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A Note on the Geography of Symplectic Manifolds

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1. Introduction

Based on recent developments in gauge theory – the introduction of Seiberg-Witten invariants and results of Taubes – our understanding of the differential topology of symplectic manifolds improved by a margin in the past year. In this note we would like to discuss some existence problems of minimal simply connected symplectic manifolds; in particular we would like to compare the "geography" of symplectic manifolds and complex surfaces.

Let us first briefly recall the geography of simply connected compact complex surfaces. Since X is simply connected, b^+ is odd (by the Noether formula $12 \mid c_1^2(X) + c_2(X)$), and the holomorphic Euler characteristic $\chi(X)$ is $\frac{1+b^+}{2}$. Also note that $c_1^2(X) = 3\sigma(X) + 2e(X)$, here $\sigma(X)$ denotes the signature, $e(X)$ the Euler characteristic of X.

Let us associate these two integers to a complex surface X

$$X \to (\chi(X), c_1^2(X)).$$

For example $(\chi(\mathbb{CP}^2),\ c_1^2(\mathbb{CP}^2)) = (1,9)$; $(\chi(S^2\times S^2),\ c_1^2(S^2\times S^2)) = (1,8)$ and $(\chi(E(n)),\ c_1^2(E(n))) = (n,0)$ ($E(n)$ is the regular elliptic surface with section and $e(E(n)) = 12n$, in particular $E(2)$ is the $K3$ surface). Note also that if X' is the blow up of X, then $(\chi(X'),\ c_1^2(X')) = (\chi(X),\ c_1^2(X) - 1)$.

By the classification result of Kodaira we know that a simply connected compact complex surface is either rational, elliptic or a surface of general type. If X is rational (meaning birationally equivalent to $\mathbb{CP}^2$), then $b^+ = \chi(X) = 1$, and the simply connected minimal rationals are diffeomorphic to $\mathbb{CP}^2$, $S^2 \times S^2$ or $\mathbb{CP}^2\#\bar{\mathbb{CP}}^2$ (the Hirzebruch-surfaces).

If X is minimal elliptic (so X admits a holomorphic map $\pi : X \to \mathbb{CP}^1$ with a smooth elliptic curve as a generic fiber), then $(\chi(X),\ c_1^2(X)) = (n,0)$ for some $n \in \mathbb{N}$. For surfaces of general type we know that $c_1^2(X) > 0$, and the two famous inequalities (the Noether inequality and the Bogomolov-Miyaoka-Yau inequality) give constraints for $c_1^2(X)$ in terms of $\chi(X)$:

$$2\chi(X) - 6 \leq c_1^2(X) \leq 9\chi(X).$$

Most of the points of this region (like $2\chi(X) - 6 \leq c_1^2(X) \leq 4\chi(X)$) is known to correspod to a minimal surface of general type (see [P] or [BPV] for further details).

The same geography question makes sense for symplectic manifolds as well – namely which points $(a, b) \in \mathbb{Z}^2$ can be realized as $(\chi(X) = \frac{1+b^+}{2},\ c_1^2(X) = 3\sigma(X) + 2e(X))$ of a minimal simply connected symplectic manifold X. Note that $12 \mid c_1^2(X) + c_2(X)$ holds for an almost complex manifold, and so in particular for a symplectic manifold as well, so b_X^+ is odd for a symplectic manifold X. Since blow up and blow down of a symplectic (-1)-sphere makes sense in the symplectic category, minimality can be defined for symplectic manifolds in the same way.

A simply connected complex surface is Kähler, hence symplectic; so the regions populated by complex surfaces are already covered by symplectic manifolds as well. In the following we will show, that a big part of the region under the Noether-line can be populated by minimal symplectic manifolds, more precisely if $D = \{(a, b) \in \mathbb{Z}^2 \mid 0 < b < 2a-6\}$, then

Theorem 1.1. *If $(a, b) \in D$ and b is even, then there is a minimal symplectic manifold X such that $(\chi(X),\ c_1^2(X)) = (a, b)$.*

Remark 1.2.
- *Note that – by recent result of Taubes – $c_1^2(X) \geq 0$ for a minimal symplectic manifold.*
- *Using slightly different construction Fintushel and Stern gave irreducible symplectic examples populating the region D. Our theorem represents only points $(a, b) \in D$ with even b as $(\chi(X),\ c_1^2(X))$ of a minimal symplectic manifold X, although most probably the same argument works for every point in D.*
- *The region D above was already populated by examples of Gompf ([G]) which were symplectic, but it is not clear yet whether those examples are minimal – although they very likely are.*
- *Note also that the examples given by Theorem 1.1 do not carry complex structure.*

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2. Donaldson series

Let us briefly recall the rudiments of Donaldson series (see also [KM], [DK]).

For a simply connected manifold X with $b^+ \geq 3$ and odd an analytic function

$$\mathbb{D}_{X,c} : \mathrm{H}_2(X; \mathbb{R}) \to \mathbb{R}$$

can be defined. The definition of $\mathbb{D}_{X,c}$ uses the ASD equation for connections on auxiliary principal SO(3)-bundles P over X with $w_2(P) \equiv c$ $(mod\ 2)$ ($c \in \mathrm{H}^2(X; \mathbb{Z})$ is fixed).

