On the “scattering law” for Kasner parameters in the model with one-component anisotropic fluid

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Abstract

A multidimensional cosmological type model with 1-component anisotropic fluid is considered. An exact solution is obtained. This solution is defined on a product manifold containing n Ricci-flat factor spaces. We singled out a special solution governed by the function *cosh*. It is shown that this special solution has Kasner-like asymptotics in the limits $\tau \to +0$ and $\tau \to +\infty$, where τ is a synchronous time variable. A relation between two sets of Kasner parameters α_∞ and α_0 is found. This formula (of “scattering law”) is coinciding with that obtained earlier for the S-brane solution (when scalar fields are absent).

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1 Introduction

In this paper we continue our investigations (started in [1]) of multidimensional solutions defined on product of several Ricci-flat factor spaces which have two Kasner-like asymptotical regions.

Here we recall that Kasner-like solutions with a chain of n Ricci-flat factor-spaces $(M_i, g^{(i)})$ have the following form [2]

$$g = w d\tau \otimes d\tau + \sum_{i=1}^{n} A_i^2 \tau^{2\alpha^i} g^{(i)}, \tag{1.1}$$

where $w = \pm 1$, $\tau > 0$,

$$\sum_{i=1}^{n} d_i \alpha^i = 1, \tag{1.2}$$

$$\sum_{i=1}^{n} d_i (\alpha^i)^2 = 1, \tag{1.3}$$

and for any $i = 1, \ldots, n$ ($n \geq 2$): $A_i > 0$ is constant, $g^{(i)}$ is a Ricci-flat metric defined on the manifold M_i (for $w = -1$ see [2]).

These solutions with non-Milne-type sets of Kasner parameters are singular since the Riemann tensor squared is divergent as $\tau \to +0$ [3]. For Milne-type sets of parameters, i.e. when $d_i = 1$ and $\alpha^i = 1$ for some i ($\alpha^j = 0$ for all $j \neq i$) the metric is regular as $\tau \to +0$, when either i) $g^{(i)} = -w dy^i \otimes dy^i$, $M_i = \mathbb{R}$ ($-\infty < y^i < +\infty$), or ii) $g^{(i)} = w dy^i \otimes dy^i$, M_i is circle of length L_i ($0 < y^i < L_i$) and $A_i L_i = 2\pi$ (i.e. when the cone singularity is absent).

In this paper we consider an exact cosmological type solution with 1-component "perfect" fluid (Section 2). (For earlier publications on multidimensional cosmological models with perfect fluid see [4]-[14] and references therein.) This solution is defined on a product manifold containing n Ricci-flat factor spaces. It is derived in the Appendix. For $w = -1$ it was found in [11, 12, 13] and generalized in [14] for the case when a scalar field was added. A special case of this solution with a Λ-term component was obtained in [15] (see also [16] for scalar field generalization).

We write the solution in a so-called "minisuperspace-covariant" form that significantly simplifies the forthcoming analysis. In Section 3 we single out a special solution governed by the *cosh* function. We show that this solution has a Kasner-like asymptotics in both limits $\tau \to +0$ and $\tau \to +\infty$, where τ is the synchronous time variable. We also find a relation between two sets of Kasner parameters $\alpha_\infty = (\alpha^i_\infty) \in \mathbb{R}^n$ and $\alpha_0 = (\alpha^i_0) \in \mathbb{R}^n$:

$$\alpha^i_\infty = \frac{\alpha^i_0 - 2U(\alpha_0)U^i(U,U)^{-1}}{1 - 2U(\alpha_0)(U,U^\Lambda)(U,U)^{-1}}, \tag{1.4}$$

$i = 1, \dots, n$. Here $U = (U_i)$ is a co-vector corresponding to the fluid component, $\bar{U} = (U^i)$ is dual vector and U^Λ is a co-vector, corresponding to the Λ-term. All these vectors and the scalar product $(.,.)$ are defined below (see Section 2). Here $U(\alpha_0) = U_i\alpha^i_0 > 0$ and $U(\alpha_\infty) = U_i\alpha^i_\infty < 0$.

A relation analogous to (1.4) ("scattering law" formula) was obtained earlier for S-brane solution with one brane in [1]. We note that in [1] the geometrical sense of the scattering law was clarified for $n > 2$. Namely, the scattering law transformation for a brane U-vector (obeying $(U,U^\Lambda) < 0$) was expressed in terms of a function mapping a "shadow" part of the Kasner sphere S^{n-2} onto "illuminated" one. The shadow and illuminated parts of the Kasner sphere were defined w.r.t. a point-like source of light located outside the Kasner sphere S^{n-2}. (For details of this geometrical construction see [1]).

