

The Schrödinger picture of standard cosmology

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We consider a time independent Schrödinger type equation derived from the equations of motion that drives a single scalar field in a standard cosmology model for inflation in a flat space-time with a Friedman-Robertson-Walker (FRW) metric with a cosmological constant. We find that all the 1-dimensional bound state solutions of quantum mechanics lead to at least one exact solution for the dynamical equations of standard cosmology, and that these solutions resemble the most recurrent inflationary solutions found in the literature. The analogies derived from this approach may be used to realize a deeper understanding of the dynamics of the model.

Keywords: Cosmology; Inflation; FRW; Schrödinger.

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1 Introduction

The current approach of inflation is that it occurs at some stage in the early universe and that its source is one or several scalar fields [1]-[2]. The different models of inflationary cosmology consider as a general feature a rapid growth of the size of the universe at some stage in the early universe [3]-[5], this simple definition of inflation may set the initial conditions for the large scale structure of the universe. In this approach one must consider an arbitrary functional form for the scalar field (SF) potential $V(\phi)$, since there is no unique prescription or phenomenology that could help to determine it.

Following this approach, let us consider a homogeneous and isotropic Universe, i.e., a model in a FRW background with a scalar homogenous field $\phi(t)$ minimally coupled to gravity and nonzero cosmological constant

$$\int d^4x \sqrt{-g} \left[R + \Lambda + \frac{1}{2}(\nabla\phi)^2 + V(\phi) \right], \quad (1)$$

where $(\nabla\phi)^2 = g^{\mu\nu}\partial_\mu\partial_\nu$ and $V(\phi)$ is the potential energy of the field. In order to describe the dynamics of the scalar field during inflation the usual treatment is performed [1]-[6], finally leading to the pair of equations

$$3H^2 = \frac{1}{2}\dot{\phi}^2 + V(\phi) + \Lambda \quad (2)$$

$$\ddot{\phi} + 3H\dot{\phi} = -\frac{dV(\phi)}{d\phi}, \quad (3)$$

where dot means derivative with respect to time, and we set $M_{Pl} = 1$, $\hbar = c = 1$. The time derivative of eq.(2) is related to eq.(3) through the momentum equation

$$\dot{H} = -\frac{1}{2}\dot{\phi}^2. \quad (4)$$

With the use of eqs. (3) and (4) the dynamics of the model may be described by the single equation:

$$3H^2 + \dot{H} = V(\phi) + \Lambda. \quad (5)$$

which can be recognized as a Riccati equation for the Hubble parameter $H(t)$. The Riccati equation has been one of the most useful equations of mathematical physics, specially in supersymmetric quantum mechanics (SUSY QM.) Its appearance here immediately suggests a QM approach to inflationary cosmology based on the second order differential equation derived from it. This has been proposed earlier, the ansatz being to replace x for t and to assign $a(t) = \psi(x)$, however leading to a nonlinear Schrödinger equation [7]. In this article we propose an alternative transformation which leads to a linear Schrödinger equation, where the SF potential $V(\phi)$ can be easily interpreted as the QM potential $U(x)$, a simple scheme whose consequences seem not to have been explored up to now. By simple algebraic comparisons we end up with a powerful method to derive particular exact solutions that may be useful for understanding the inflationary period. A similar approach for the case of cosmology with a perfect fluid has been proposed earlier, but in the context of classical mechanics ([11, 12].) However, the fact that in our approach no restriction is made on the form of the SF potential allows us to probe a deeper connection between QM and inflationary cosmology.

2 The Schrödinger picture of Standard Cosmology

2.1 A Schrödinger type equation for inflation

By defining $\psi(t)$ through

$$H \equiv \frac{1}{3} \frac{\dot{\psi}(t)}{\psi(t)}, \quad (6)$$

the Riccati equation (5) can be transformed into the one dimensional Schrödinger equation

$$\left[-\frac{d^2}{dt^2} + 3V(t) \right] \psi(t) = -3\Lambda \psi(t), \quad (7)$$

we shall consider solutions to eq.(7) based only on the fact that the Hubble parameter $H(t)$ cannot be a singular function, implying that $\psi(t)$ has to be an at least C^1 class function without zeros, but without any other restriction. For example, we may consider all ground state solutions of known exactly solvable bound state problems in QM as solutions to eq.(7). Hence, an immediate equivalence arises between the SF potential $V(\phi)$ and the cosmological constant Λ , with the QM potential $U(x)$ and ground state energy eigenvalue E_g ,¹ respectively,

$$3V(\phi(t)) + 3\Lambda \leftrightarrow 2U(x) - 2E_g \quad (8)$$

¹ For simplicity we shall use $m = 1$ for the Schrödinger particle.

