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The minimal genus of an embedded surface of non-negative square in a rational surface

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1. Introduction

The long-standing conjecture of Thom on the minimal genus of an embedded surface in $\mathbf{CP}^2$ carrying a given homology class was resolved in the fall of 1994 by Kronheimer-Mrowka [KM94] and Morgan–Szabo–Taubes (to appear). Upon hearing the argument used in [KM94], I saw how to extend that proof to all rational surfaces, provided that the self-intersection of the homology class in question is non-negative. This note contains that extension. I subsequently learned that the paper of Morgan–Szabo–Taubes will include a more general result applying to any Kähler surface with $b_+^2 = 1$. Their method is less computational than that presented here.

For the purposes of the paper, a rational surface will be a 4-manifold diffeomorphic to $S^2 \times S^2$ or to $\mathbf{CP}^2 \# n\overline{\mathbf{CP}}^2$ and will be denoted by X. We will make no notational distinction between an embedded surface in X and the homology class which it carries. Choose a basis $\{S_0, S_1, \ldots, S_n\}$ for the homology of X_n. Here S_0 is the complex line (with its usual orientation) in $\mathbf{CP}^2$ and the other S_i are the exceptional curves in the $\overline{\mathbf{CP}}^2$'s, oriented so that $-S_i$ is a complex curve. Denote by H the Poincaré dual of S_0, and by E_i the Poincaré dual of $-S_i$, so that $E_i \cdot S_i = 1$. These classes form a basis of $H_2(X)$, so we may write (in homology) $\Sigma = \sum a_i S_i$, and will denote by $|\Sigma|$ the class $\sum |a_i| S_i$.

Theorem 1.1. *Let X be a rational surface with canonical class K_X, and let Σ be an embedded surface with self-intersection $\Sigma \cdot \Sigma > 0$. Then the genus $g(\Sigma)$ satisfies*

$$2g - 2 \geq K_X \cdot |\Sigma| + \Sigma \cdot \Sigma \tag{1}$$

If $\Sigma \cdot \Sigma = 0$, then inequality (1) holds if, in addition $K_X \cdot |\Sigma| \geq 0$.

Corollary 1.2. *A complex curve in X minimizes the genus in its homology class.*

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Proof of Corollary: A complex curve in X_n, other than one of the exceptional curves $-S_i$, must have positive intersection with S_0 and all of the $-S_i$. So the coefficients all are non-negative, and $|\Sigma| = \Sigma$. For classes of positive square, the corollary thus follows directly from the adjunction formula, which states that equality holds in (1) for Σ a complex curve. For Σ of square 0, the adjunction formula again implies that (apart from a few cases where Σ is a rational curve), the inequality $K_X \cdot |\Sigma| \geq 0$ holds, so that again Σ is genus-minimizing. □

Because the manifolds $S^2 \times S^2$ or $\mathbf{CP}^2 \# \overline{\mathbf{CP}}^2$ admit orientation reversing diffeomorphisms, the theorem applies as well to classes of negative self intersection. As a corollary, we thus get the minimal genus for any homology class in those manifolds:

Corollary 1.3. *The minimal genus of a surface in $S^2 \times S^2$ carrying the homology class (a, b) in the obvious basis, with $ab \neq 0$, is $(|a| - 1)(|b| - 1)$. The minimal genus of a surface in $\mathbf{CP}^2 \# \overline{\mathbf{CP}}^2$ homologous to $a_0 S_0 + a_1 S_1$, assuming $|a_0| > |a_1|$, is given by*

$$\frac{(|a_0| - 1)(|a_0| - 2)}{2} - \frac{|a_1|(|a_1| + 1)}{2}$$

If $|a_0| < |a_1|$, then the genus is given by the same formula with the roles of the a_i reversed.

The statements about $S^2 \times S^2$ are proved via the diffeomorphism of $S^2 \times S^2 \# \overline{\mathbf{CP}}^2$ with $\mathbf{CP}^2 \# 2\overline{\mathbf{CP}}^2$. In $S^2 \times S^2$, the classes $(a, 0)$ and $(0, b)$ are represented by embedded spheres, so the corollary determines the minimal genus in any homology class. A similar remark applies to the missing case in the corollary, i.e. the classes $n(S_0 \pm S_1)$ are all represented by embedded spheres. We do not know in general what the minimal genus is for classes of square 0 in X_n.

