On Homotopic, non–isomorphic tight contact structures on 3–manifolds

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1. Introduction

Let M be a smooth 3-dimensional manifold. A contact structure on M is a distribution ξ of tangent 2-planes locally defined by a 1-form α ($\xi = \{\alpha = 0\}$) such that $\alpha \wedge d\alpha$ is nowhere vanishing.

On a generically embedded surface $S \subset M$, ξ induces a line field which integrates to a foliation S_ξ with isolated singularities at points where S is tangent to ξ, called the *characteristic foliation* [Ae]. A contact structure is called *overtwisted* if there is an embedded disc $D \subset M$ such that its characteristic foliation contains a closed orbit with exactly one singular point inside it. Otherwise the contact structure is called *tight* ([El3],[Ae]).

By a classical result of Gray [Gr] two contact structures ξ_0 and ξ_1 that are homotopic through a family of contact structures are actually isotopic. Eliashberg proved that two overtwisted contact structures which are homotopic as 2–plane fields are homotopic through contact structures. Thus, the classification of overtwisted contact structures up to isotopy coincides with their homotopy classification as 2–plane fields. On the other hand, Eliashberg's work implies that there are only finitely many Euler classes of tight contact structures on a given 3–manifold M. In fact, given a tight oriented contact structure ξ on M and a smoothly embedded 2–dimensional surface $S \subset M$, if $e(\xi)$ denotes the Euler class of ξ as a 2–plane bundle over M, Eliashberg's result says that either (i) the genus of S is equal to zero and $e(\xi) \cdot [S] = 0$, or (ii) $|e(\xi) \cdot [S]| \leq -\chi(S)$.

It was conjectured (see e.g. 10.3 in [El2]) that on a given 3–manifold all tight contact structures having the same Euler class are isotopic. This conjecture was disproved by

Giroux [Gi1, Gi2] who showed that T^3 has at least countably many homotopic non-isomorphic tight contact structures.

In this paper we describe new possible counterexamples to the conjecture for an infinite family of homology 3–spheres, and we use Seiberg–Witten theory to prove a result (theorem 4.2) which seems to support our belief that they are honest counterexamples. The construction goes roughly as follows. Our homology 3–spheres are oriented boundaries of different Stein manifolds. Since Stein manifolds have naturally induced tight contact structures on their boundaries this produces the contact structures. The particular choice of the Stein manifolds allows us to prove that the resulting contact structures are homotopic as 2–plane fields. In the final argument we show that either there exist certain exotic symplectic structures on collars around the boundary of our Stein manifolds, or an isomorphism between the contact structures cannot exist. To do this, we prove that if the first case does not occur and such a contactomorphism exists, then a certain cut–and–paste operation is possible within the symplectic category. The cut–and–paste operation would produce closed manifolds with symplectic structures which are not allowed by recent results in Seiberg–Witten theory, in particular the remarkable new results of Taubes concerning symplectic 4–manifolds [Ta2]. We finish the paper by motivating and then stating the conjecture that our contact structures are all non–isomorphic.

2. Notation and background material

The standard contact structure ξ_0 on S^3 is defined as the field of complex 2–planes tangent to the unit sphere in $\mathbb{C}^2$. When restricted to $\mathbb{R}^3 \cong S^3 \setminus p \subset S^3$ it is, in properly chosen coordinates, given by the kernel of the 1–form $\alpha = xdy + dz$ (cf. [Be, Er]). The planes in this field are all parallel to the x-axes and the slope of their projection in the yz–plane is $-x$.

A knot K in a contact 3-manifold M is *Legendrian* if it is tangent to the contact plane at every point. The projection of a generic Legendrian knot $K \subset \mathbb{R}^3$ to the yz–plane is a *front* C_K, i.e. a piecewise–smooth immersed curve with finitely many singularities which are either ordinary double points or horizontal cusps, corresponding to the points on K where the tangent vector is parallel to the x–axis. Thus, the cusps are exactly the local extrema of $y|C_K$. The knot K can be uniquely reconstructed from C_K: since $\dfrac{dz}{dy} = -x$ the slope of the tangent lines to C_K give the x–coordinate of the corresponding point on K. In particular, the diagram obtained from a front C by letting the over–arc at any double point be the one with the most negative slope is a knot diagram for the Legendrian knot K_C reconstructed from C.