This is indeed a very simple proposal, which shows that all the known exactly solvable stationary problems of 1-dimensional QM must provide at least one exact solution to the cosmological Schrödinger type equation. The general algebraic procedure is very simple: for any given QM problem, use the ground state eigenfunction $\psi_g(x)$ and eq.(6) to find $H(t)$; then, use eq.(4) to find $\phi(t)$, which together with eq.(8) defines $V(\phi)$.

For example, in the QM case of the simple harmonic oscillator (SHO), where $U(x) = \omega^2 x^2/2$, the Schrödinger equation

$$\left[-\frac{d^2}{dx^2} + 2 \left(\frac{\omega^2}{2} x^2 - E_n \right) \right] \psi_n(x) = 0. \quad (9)$$

has the wave functions and energy eigenvalues given by

$$\psi_n(x) = \sqrt{\frac{1}{2^n n!} \sqrt{\frac{\omega}{\pi}}} e^{-\frac{\omega}{2} x^2} H_n(\sqrt{\omega} x) \quad E_n = \left(n + \frac{1}{2} \right) \omega \quad (10)$$

where $H_n(y)$ are the Hermite polynomials. In the case $n = 0$, the Hermite polynomial is $H_0(\sqrt{\omega} x) = 1$, with energy and wave function

$$E_0 = \frac{1}{2} \omega \quad \psi_0(x) = \sqrt[4]{\frac{\omega}{\pi}} e^{-\frac{\omega}{2} x^2}. \quad (11)$$

To construct the corresponding cosmological variables, we replace x by t in eq.(11), and use eq.(6) to find the associated Hubble's parameter

$$H(t) = -\frac{\omega}{3} t \quad (12)$$

and with the use of eq.(4) we obtain the expression of the scalar field

$$\phi(t) = \sqrt{\frac{2\omega}{3}} t \quad (13)$$

Finally, we use eqs.(8) and (13) to find the SF potential $V(\phi)$ and the constant Λ ,

$$V(\phi) = \lambda \phi^2, \quad \Lambda = -\frac{2}{3} \lambda \quad (14)$$

where $\lambda = \frac{\omega}{2}$. As one can see, the scalar field potential derived from the SHO potential turns out to be $\lambda \phi^2$. Surprisingly, one of the most useful and basic potentials of QM transforms into one of the most useful potentials in this cosmological model (see [1], [13], and references there in.) It is even more surprising that other typical QM potentials resemble typical scalar field potentials in standard cosmology; for example, compare the results in [8] and [14] to the ones obtained in Table 1.

2.2 Inflationary solutions from QM central force problems

In Table 1 we have left some free parameters which can be used to fix the cosmological variables strength, even though their overall behavior is already determined. For example, the cosmological solutions above may be set to correspond to the half plane solutions of the Schrödinger problem, with the initial condition $a(0) \neq 0$ if $t = 0$ when $x = 0$, and it is not possible to have as initial condition $a = 0$. Since in the half plane not only the QM ground state function is nodeless, but also the first excited state function is, we end up with two exact standard cosmology solutions with different initial conditions derived from each 1-dimensional QM problem in the half plane $t \geq 0$, the second one with the initial condition $a(0) = 0$.

Following this discussion, it is obvious that all ground state solutions to the radial problem

$$-\frac{d^2 u_{nl}}{dr^2} + 2 \left[U(r) + \frac{l(l+1)}{2r^2} - E_{nl} \right] u_{nl} = 0 \quad (15)$$

are useful to derive exact solutions to the SF equations. Moreover, since all eigenfunctions of the radial problems with angular momentum $l = n - 1$ are nodeless, we end up with an infinity of exact solutions to the SF dynamics equations, parameterized by a cosmological constant that belongs to the discrete set $\Lambda_n = -\frac{1}{3}E_n$, with the SF potential always including the centrifugal barrier term $n(n-1)/t^2$, and with the initial condition $a(0) = 0$.