2. Basic results

Our proof is a straightforward extension of the method of Kronheimer-Mrowka, which is in turn based on the new 4-manifold invariants derived from the Seiberg-Witten equation [Wit94]. We extract the basics of those invariants from the paper [KM94], to which we refer the reader for additional details.

Suppose now that X is a 4-manifold diffeomorphic to $\mathbf{CP}^2 \# n\overline{\mathbf{CP}}^2$, and that X has been equipped with a Riemannian metric g. Let L be a complex line bundle on X which is the dual of K_X, so that $c_1(L) = 3H - E = 3H - \sum E_i$ in the notation above. Let ω_g be a self-dual harmonic 2-form, normalized so that the cohomology class $[\omega_g]$ which it carries lies in the same component of the positive cone in $H^2(X, \mathbf{R})$ as the hyperplane class H. Let $W \subset H^2(X)$ be the 'wall' defined by the condition $x \cup c_1(L) = 0$. If the metric g satisfies the genericity condition that $[\omega_g] \notin W$, then Kronheimer-Mrowka define $n(g)$ as the number of points (counted modulo 2) in the 0-dimensional solution space to the perturbed Seiberg-Witten equations. If g_t is a path of metrics with g_0, g_1 generic, so that the corresponding path $[\omega_{g_t}]$ is transverse to W, then $n(g)$ changes by the intersection number of W and $[\omega_{g_t}]$. Finally, for X a rational surface, they calculate that $n(g) = 1$ when $c_1(L) \cup [\omega_g]$ is negative.

3. Proof of the theorem

Suppose first that Σ is a surface in X_n with $\Sigma \cdot \Sigma \geq 0$, for which the inequality 1 fails to hold. In brief, Σ is a counterexample to theorem 1.1. Express $[\Sigma]$ as $\sum_{i=0}^n a_i S_i$, and notice that if any of the a_i is negative, then there is another counterexample Σ' with $a_i' = -a_i$. For, there is an orientation preserving diffeomorphism $\varphi : X \to X$ with $\varphi_*(S_i) = -S_i$, and $\varphi_*(S_j) = S_j$ for $i \neq j$. Let $\Sigma' = \varphi(\Sigma)$; then it is readily seen to be a counterexample as well. So we may as well assume that all the $a_i \geq 0$. We also make the remark, following [KM, Lemma 7.7] that if Σ is a counterexample to theorem 1.1, then the homology class $r\Sigma$ (for any positive r) also contains a counterexample.

With these preliminary observations in hand, suppose that Σ is a counterexample, with $\Sigma \cdot \Sigma = m \geq 0$, and form the class $\widetilde{\Sigma} = \Sigma + \sum_{i=n+1}^{n+m} S_i$ in $X_n \# m\overline{\mathbf{CP}}^2$. Evidently $\widetilde{\Sigma}$ has the same genus as Σ, and self-intersection 0. Choose a sequence of metrics g_R with increasingly long cylinders $Y \times [-R, R]$, where $Y \cong S^1 \times \widetilde{\Sigma}$ is the boundary of the tubular neighborhood of $\widetilde{\Sigma}$. Normalize the corresponding harmonic forms ω_R so that the $[\omega_R] \cup H = 1$.

Lemma 3.1. *Suppose that $\Sigma \cdot \Sigma > 0$, or that $\Sigma \cdot \Sigma = 0$ and $K_X \cdot \Sigma > 0$. If Σ is a counterexample, then there is a counterexample Σ' in the class $r[\Sigma]$, $(r \geq 1)$ so that applying the above construction to Σ', then R sufficiently large implies that $c_1(L) \cup [\omega_R]$ is negative. (Here $L = -K_{X_n \# m\overline{\mathbf{CP}}^2}$).*

Proof. In homology, Σ may be written as $\sum_{i=0}^n a_i S_i$; recall that we have assumed that all the $a_i \geq 0$. Because Σ has non-negative square, we must have that $a_0 \geq 1$. Also, $[\omega_R] = H + \sum_{i=1}^{n+m} x_i E_i$, where the coefficient 1 of H is due to our normalization. Since $\omega_R \cup \omega_R > 0$, we must have that $\sum x_i^2 < 1$. Now

$$\begin{aligned} [\omega_R] \cup c_1(L) &= [\omega_R] \cdot \widetilde{\Sigma} + [\omega_R] \cup (3 - a_0)H + [\omega_R] \cup \sum_{i=1}^n (a_i - 1)E_i \\ &= [\omega_R] \cdot \widetilde{\Sigma} + (3 - a_0) - \sum_{i=1}^n x_i(a_i - 1) \end{aligned}$$