Given a generic Legendrian knot $K \subset \mathbb{R}^3 \subset S^3$, its Thurston–Bennequin invariant $\mathrm{tb}(K)$ (see e.g. [El2]) can be defined using such a projection in the following way. Let w denote the writhe of K, namely the algebraic number of self–crossings. Let c denote the number of cusps. Then, the number $\mathrm{tb}(K) = w - \frac{1}{2}c$ can be shown to depend only on the Legendrian isotopy type of K. Similarly, for an oriented generic Legendrian knot K, the *rotation number* $\mathrm{r}(K)$ can be defined by $\mathrm{r}(K) = \frac{1}{2}(d - a)$, where a is the number

of ascending cusps and d the number of descending cusps in C_K. $r(K)$ changes sign if the orientation of K is reversed, but it is an invariant of the oriented Legendrian isotopy class of K.

In figure 1 the diagram of a Legendrian right–handed trefoil is shown, and tb and r are computed.
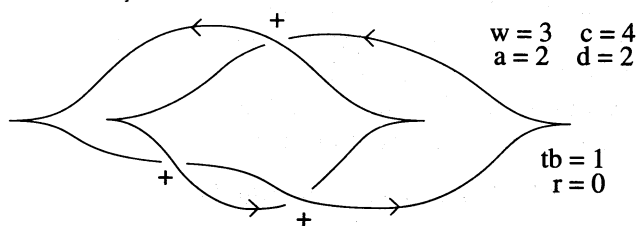

$$w = 3 \quad c = 4$$
$$a = 2 \quad d = 2$$

$$tb = 1$$
$$r = 0$$

FIGURE 1

Recall that a symplectic form on a smooth 4-manifold X is a closed, non-degenerate 2-form ω. If X has also an almost complex structure J, ω is called J-*positive* if $\omega(v, Jv) > 0$ for every tangent vector v. An almost complex structure J is called ω-*compatible* if ω is J-positive and $\omega(Jv, Jv') = \omega(v, v')$. The space of almost complex structures compatible with a given symplectic form is well known to be contractible [Ae], so the first Chern class of the almost complex structure is uniquely determined by the symplectic structure, and sometimes called the first Chern class of ω.

A vector field Θ on a symplectic manifold is called *contracting* if the Lie derivative $\mathcal{L}_\Theta \omega = \nu \omega$ for a negative (locally constant) function ν. For a domain W with smooth boundary M contained in (X, ω) we say that W is $\omega - convex$ if there is a contracting vector field which is transverse to M and pointing into W. An elementary argument shows that there is a collar U inside W around M and an orientation–preserving diffeomorphism $\varphi : U \to M \times \mathbb{R}$ such that $\varphi(M) = M \times \{0\}$, $\varphi_*(\Theta) = -\frac{d}{dt}$, $\varphi^* (d(e^t \alpha)) = \omega|U$, and $\varphi^*(\alpha)|_M = i_\Theta \omega|_M$, where $t \in \mathbb{R}$, $\alpha \in \Omega^1(M)$.

Let X be a smooth 4-manifold with an almost complex structure $J : TX \to TX$. Any smooth hypersurface $M \subset X$ has a canonically induced distribution of J–invariant tangent 2-planes $\xi = TM \cap J(TM)$. Suppose that M is defined as the zero set of a smooth function $f : X \to \mathbb{R}$, with $df|_M \neq 0$. Then ξ is the kernel of the 1-form $\alpha = -J^* df$. M is called J-*convex* if the quadratic form $-d\alpha(v, Jv)$ restricted to ξ is everywhere positive definite. Note that X doesn't necessarily have to be closed. For example, it makes sense to talk about almost complex manifolds with J–convex boundary. When J is integrable and M is J-convex the induced contact structure ξ is called *holomorphically fillable*. By a theorem of Gromov [Gro] a holomorphically fillable contact structure is tight.

39

3. Legendrian knots and Stein manifolds

A *Stein manifold* is a complex manifold which can be embedded as a proper complex submanifold of $\mathbb{C}^n$. Any Stein manifold X admits a (strictly) plurisubharmonic function ϕ, namely a function which is strictly subharmonic on any holomorphic disk in X. If we denote by J multiplication by $\sqrt{-1}$ in TX and by J^* its dual, the 2-form $\omega_\phi = dJ^* d\phi$ is nondegenerate and closed, hence it defines a symplectic structure. X is therefore Kähler with a Kähler metric defined by $g_\phi(v, v') = \omega_\phi(v, Jv')$. The gradient vector field of ϕ is contracting for the symplectic form ω_ϕ, and if we look at the domain $W = \{\phi \leq c\}$ its boundary is J–convex and ω_ϕ-convex (see [EG]). The contact structure $\xi = TM \cap J(TM)$ is therefore holomorphically fillable, hence tight.