As an example, let us consider the Hydrogen like potential $U(r) = -Zq^2/r$ where all the nodeless eigenfunctions are given by

$$u_{n,n-1}(r) \propto r^n e^{-Z\alpha r/n} .$$

where the fine structure constant α and the atom number Z , are constants only used to determine the strength of the cosmological variables. For the ground state ($n = 1$) the cosmological variables are

$$\begin{aligned} H(t) &= \frac{1}{3t} - \frac{Z\alpha}{3} \\ \phi(t) &= \sqrt{\frac{2}{3}} \ln t \\ V(\phi) &= -\lambda e^{-\sqrt{3/2}\phi} \\ \Lambda &= \frac{\lambda Z}{2} \end{aligned} \quad (16)$$

where $\lambda = 2Z\alpha^2/3$, while for $n > 1$, they become

$$\begin{aligned} H(t) &= \frac{n}{3t} - \frac{Z\alpha}{3n} \\ \phi(t) &= \sqrt{\frac{2n}{3}} \ln t \end{aligned} \quad (17)$$

	Schrödinger picture	Standard Cosmology
SHO	$\psi_0(x) = \sqrt[4]{\frac{\omega}{\pi}} e^{-\frac{\omega}{2}x^2}$ $E_0 = \frac{1}{2}\omega$ $U(x) = \frac{1}{2}\omega^2 x^2$	$H(t) = -\frac{2\lambda}{3}t, \quad \phi(t) = 2\sqrt{\frac{\lambda}{3}}t$ $\Lambda = -\frac{2}{3}\lambda$ $V(\phi) = \lambda\phi^2, \quad \lambda = \frac{\omega}{2}$
MOR	$\psi_0(x) = 8e^{-2e^{-\alpha x}} e^{-\frac{3}{2}\alpha x}$ $E_0 = -\frac{9}{8}\alpha^2$ $U(x) = 2\alpha^2(e^{-2\alpha x} - 2e^{-\alpha x})$	$H(t) = \frac{32}{3}\sqrt{\frac{\lambda}{3}} \left(e^{-16\sqrt{\frac{\lambda}{3}}t} - \frac{3}{4} \right), \quad \phi(t) = -\frac{4}{\sqrt{3}}e^{-4\sqrt{\frac{\lambda}{3}}t}$ $\Lambda = 16\lambda$ $V(\phi) = \lambda\phi^4 - \frac{32}{3}\lambda\phi^2, \quad \lambda = \frac{3\alpha^2}{64}$
PTT	$\psi_0(x) = \frac{1}{2\lambda} \cos^\lambda(\alpha x)$ $E_0 = \frac{\alpha^2\lambda^2}{2}$ $U(x) = \frac{\alpha^2}{2} \frac{\lambda(\lambda-1)}{\cos^2(\alpha x)}$	$H(t) = -\frac{\alpha\lambda}{3} \tan(\alpha t), \quad \phi(t) = \sqrt{\frac{2\lambda}{3}} \ln \sqrt{\frac{1+\sin(\alpha t)}{1-\sin(\alpha t)}}$ $\Lambda = -\frac{\lambda^2\alpha^2}{3}$ $V(\phi) = \frac{\alpha^2}{3} \lambda(\lambda-1) \cosh^2\left(\sqrt{\frac{3}{2\lambda}}\phi\right)$
PTH	$\psi_0(x) = \frac{1}{\cosh \alpha x}$ $E_0 = -\frac{\alpha^2}{2}$ $U(x) = \frac{-\alpha^2}{\cosh^2(\alpha x)}$	$H(t) = -\frac{\alpha}{3} \tanh(\alpha t), \quad \phi(t) = \sqrt{\frac{2}{3}} \arcsin(\tanh(\alpha t))$ $\Lambda = \frac{\alpha^2}{3}$ $V(\phi) = -\frac{2\alpha^2}{3} \cos^2\left(\sqrt{\frac{3}{2}}\phi\right)$
HA	$u_{n,n-1}(r) = \frac{(2\alpha)^{3/2}}{\sqrt{2n^4(2n-1)!}} \left(\frac{2\alpha r}{n}\right)^n e^{-\alpha r/n}$ $E_n = -\frac{\alpha^2}{2n^2}$ $U_{eff}(r) = -\frac{\alpha}{r} + \frac{n(n-1)}{2r^2}$	$H(t) = \frac{n}{3t} - \frac{\alpha}{3n}, \quad \phi(t) = \sqrt{\frac{2n}{3}} \ln t$ $\Lambda = \frac{\alpha^2}{3n^2}$ $V(\phi) = \frac{n(n-1)}{3} \left[e^{-2\sqrt{\frac{3}{2n}}\phi} - \frac{2\alpha}{n(n-1)} e^{-\sqrt{\frac{3}{2n}}\phi} \right]$

