tau(y)x$. The definition of (σ,τ)- Lie iedal was given in [4] as follows: (i) U is called a (σ,τ)-right Lie ideal of R if $[U,R]_{\sigma,\tau} \subset U$. (ii) U is called (σ,τ)- left Lie ideal of R if $[R,U]_{\sigma,\tau} \subset U$. (iii) U is called a (σ,τ)-Lie ideal of R if U is both a (σ,τ)-right Lie ideal and (σ,τ)-left Lie ideal of R. Every Lie ideal of R is a (1,1)-right(left)Lie ideal of R, where $1 : R \to R$ is the identity map. Let $R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} |a,b,c,d, \in I \right\}, U = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} |x, y\epsilon I \right\} \subset R, \sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$ and $\tau \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$. Then σ and τ are automorphisms of R, and U is a (σ,τ)-left Lie ideal or R but not Lie ideal of R.

Let d be a nonzero derivation of R. The following results have beeen proved for Lie ideals in [1] and [3]. (i) If $d(U) \subset Z$ then $U \subset Z$.(ii) If $d^2(U) \subset Z$ then $U \subset Z$. In this paper we generalized the above results in prime rings with (σ,τ)-Lie ideals.

Throught R will represent a prime ring σ and τ automorphisms of R, d a nonzero derivation of R such that $d\sigma = \sigma d, d\tau = \tau d$ and Z the center of R. Further, we shall often use the relations:

$$\begin{aligned} [xy,z]_{\sigma,\tau} &= x[y,z]_{\sigma,\tau} + [x,\tau(z)]y = x[y,\sigma(z)] + [x,z]_{\sigma,\tau}y \quad \text{and} \\ [x,yz]_{\sigma,\tau} &= \tau(y)[x,z]_{\sigma,\tau} + [x,y]_{\sigma,\tau}\sigma(z). \end{aligned}$$

Results

Lemma *Let R be a prime ring, U a nonzera (σ,τ)-left Lie ideal. If $[R,U]_{\sigma,\tau} \subset Z$ then $U \subset Z$.*

Proof. For all $x \in R, u \in U, [x,u]_{\sigma,\tau} \in [R,U]_{\sigma,\tau}$. We replace x by $x\sigma(\mathrm{u})$, then we get $[x,u]_{\sigma,\tau}\sigma(u) \in [R,U]_{\sigma,\tau}$ and so $[x,u]_{\sigma,\tau}\sigma(u) \in Z$ for all $x \in R, u \in U$. Thus, since $[x,u]_{\sigma,\tau} \in Z$ we get $[x,u]_{\sigma,\tau} = 0$ for all $x \in R$, or $\sigma(u) \in Z$. If $\sigma(u) \in Z$, then $u \in Z$. If $[x,u]_{\sigma,\tau} = 0$ for all $x \in R$, then replacing x by $xy, y \in R$ we get $o = R[R,\sigma(u)]$ and so by the primeness of R ve obtain $u \in Z$. Thus $U \subset Z$ is obtained. □

Lemma *Let R be a prime ring with char $R \neq 2, U$ a nonzero (σ,τ)-Lie ideal of R, d a nonzero derivation of R. If $d(U) \subset Z$ then $U \subset Z$.*

Proof. Since $d(U) \subset Z$, for all $x \in R, u \in U$ we have,

$$Z \ni d([x,u]_{\sigma,\tau}) = [d(x),u]_{\sigma,\tau} + [x,d(u)]_{\sigma,\tau} \tag{1}$$

In (1), if we replace x by $xd(v), v \in U$ and using $d(U) \subset Z$ and (1) we get $[x,u]_{\sigma,\tau}d^2(v) \in Z$ for all $x \in R, u,v \in U$. Thus, since $d^2(v) \in Z$, for all $x \in R, u,v \in U$, we get $[x,u]_{\sigma,\tau} \in Z$ or $d^2(v) = 0$. That is $d^2(U) = 0$ or $[R,U]_{\sigma,\tau} \subset Z$. If $d^2(U) = 0$ then by [2, Theorem 2] we get $U \subset Z$. If $[R,U]_{\sigma,\tau} \subset Z$, then by lemma 1 we have $U \subset Z$. □

Lemma *Let R be a prime ring with char $R \neq 2,3$ U a nonzero $(\sigma,\tau)-$ Lie ideal of R, d a nonzero dervation of R such that $d(U) \subset U, d^2(U) \subset Z$. If $d^3(U) = 0$ then $U \subset Z$.*

Proof. Assume that $U \not\subset Z$. Since $[x,u]_{\sigma,\tau} \in U$, replacing x by $\tau(u)x$ we have $\tau(u)[x,u]_{\sigma,\tau} \in U$. For all $x \in R$ and $u \in U$, $0 = d^3(\tau(u)[x,u]_{\sigma,\tau}) = 3(\tau(d^2(u))d([x,u]_{\sigma,\tau}) + \tau(d(u))d^2([x,u]_{\sigma,\tau}))$ and so,

$$\tau(d^2(u))d([x,u]_{\sigma,\tau}) + \tau(d(u))d^2([x,u]_{\sigma,\tau}) = 0 \tag{2}$$