To construct our examples we shall need the following result, which was pointed out to us by Bob Gompf, and is implicitly contained in [El1].

Theorem 3.1 ([El1]). *Let W be a smooth 4–manifold with boundary having a handlebody decomposition $B^4 \bigcup_i H_i^2$ with only 2–handles. Suppose that there exists a Legendrian link $\mathcal{L} = \bigcup_i K_i$ in the framed isotopy class of the union of the attaching circles of the 2–handles such that $\mathrm{fr}(K_i) = \mathrm{tb}(K_i) - 1$ for all i. Then, on W there is an almost complex structure which restricts to a Stein structure on the interior of W and such that ∂W is J–convex. Moreover, let h_i denote the 2–homology class supported by H_i^2. Then, $\langle c_1(W), h_i \rangle = \mathrm{r}(K_i)$.*

Notice that the contact structure induced on ∂W by the complex structure on W is holomorphically fillable, and therefore by Gromov's theorem it is tight.

Let W_n^0 and W_n^1 be the smooth 4–manifolds obtained by attaching 2–handles to the 4–ball according to the framed links of figures 2(a) and 2(b) respectively (see e.g. [Ki]). A simple sequence of Kirby moves [Go] shows that $\partial W_n^0 \cong \partial W_n^1$. On the other hand, the intersection form of W_n^0 is even if and only if n is even, while the intersection form of W_n^1 is always even. Moreover, since the determinants of their intersection matrices are ± 1, their common boundary is an integral homology 3–sphere. Thus, W_n^0 and W_n^1 are homeomorphic if and only if n is even [Fr].

The two links of figure 2 can be realized by the Legendrian links illustrated in figure 3. Let T, S (T', S' respectively) the generators of $H_2(W_n^0; \mathbb{Z})$ ($H_2(W_n^1; \mathbb{Z})$) given by Seifert surfaces of the knots union the cores of the 2-handles, so that T (T') corresponds to the 0–framed knot. The links of figure 3 satisfy the hypothesis of theorem 3.1 for $n \geq 2$, hence W_n^0 and W_n^1 are Stein manifolds with tight contact structures on their boundaries. Let us call these contact structures ξ_n^0 and ξ_n^1. Using theorem 3.1 we can easily compute $c_1(W_n^0) = (2 - n) \mathrm{PD}(T)$ and $c_1(W_n^1) = 0$.

The statement of the following proposition was pointed out to us by Bob Gompf. A simple proof can be given using some obstruction theory. Let $\chi(X)$ and $\sigma(X)$ denote, respectively, the Euler characteristic and the signature of a 4–manifold X.

Proposition 3.2. *Let M be an oriented integral homology 3–sphere, X_i, $i = 1, 2$ two almost complex 4–manifolds with boundaries orientation–preservingly isomorphic to M. Let ξ_i, $i = 1, 2$ be the 2–plane fields induced on M by the almost complex structures on*
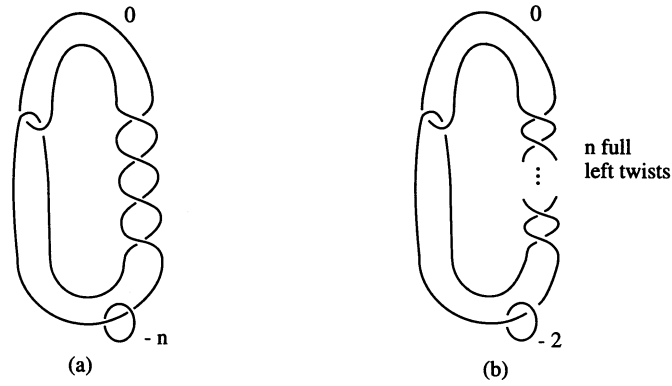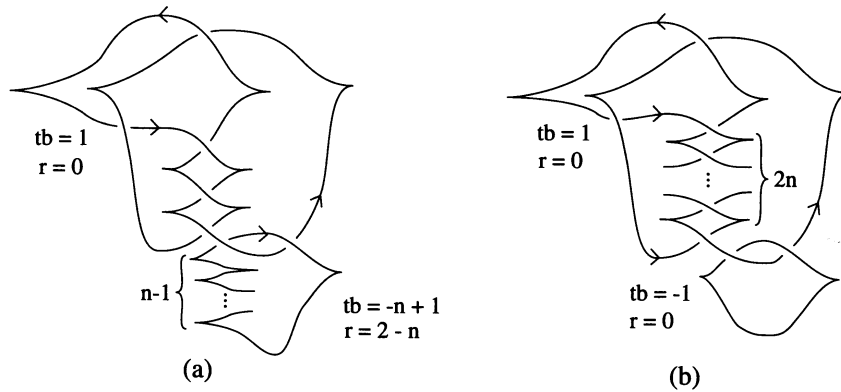
FIGURE 2