Table 1: Standard Cosmology exact solutions from six Schrödinger problems: simple harmonic oscillator (SHO), Morse potential (MOR), Trigonometric Pöschl Teller (PTT), Hyperbolic Pöschl Teller (PTH), and Hydrogen Atom (HA).

$$V(\phi) = \frac{n(n-1)}{3} \left[e^{2\sqrt{3/2n}\phi} - \frac{2Z\alpha}{n(n-1)} e^{-\sqrt{3/2n}\phi} \right]$$

$$\Lambda = \frac{Z^2\alpha^2}{3n^2}$$

Hence, for $n = 1$ the SF potential becomes an exponential potential, while for $n > 1$, if we choose $Z\alpha = n(n-1)$ it becomes a Morse potential. The appearance of this Morse potential is a very interesting feature of this model, since this potential is very slowly varying for $t \rightarrow \pm\infty$, has a very soft minimum, and later exponentially grows for $t \rightarrow \mp\infty$, the sign depending on the parameters. Therefore, this potential has all the desired features to allow a SF $\phi(t)$ slowly roll to the minimum of the SF potential $V(\phi)$ [9]. All these results are also included in Table 1.

2.3 QM and Standard Cosmology analogies

Looking at Table 1, the Schrödinger picture of standard cosmology seems to be a fruitful approach to the construction of exact solutions to the inflationary equations (2,3), since all potentials from these known QM problems resemble known SF potentials. It may seem that there must exist further analogies between these two models of the micro and macro cosmos than just an algebraic resemblance.

In the present analogy $\psi(t) = a^3(t)$ describes the way the universe volume is expanding since in the Schrödinger picture, $\psi(x)$ is related to probability conservation, hence the equivalence proposed here points to energy density conservation in this expanding universe. On the other hand, the only constant term in the QM problem is the energy E , which therefore determines the cosmological constant Λ of eq.(7), which could be associated with the vacuum energy density. Therefore, the sign of Λ is completely determined by the corresponding QM problem from which the SF solution is derived, becoming an immediate check for the dynamical characteristics that one wants to determine with the proposed SF potential.

With respect to the scalar field $\phi(t)$, following eqs.(4), (5) and (8), we can see that wherever $\dot{a}(t) \simeq 0$,

$$\phi(t) \simeq \int^t dy \sqrt{2(E - U(y))} \quad (18)$$

which resembles the action $S(x)$ of the quantum theory.

2.4 Slow Roll and WKB approximations

One further analogy deserves special attention. Beginning with the slow roll approximation condition

$$\left| \frac{V'}{V} \right|^2 < 1$$

where we should do the substitution $V \rightarrow V + \Lambda$ to comply with eq.(5), we can see that since $\ddot{a}/a > 0$ implies that $V + \Lambda > \dot{\phi}^2$, we have that

$$|\dot{V}|^2 = |\dot{\phi}V'|^2 < |V + \Lambda| |V'|^2 < |V + \Lambda|^3$$

In our QM analogy, we would have to do the substitutions $\frac{dV}{dt} \rightarrow \frac{dU}{dx}$ and $|V + \Lambda| \rightarrow |E - U|$, giving

$$\left| \frac{dU}{dx} \right|^2 < |E - U|^3$$

which is just the WKB approximation

$$\left| \frac{d^2W}{dx^2} \right| < \left| \frac{dW}{dx} \right|^2$$

of the stationary problem, where $W(x) = \pm \sqrt{2(E - U)}$ is Hamilton's principal function.

3 An ever expanding universe

All the QM problems considered here lead to two possible initial conditions for the scale factor, $a(0) = 0$ or $a(0) > 0$, but with only one final condition, $a(t) \rightarrow 0$ as $t \rightarrow \infty$, if eqs.(2,3) may be used for all the half plane $t \geq 0$. If this Big Crunch could not be attainable, as observations seem to predict, our simple approach may still be useful to describe the expected dynamics.

If $a(t)$ is always increasing, then a QM bound state is not the right solution. However, as is depicted in Ref.[10], the Schrödinger equation (7) has an infinite number of wave functions that diverge to $\pm\infty$ as $t \rightarrow \infty$, for the continuum set of energies E , with $E_n < E < E_{n+1}$, as is depicted in Fig.(1) for the case of the SHO.