The relation (1.4) appears also when the billiard approach to multicomponent anisotropic fluid is considered [18, 19, 20, 21]. It may be shown (as it was done in [23, 24] for S-brane solutions) that after the collision with a billiard wall (corresponding to the fluid component) the set of Kasner parameters, is defined by the Kasner set before the collision through the formula analogous to (1.4), see [26]. For the billiard approach in models with scalar field and fields of forms see [22, 23, 25, 26] and refs. therein.

2 Model with anisotropic fluid and its exact solution

2.1 The set-up

Now, we consider a cosmological type solution to Einstein equations with an anisotropic (perfect) fluid matter source

$$R^M_N - \frac{1}{2}\delta^M_N R = k^2 T^M_N \tag{2.1}$$

defined on D-dimensional manifold

$$M = \mathbb{R}_{.} \times M_1 \times M_2 \times \ldots \times M_n, \tag{2.2}$$

with block-diagonal metric

$$g = we^{2\gamma(u)} du \otimes du + \sum_{i=1}^{n} e^{2\beta^i(u)} g^{(i)}. \tag{2.3}$$

Here $\mathbb{R}_{.} = (u_-, u_+)$ is an interval, $w = \pm 1$ and $n \geq 2$. Manifold M_i with the metric $g^{(i)}$ is a Ricci-flat space of dimension d_i: $R_{m_i n_i}[g^{(i)}] = 0$, $i = 1, 2, \ldots, n$, and κ^2 is a multidimensional gravitational constant.

Energy-momentum tensor of anisotropic fluid is adopted in the following form:

$$(T^M_N) = \mathrm{diag}(-\hat{\rho}, \hat{p}_1 \delta^{m_1}_{k_1}, \ldots, \hat{p}_n \delta^{m_n}_{k_n}), \tag{2.4}$$

where $\hat{\rho}$ and $\hat{p}_i$ are "density" and "pressures", respectively, depending upon radial variable u.

In the cosmological case when $w = -1$ and all metrics $g^{(i)}$ have Euclidean signatures, $\hat{\rho} = \rho$ is a density and $\hat{p}_i = p_i$ is a pressure in i-th space. For static configurations with $w = 1$, $g^{(1)} = -dt \otimes dt$ and all metrics $g^{(i)}$, $i > 1$, having Euclidean signatures, the physical density and pressures are related to the effective ("hat") ones by formulas: $\rho = -\hat{p}_1$, $p_u = -\hat{\rho}$, $p_i = \hat{p}_i, \quad (i \neq 1)$, where p_u is the pressure in u-th direction.

We also impose the following equation of state

$$\hat{p}_i = \left(1 - \frac{2U_i}{d_i}\right) \hat{\rho}, \tag{2.5}$$

where U_i are constants, $i = 1, 2, \ldots, n$.

In what follows we use a scalar product

$$(U, U') = G^{ij} U_i U'_j = \sum_{i=1}^{n} \frac{U_i U'_i}{d_i} + \frac{1}{2 - D} (\sum_{i=1}^{n} U_i)(\sum_{j=1}^{n} U'_j), \tag{2.6}$$

for $U = (U_i), U' = (U'_i) \in \mathbb{R}^n$, where

$$G^{ij} = \frac{\delta^{ij}}{d_i} + \frac{1}{2 - D} \tag{2.7}$$

are components of dual minisuperspace metric. Recall that $(G^{ij}) = (G_{ij})^{-1}$, where

$$G_{ij} = d_i \delta_{ij} - d_i d_j, \tag{2.8}$$

are components of minisuperspace metric [17].

We also define a co-vector

$$U^{\Lambda} = (d_i), \tag{2.9}$$

corresponding to the Λ-term and the vector $\bar{U} = (U^i)$

$$U^i = G^{ij} U_j = \frac{U_i}{d_i} + \frac{1}{2-D} \sum_{j=1}^{n} U_j, \tag{2.10}$$

which is dual to U.

2.2 Exact solution

Here we consider an exact cosmological solution to Hilbert-Einstein equations (2.1) defined on the manifold (2.2). We impose the following restriction on the U-vector in (2.5)

$$K = (U,U) = \sum_{i=1}^{n} \frac{U_i^2}{d_i} + \frac{1}{2-D} (\sum_{i=1}^{n} U_i)^2 \neq 0. \tag{2.11}$$

(The case $K = 0$ will be considered in a separate publication.)

The solution has the following form (see Appendix C)

$$g = |f(u)|^{-2h(U,U^{\Lambda})} \exp(2c^0 u + 2\bar{c}^0) w du \otimes du + \sum_{i=1}^{n} |f(u)|^{-2hU^i} \exp(2c^i u + 2\bar{c}^i) g^{(i)}, \tag{2.12}$$

$$k^2 \hat{\rho} = -wA |f(u)|^{2h(U,U^{\Lambda})-2} \exp(-2c^0 u - 2\bar{c}^0), \tag{2.13}$$

where $w = \pm 1$, $h = K^{-1}$, $g^{(i)}$ is a Ricci-flat metric on M_i, and

$$(U, U^{\Lambda}) = \frac{\sum_{i=1}^{n} U_i}{2-D}, \tag{2.14}$$

$i = 1, \ldots, n$.