FIGURE 3

X_i. ξ_1 and ξ_2 are homotopic as 2-plane fields if and only if

$$c_1(X_1)^2 - 2\chi(X_1) - 3\sigma(X_1) = c_1(X_2)^2 - 2\chi(X_2) - 3\sigma(X_2).$$

Corollary 3.3. For every $n \geq 2$ ξ_n^0 and ξ_n^1 are homotopic as 2-plane fields.

4. Main theorem

Recall that for a smooth 4-manifold X the Seiberg-Witten monopole equations ([SW1, SW2, W]) give rise to invariants of the differentiable structure of X. When X is Kähler, the set of *Seiberg-Witten basic classes* SWB $\subset H^2(X, \mathbb{Z})$ is invariant under change of sign and contains the canonical class. Taubes proved [Ta1] that, more generally, the canonical class of a symplectic 4-manifold X belongs to the set of its Seiberg-Witten

41

basic classes. Moreover, Seiberg-Witten basic classes satisfy the *generalized adjunction inequality* [KM, MST]: for $c \in$ SWB and a surface S embedded in X such that $S \cdot S \geq 0$ the genus of S is bounded from below, as $2g(S) - 2 \geq c \cdot S + S \cdot S$.

The following lemma is needed for the proof of the main theorem.

Lemma 4.1. *Let X be a smooth, closed 4–manifold containing a smoothly embedded copy of the Milnor fiber M_n. Then, any Seiberg–Witten basic class K restricts trivially to $H^2(M_n; \mathbb{Z}) \subset H^2(X; \mathbb{Z})$.*

Proof. M_n can be described by the framed link of figure 4(a) or, after cancelling two $1 - -2$–handle pairs, by the framed link of figure 4(b) (see e.g. [Go]). It is clear from the cyclic and symmetric nature of the first link that there is a diffeomorphism taking any of the (-2)–spheres in one half of the picture in figure 4(b) to any other (-2)–sphere in the same half.
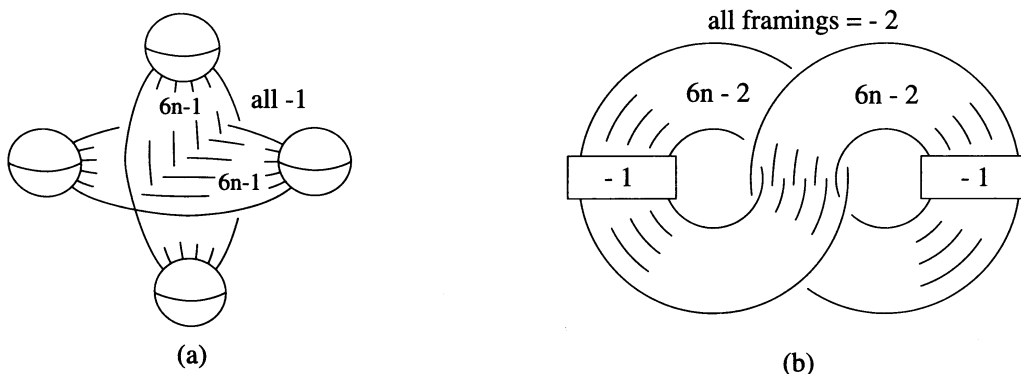

(a)

(b)

FIGURE 4

Inside the link of figure 4(b) we can clearly find the configuration in figure 5(a).

We can realize the class $T = a + b + c + A + B + C$ by band connect–summing as in figure 5(b). It is not hard to see that the pair T, S of figure 5(b) isotopes to the link of W_1^1. Hence we have a 0–torus T intersecting transversally the (-2)–sphere S in one point.