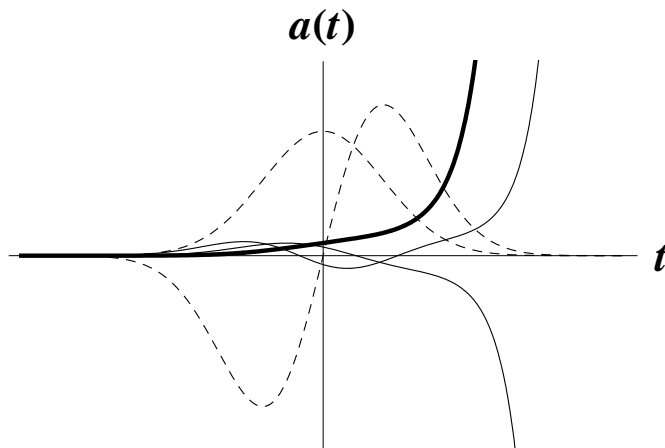


Figure 1: Different forms of $a(t)$ obtained from one QM problem.

In Fig.(1) the dashed curves correspond to $a(t)$ obtained from the ground state and first excited state eigenfunctions, $\psi_0(x)$ and $\psi_1(x)$. These two curves have $a(t) \rightarrow 0$ as $t \rightarrow \infty$. On the other hand, the solid curves draw the scale factor for three different cosmological constants, derived from the energy eigenvalues, $E < E_0$, $E_0 < E < E_1$ and $E_1 < E < E_2$, whose wave functions diverge to $+\infty$, $-\infty$ and $+\infty$ again, the first one without nodes and the other with increasing number of nodes. Therefore, only the wave function with $E < E_0$ could lead to a physical solution $a(t) > 0$ for all t , describing a cosmological solution for an ever expanding universe.

4 Conclusion

In this article we present a simple ansatz that links two completely separate models, one being a cosmological model for inflation of the macrocosmos, and the other a quantum model for the microcosmos. Surprisingly, we have found that the most common potential functions in these two models map from one to the other, with probability conservation in

QM reflecting into energy density conservation in the cosmological model, a cosmological constant determined by the energy eigenvalue in the former, and the WKB approximation reflecting into the slow roll approximation. All these analogies define deep connection between the two models, with respect to their dynamical behaviors. Finally, our initial proposal was to look at the bound state solutions of the QM problems to describe a non singular Hubble variable for the cosmological model, but since present observations seem to indicate that the universe is not receding but on the contrary accelerates with time, we may have to recur to the unbounded solutions of the Schrödinger problem, supporting even more the validity of the simple analogy developed in this work.

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References

- [1] Bruce A. Bassett, Shinji Tsujikawa, David Wands, *Rev. Mod. Phys.* **78** (2006) 536.
- [2] Edmund J. Copeland, M. Sami and Shinji Tsujikawa, *Int. J. Mod. Phys. D.* **15** (2006) 1753-1936.
- [3] Andrew R. Liddle and David H. Lyth, “*Cosmological Inflation and Large Scale Structure*”, Cambridge University Press (2000).
- [4] Andrei Linde, “*Particle Physics and Inflationary Cosmology*”, Harwood Academic Publishers, (1990).
- [5] Edmund J. Copeland, Edward W. Kolb Andrew R. Liddle and James E. Lidsey, *Phys. Rev. D* **48** (1993) 2529-1993.
- [6] Andrew R. Liddle, Paul Parsons and John D. Barrow, *Phys. Rev. D.* **50** (1994) 7222-7232.
- [7] Rachael M. Hawkins and James E. Lidsey, *Phys. Rev. D.* **66** (2002) 023523.
- [8] V. V. Kiselev and S. A. Timofeev, “Quasi-attractor dynamics of $\lambda-\phi^4$ -inflation,” arXiv:0801.2453 [gr-qc].
- [9] M. A. Reyes, “On exact solutions to the scalar field equations in standard cosmology,” arXiv:0806.2292 [gr-qc].
- [10] B. Mielnik and M. A. Reyes, *J. Phys. A* **29**, 6009 (1996).

- [11] J. A. S. Lima, J. A. M. Moreira and J. Santos, *Gen. Rel. Grav.* **30**, (1998) 425 .
- [12] J. A. S. Lima, *Am. J. Phys.* **69** (2001) 1245
- [13] L. A. Urena-Lopez and M. J. Reyes-Ibarra, *Int. J. Mod. Phys. D* **18**, 621 (2009).
- [14] T. Matos, J. R. Luevano, I. Quiros, L. A. Urena-Lopez and J. A. Vazquez, *Phys. Rev. D* **80**, 123521 (2009).