The argument in Lemma 10 of [KM94] shows that $[\omega_R] \cdot \widetilde{\Sigma} \to 0$ as $R \to \infty$. So it suffices to show that the conditions $a_0^2 - \sum_{i=1}^n a_i^2 > 0$ and $\sum_{i=1}^n x_i^2 < 1$ imply that

$$(3 - a_0) - \sum_{i=1}^n x_i(a_i - 1)$$

is negative. To approach this, maximize the above expression (as a function of $x_1, \ldots, x_n$) with the constraint $\sum_{i=1}^n x_i^2 = 1$, in the hopes that it will be negative. The maximum value is readily found to be

$$3 - a_0 + \sqrt{\sum_{i=1}^{n} (a_i - 1)^2}$$

Assuming, as we may by taking a multiple of Σ, that $a_0 > 3$, this maximum is negative if

$$(a_0 - 3)^2 > n + \sum_{i=1}^{n} a_i^2 - 2\sum_{i=1}^{n} a_i$$

i.e., if

$$a_0^2 - \sum_{i=1}^{n} a_i^2 - 6a_0 + 2\sum_{i=1}^{n} a_i > n - 9 \qquad (*)$$

Unfortunately it is not always true that $(*)$ holds, even if a_0 is very large. (Take, for example $n = a_0 - 1, a_1 = a_0 - 2$, and the other $a_i = 2$). However, if the a_i are all replaced by ra_i, then the left-hand side of $(*)$ becomes

$$r^2(a_0^2 - \sum_{i=1}^{n} a_i^2) + r(-6a_0 + 2\sum_{i=1}^{n} a_i)$$

So if either $\Sigma \cdot \Sigma = a_0^2 - \sum_{i=1}^n a_i^2 > 0$, or $\Sigma \cdot \Sigma = 0$ and $K_X \cdot \Sigma = -3a_0 + \sum_{i=1}^n a_i > 0$, then$(*)$ will hold for some $r > 1$. So if Σ were a counterexample with positive square, or with 0 square and for which $\sum_{i=1}^n a_i > 3a_0$, let Σ' be a counterexample in the homology class $r[\Sigma]$, where r is chosen so that

$$(3 - ra_0) + \sum x_i(ra_i - 1) < 0$$

for all $\{x_i\}$ satisfying $\sum x_i^2 < 1$. □

Proof of Theorem 1.1: Suppose that $\Sigma \cdot \Sigma > 0$, or that $\Sigma \cdot \Sigma = 0$ and $K_X \cdot |\Sigma| > 0$. Assume, perhaps replacing Σ by some large positive multiple, that Σ is a counterexample in the homology class $a_0 S_0 + \sum a_i S_i$, chosen so that all the $a_i \geq 0$ and that lemma 3.1 applies to Σ. Starting from the fact that the invariant $n(g) = 1$ for any metric g such that $c_1(L) \cup [\omega_g] < 0$, Kronheimer-Mrowka show that

$$c_1(L) \cdot \widetilde{\Sigma} \geq -(2g - 2)$$

But

$$\begin{aligned} c_1(L) \cdot \widetilde{\Sigma} &= c_1(L) \cdot \Sigma + c_1(L) \cdot \sum_{i=n+1}^{n+m} S_i \\ &= -K_X \cdot \Sigma - \Sigma \cdot \Sigma \end{aligned}$$

So

$$2g - 2 \geq K_X \cdot \Sigma + \Sigma \cdot \Sigma$$

This leaves only the possible exception that $\Sigma \cdot \Sigma = 0$, and that $K_X \cdot \Sigma = 0$. In this case, we must also suppose that Σ is non-trivial in homology. The content of inequality 1 in this instance is merely that Σ is not represented by an embedded sphere. But if it were, we proceed by doing surgery on Σ, resulting in a definite manifold X'. The condition that $K_X \cdot \Sigma = 0$ means that Σ is orthogonal to the characteristic class $3S_0 + \sum_{i \geq 1} S_i$. Thus $H_2(X')$ has a characteristic element of square less (in absolute value) than its rank, which readily implies that the intersection form is non-standard, contradicting Donaldson's theorem [Don87, DK90]. □

The condition that there exist a 'short' characteristic element, as in the last paragraph of the proof, is precisely the ingredient necessary to use the monopole equations in a simplified proof of Donaldson's theorem.

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