The generalized adjunction formula for Seiberg–Witten basic classes implies that $K \cdot T = 0$ and $K \cdot (S + T) = 0$ since both classes are represented by 0–tori. Hence, $K \cdot S = 0$. Arguing by symmetry as suggested above, one can easily conclude from this that K restricts trivially to the Milnor fiber. $\square$

Let E_n be a smooth Kähler simply connected elliptic surface with geometric genus $n - 1$. By a result of Gompf [Go] there is a smooth decomposition $E_n = N_n \bigcup M_n$, where N_n is the nucleus introduced in section 3, realized here as a regular neighborhood of the
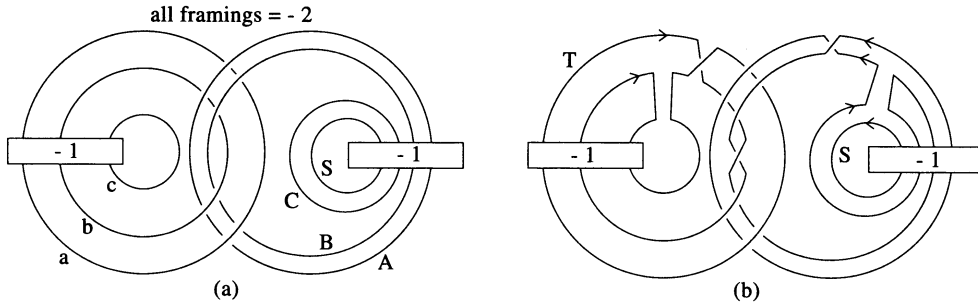
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FIGURE 5

union of a cusp fiber with a section of the elliptic fibration, while the complement M_n is diffeomorphic to the Milnor fiber of the Brieskorn singularity $(2, 3, 6n - 1)$.

It is an easy exercise in Kirby calculus [Ki] to show that the 4–manifold W_n^0 defined in section 3 embeds smoothly inside $N_n \# \overline{\mathbb{CP}}^2$, and therefore that there is a smooth embedding $i : W_n^0 \hookrightarrow \widehat{E_n} = E_n \# \overline{\mathbb{CP}}^2$ (cf. lemma 2.2 in [Li]). This embedding sends the class T (see section 3) to the pull–back in $\widehat{E_n}$ of the class F of a generic fiber in E_n, and S to the pull–back of the class of a section. It follows that there is an orthogonal decomposition $H^2(\widehat{E_n}) = H^2(i(W_n^0)) \oplus \langle \alpha \rangle \oplus H^2(M_n)$, where α is the exceptional class. Denote by ω^i, $i = 1, 2$, the Kähler forms induced on W_n^i by their Stein structures. Let ω denote a Kähler form on $\widehat{E_n}$.

Theorem 4.2. *For every odd $n \geq 2$, one of the following holds:*

1. *There is no smooth embedding $j : W_n^0 \hookrightarrow \widehat{E_n}$, j homotopic to i, such that it is possible to find two neighborhoods $V^0, V^1 \subset W_n^0$ of ∂W_n^0, and a diffeomorphism $\varphi : V^0 \to V^1$ such that $\varphi^*(j^*(\omega)) = \omega^0$.*

2. *ξ_n^0 and ξ_n^1 are non–isomorphic contact structures.*

Proof. Suppose the first case does not hold. Then, there exist a smooth embedding j, two neighborhoods V^0 and V^1, and a diffeomorphism $\varphi : V^0 \to V^1$ as in the statement. Up to cutting and regluing symplectically $j(W_n^0)$ back into $\widehat{E_n}$, we may assume that $(j^*)^{-1}(\omega^0)$ extends to a symplectic form $\tilde{\omega}$ on the closed manifold $\widehat{E_n}$ so that ∂W_n^0 is $\tilde{\omega}$–convex. Next, we shall argue that if there is a diffeomorphism $\phi : \partial W_n^0 \to \partial W_n^1$ which sends ξ_n^0 to ξ_n^1, then one can cut $j(W_n^0)$ out of $\widehat{E_n}$ and glue back in W_n^1 so that the resulting manifold is symplectic.