In (2), we replace u by $d(u)$, then for all $x \in R, u \in U$ we get $0 = \tau(d^2(u))d^2([x,d(u)]_{\sigma,\tau})$. Thus, since $d^2(u) \in Z$ we have that

$$d^2(u) = 0 \,\mathrm{or}\, d^2([x,d(u)]_{\sigma,\tau}) = 0 \,\mathrm{for\ all}\, x \in R. \tag{3}$$

If $d^2([x,d(u)]_{\sigma,\tau}) = 0$ for all $x \in R$; replacing x by $x\sigma(d(u))$ in the last equation and using char $R \neq 2$ we obtain

$$d([x,d(u)]_{\sigma,\tau}) = 0, \text{ for all}\, x \in R. \text{ or } d^2(u) = 0.$$

If $d([x,d(u)]_{\sigma,\tau})=0$ for all $x\in R$; then from the relation $0=d([x\sigma(d(u)),d(u)]_{\sigma,\tau})$ we have, $[x,d(u)]_{\sigma,\tau}=0$ for all $x\in R$ or $d^2(u)=0$. Let $[x,d(u)]_{\sigma,\tau}=0$, for all $x\in R$. If we replace x by $xy, y\in R$ then we see that $d(u)\in Z$. Thus in the second case (3), we have $d(u)\in Z$. Consequently by (3) we obtain that $d^2(u)=0$ or $d(u)\in Z$. Let $K=\{u\in U|d^2(u)=0\}$ and $L=\{u\in U|d(u)\in Z\}$.K and L are additive subgroups of U and $U=K\cup L$. Since $d\neq 0$ and $U\not\subseteq Z$, by [2, Theorem 2], $U\neq K$. Thus, by Brauer Trick we have $U=L$. But, if $U=L$ then $d(U)\subset Z$ and so $U\subset Z$ by Lemma 2. This is a contradiction. Therefore we have $U\subset Z$. □

Theorem *Let R be a prime ring with char $R\neq 2,3, U$ a nonzero $(\sigma,\tau)-$ Lie ideal of R, d a nonzero derivation of R such that $d(U)\subset U$ and $d^2(U)\subset Z$ then $U\subset Z$.*

Proof.

Let $d(U)\subset U$ and $d^2(U)\subset Z$. Then, for all $x\in R, u\in U$

$$Z\ni d^2([x,u]_{\sigma,\tau})=[d^2(x),u]_{\sigma,\tau}+2[d(x),d(u)]_{\sigma,\tau}+[x,d^2(u)]_{\sigma,\tau} \quad (4)$$

In (4), replacing x by $xd^2(v), v\in U$ and using $d^2(U)\subset Z$ we get

$$2d^3(v)d([x,u]_{\sigma,\tau})+[x,u]_{\sigma,\tau}d^4(v)\in Z, \text{ for all } x\in R, u,v\in U \quad (5)$$

In the last equation, if we take $xd^2(w), w\in U$ instead of x and use that char $R\neq 2$, then we have $d^3(v)d^3(w)[x,u]_{\sigma,\tau}\in Z$, for all $x\in R, u,v,w\in U$. Since $d^2(U)\subset Z$, thus we have $d^3(U)=0$ or $[R,U]_{\sigma,\tau}\subset Z$. If $d^3(U)=0$, then $U\subset Z$ by Lemma 3. If $[R,U]_{\sigma,\tau}\subset Z$, then $U\subset Z$, by Lemma 1. Cosequently we obtain that $U\subset Z$. □

References

[1] Awtar, R.: Lie Structure in Prime Rings with Derivation. Publ. Math. Debrecen 31, 209-215 (1984) 267 (1981).

[2] Aydın, N., Soytürk, M. : (σ,τ)-Lie Ideals in Prime Rings with Derivation. To appear in DOĞA Turkish Journal of Mathematics.

[3] Bergen, J., Herstein, I. N., and Kerr, J. W. : Lie Idelas and Derivations of Prime Rings. J. Algebra 71, 259-267 (1981).

[4] Kaya, K.: (σ,τ)-Right Lie Ideals in Prime Rings. Proc. 4. National Math. Symposium Antakya (1991).

TÜREVLİ ASAL HALKALARDA (σ,τ)- LE İDEALLER

Özet

Bu makalede aşağıdaki sonuçlar ispatlanmıştır. R, char $R \neq 2,3$ olan bir asal halkla, U, R nin sıfırdan farklı bir ideali, σ ve τ R nin iki otomorfizmi ve $0 \neq d : R \to R, d\sigma = \sigma d, \tau d = d\tau$ olacak şekilde R nin bir türevi olsun 1) Z, R nin merkezi olmak üzere $d(U) \subset Z$ ise $U \subset Z$. 2) $d(U) \subset U$ ve $d^2(U) \subset Z$ ise $U \subset Z$ dir.

Muharrem SOYTÜRK,
Cumhuriyet Üniversitesi,
Fen-Edebiyat Fakültesi,
Matematik Bölümü,
58140, Sivas-TURKEY

Received 28.11.1994