Remark 2.1. *To define $\mathbb{D}_{X,c}$ one needs an additional property of X – it has to be of simple type (see [KM]). Also the definition of $\mathbb{D}_{X,c}$ needs a choice of a homology orientation of X (see [D]).*

The beautiful structure theorem of Kronheimer-Mrowka and Fintushel-Stern ([KM], [FS1]) states that

$$\mathbb{D}_{X,c} = \exp(\frac{Q}{2}) \cdot \sum_{i=1}^{s} a_i e^{K_i}$$

where $a_i \in \mathbb{Q} \setminus \{0\}$ and $K_i \in \mathrm{H}^2(X;\mathbb{Z})$ $(i = 1, ..., s)$. $\{K_i\}_{i=1}^s$ is the set of (KM)-basic classes of the manifold X, these classes satisfy the following properties:

- $K_i \equiv w_2(X)$ $(mod\ 2)$;
- if K is a basic class, then $-K$ is a basic class as well;
- if $\Sigma \subset X$ is a smoothly embedded surface with $[\Sigma]^2 \geq 0$ and genus $g(\Sigma)$, then for any basic class K

$$2g(\Sigma) - 2 \geq [\Sigma]^2 + |K([\Sigma])|.$$

Theorem 2.2. *(Blow up formula)*
If $\{K_i\}_{i=1}^s$ is the set of basic classes for X, then $\{K_i \pm E\}_{i=1}^s$ is the set of basic classes for $X \# \bar{\mathbb{CP}}^2$ (E is the Poincare dual of the exceptional fiber). □

More generally if $X = X_1 \# X_2$ where $b^+(X_2) = 0$ (so the intersection form of X_2 is $n\langle -1 \rangle$ spanned by $\{e_1, ..., e_n\}$), then the set of basic classes of X is $\{K_i \pm E_1 \pm ... \pm E_n\}$ where $\{K_i\}$ is the set of basic classes of X_1 (and E_i is the Poincare dual of e_i).

By the connected sum theorem of Donaldson we know, that if X has non-zero series, then X cannot admit a decomposition $X = X_1 \# X_2$ with $b^+(X_i) > 0$ $(i = 1, 2)$. A decomposition with $b^+(X_2) = 0$ however is possible, so irreducibility doesn't follow directly from the non-vanishing of the invariants.

Proposition 2.3. *Assume that the set of basic classes $\{K_i\}_{i=1}^s$ of the manifold X satisfies*

$$(K_i - K_j)^2 \neq -4 \ for\ all\ 1 \leq i, j \leq s.$$

In this case X is irreducible.

Proof: The existence of basic classes insure, that $\mathbb{D}_X \neq 0$, so if X is reducible, then $X = X_1 \# X_2$ with $b^+(X_2) = 0$ is the only possibility. By the previous remark however in this case there are basic classes K_i, K_j, such that $K_i - K_j = 2E_1$, so $(K_i - K_j)^2 = -4$ contradicting our assumption. □

Assume that the manifold X has only 2 basic classes $\pm K \in \mathrm{H}^2(X;\mathbb{Z})$ and $K^2 > 0$. Assume also that X contains a torus f with square 0 lying in a cusp neighborhood. In this case one can take the fiber sum of X with the reular elliptic surface $E(n)$ along f.

Proposition 2.4. *$X \#_f E(n)$ is an irreducible manifold.*

Proof: Applying the computations presented in [S] (Proposition 3.3), the set of basic classes of $X \#_f E(n)$ is

$$\{\pm K + k \cdot F \mid k \equiv n \ (mod\ 2),\ |k| \leq n\}$$

(F is the Poincare dual of the homology class represented by f). The difference of two basic classes is either $k_1 \cdot F$ or $\pm(2K + k_2 \cdot F)$; the squares of these elements are at least 0 so by Proposition 2.3 $X\#_f E(n)$ is irreducible. □

3. Irreducible symplectic manifolds

Let us take the set Ξ of simply connected symplectic manifolds X having the following properties:

1. X has exactly two basic classes ($\pm K$) and $K^2 > 0$;
2. X contains a torus f with $f^2 = 0$ such that f is lying in a cusp neighborhood and f is a symplectic or lagrangian submanifold of X.

By the construction of Gompf $X\#_f E(n)$ is symplectic; by Proposition 2.4 it is irreducible as well. Note that $(\chi(X\#_f E(n)),\ c_1^2(X\#_f E(n))) = (\chi(X) + n,\ c_1^2(X))$. So to prove Theorem 1.1 we only have to show, that for every even $b > 0$ Ξ contains an element X such that $(\chi(X),\ c_1^2(X)) = (a, b)$ with $b \geq 2a - 6$. As Fintushel and Stern observed ([FS]), complete intersections, Moishezon surfaces and Salvetti surfaces are elements of Ξ (note that in these cases the torus f is a lagrangian submanifold). Also by analyzing the effect of rational blowdown, Fintushel and Stern realized ([FS2]) that surfaces on the Noether-line $c_1^2 = 2\chi - 6$ (the Horikawa surfaces) can be constructed by blowing down rationally elliptic surfaces $E(n)$. Since $E(n)$ contains lagrangian tori disjoint from the configurations one blows down to get the Horikawa surfaces, we have

Theorem 3.1. *The Horikawa surfaces constructed by rationally blowing down the elliptic surfaces $E(n)$ are in Ξ.*

In this way we have an element of Ξ with $c_1^2 = 2\chi - 6$ for every even c_1^2, and this proves Theorem 1.1.

Remark 3.2.
- *By performing a logarithmic transformation of multiplicity 2 on f – which is known to be a symplectic operation – we can turn a spin manifold into a non-spin one; the resulting manifold remains irreducible.*
- *Most probably the surfaces on the "next Horikawa line" $c_1^2 = 2\chi - 5$ contain also the required symplectic or lagrangian torus in the cusp neighborhood, so we can relax the assumption on the parity of b in Theorem 1.1. This issue will be discussed elsewhere.*

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