Let us observe the following. By the ω^i–convexity of ∂W_n^i inside W_n^i there are two collars $U^i \subset W_n^i$, $i = 0, 1$ around ∂W_n^i and diffeomorphisms $\varphi_i : U_i \to \partial W_n^i \times \mathbb{R}$ such that $\varphi_i^*(d(e^t \alpha_i))$ equals the restriction of the symplectic form ω^i on W_n^i, and $\alpha_i \in \Omega^1(\partial W_n^i)$ defines the J_i–invariant 2–plane distribution ξ_n^i. ξ_n^0 and ξ_n^1 are isomorphic if and only if there is an orientation–preserving diffeomorphism $f : \partial W_n^0 \to \partial W_n^1$ such that $f^*(\alpha_1) =$

$\lambda\alpha_0$, where $\lambda : \partial W_n^0 \to \mathbb{R}$ is a non-vanishing function. Define the diffeomorphism $F : \partial W_n^0 \times \mathbb{R} \to \partial W_n^1 \times \mathbb{R}$ by $F(p,t) = (f(p), t - \ln \lambda(p))$. Then,

$$F^*(d(e^t \alpha_1)) = d\left(F^*(e^t \alpha_1)\right) = d\left(e^{t-\ln\lambda}\lambda\alpha_0\right) = \pm d\left(e^t\alpha_0\right).$$

Up to changing ω^1 into $-\omega^1$ (and α_1 into $-\alpha_1$), we conclude that ϕ extends to a symplectomorphism between two collars around ∂W_n^0 and ∂W_n^1.

Hence, since $j^*(\tilde{\omega}) = \omega^0$, if there exists a contactomorphism ϕ then there is a symplectic structure on $X = W_n^1 \bigcup_{j \circ \varphi_0^{-1} \circ F^{-1} \circ \varphi_1} \left(\widehat{E_n} \setminus j(W_n^0)\right)$, and we may choose a compatible almost complex structure. Since $X = W_n^1 \bigcup (\widehat{N_n} \setminus j(W_n^0)) \bigcup M_n$ and j is homotopic to i, the cohomology of X decomposes as $H^2(X) = H^2(W_n^1) \oplus \langle \alpha \rangle \oplus H^2(M_n)$, with $\alpha^2 = -1$. It is evident from figure 2(b) that the generators T' and S' of $H_2(W_n^1)$ are represented by an embedded 0–torus and a (-2)–sphere respectively, intersecting geometrically in one point. Thus, $T + S$ can also be represented by an embedded 0–torus. Since by the generalized adjunction formula any Seiberg–Witten basic class K is orthogonal to both T and $T + S$, K restricts trivially to $H^2(W_n^1)$. Hence, by lemma 4.1 we can conclude that any Seiberg–Witten basic class for X is a multiple of α. Observe that (see e.g. [Go]) $\chi(X) = \chi(\widehat{E_n}) = 12n + 1$, and $\sigma(X) = \sigma(\widehat{E_n}) = -8n - 1$. Thus any Seiberg–Witten class K satisfies $K^2 = 2\chi(X) + 3\sigma(X) = -1$, which implies $K = \pm\alpha$.

By Taubes' recent results [Ta2] $c_1(X)^2 = -1 < 0$ implies $X = X' \sharp \overline{\mathbb{CP}}^2$, where X' is again symplectic. The blow–up formula for Seiberg–Witten invariants (see e.g. [FS]) tells us that $\pm c_1(X') \pm E$ are basic classes for X. Since $\pm\alpha$ are the only two possible basic classes for X, this implies $c_1(X') = 0$. Thus $w_1(X') = w_2(X') = 0$. But by Rohlin's theorem this implies $\sigma(X') \equiv 0 \bmod 16$, which is impossible if n is odd, since $\sigma(X') = -8n$. $\qquad\square$

Remark 4.3. *The proof of the theorem shows that if the embedding $i : W_n^0 \hookrightarrow \widehat{E_n}$ could be homotoped to an embedding j such that $j(\partial W_n^0)$ is ω–convex with respect to the Kähler form ω on $\widehat{E_n}$, the second case in the statement would have to hold. The authors were able to show that i can be isotoped to a holomorphic embedding j such that $j(\partial W_n^0)$ is J–convex, but are unable, at present, to achieve ω–convexity.*

Finally, here is what we think is reasonable to expect.

Conjecture 1. *For every $n \geq 2$ ξ_n^0 and ξ_n^1 are homotopic, non–isomorphic tight contact structures.*

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