The moduli function f reads

$$
\begin{aligned}
f(u) = & R\sinh(\sqrt{C}(u-u_0)),\ C>0,\ KA<0; && (2.15)\\
& R\sin(\sqrt{|C|}(u-u_0)),\ C<0,\ KA<0; && (2.16)\\
& R\cosh(\sqrt{C}(u-u_0)),\ C>0,\ KA>0; && (2.17)\\
& |2AK|^{1/2}(u-u_0),\ C=0,\ KA<0, && (2.18)
\end{aligned}
$$

where $R = |2AK/C|^{1/2}$, and C, u_0 are constants. (In (2.12) and (2.13) $f(u) \neq 0$ is assumed for all $u \in (u_-, u_+)$.)

Vectors $c = (c^i)$ and $\bar{c} = (\bar{c}^i)$ obey the following constraints:

$$U(c) = U_i c^i = 0, \qquad U(c) = U_i \bar{c}^i = 0 \tag{2.19}$$

$$CK^{-1} + G_{ij} c^i c^j = 0, \tag{2.20}$$

where $G_{ij} c^i c^j = \sum_{i=1}^n d_i (c^i)^2 - (\sum_{i=1}^n d_i c^i)^2$.

In (2.12) and (2.13) we also denote

$$c^0 = U^\Lambda(c) = \sum_{i=1}^n d_i c^i, \quad \bar{c}^0 = U^\Lambda(\bar{c}) = \sum_{i=1}^n d_i \bar{c}^i. \tag{2.21}$$

The special solution with $C = c_i = 0$ (for all i) and $w = -1$ was considered in detail in [28, 29]. For $U = U^\Lambda$ and $A > 0$ it contains a special solution with $d_i = 1$, $g^i = dy^i \otimes dy^i$ ($i = 1, ..., n$), describing either (a part of) de-Sitter space (for $w = -1$) or (a part of) anti-de-Sitter space (for $w = 1$).

Minisuperspace-covariant form of solution.

This solution is derived in Appendix C in terms of "minisuperspace-covariant" notations for functions $\gamma(u)$, $\beta^i(u)$ appearing in metric (2.3).

Solution for $\beta = (\beta^i(u))$ reads as follows:

$$\beta^i(u) = -\frac{U^i}{(U,U)} \ln|f(u)| + c^i u + \bar{c}^i, \tag{2.22}$$

where $f(u)$ was defined in (2.15)-(2.18) and

$$\gamma = \gamma_0 \equiv \sum_{i=1}^n d_i \beta^i = U_i^\Lambda \beta^i \tag{2.23}$$

and u is the harmonic variable.

3 Scattering law for Kasner parameters

Now we restrict our consideration by a special solution with $C > 0$, $K = (U,U) > 0$ and $A > 0$. In this case the solution is governed by moduli function $f(u) = R\cosh(\sqrt{C}(u - u_0))$, $u \in (-\infty, +\infty)$, and has two Kasner-like asymptotics in the limits $\tau \to +0$ and $\tau \to +\infty$, where τ is a synchronous time variable (see below).

Another case, when there are two Kasner-like asymptotical regions, takes place when $C > 0$, $K = (U,U) < 0$ and $A < 0$ (this will be a subject of a separate paper).

3.1 Kasner-like behaviour

Let us consider our solution in a synchronous time:

$$\tau = \varepsilon \int_{u_0}^{u} d\bar{u} e^{\gamma_0(\bar{u})}, \tag{3.1}$$

where $\varepsilon = \pm 1$, and

$$e^{\gamma_0(u)} = |f(u)|^{-h(U^\Lambda,U)} \exp(c^0 u + \bar{c}^0) \tag{3.2}$$

is a lapse function.

Due to

$$f \sim \frac{R}{2} \exp(\pm\sqrt{C}(u - u_0)), \tag{3.3}$$

for $u \to \pm\infty$, we get asymptotical relations for the lapse function

$$e^{\gamma_0} \sim \mathrm{const} \exp(b_\pm \sqrt{C} u), \tag{3.4}$$

as $u \to \pm\infty$, with

$$b_\pm = \mp h(U^\Lambda, U) + \frac{c^0}{\sqrt{C}}. \tag{3.5}$$

Using relations (2.21) and $h = (U,U)^{-1}$, we could rewrite parameters $b_\pm$ in a minisuperspace-covariant form:

$$b_\pm = \mp \frac{(U^\Lambda, U)}{(U,U)} + (s, U^\Lambda), \tag{3.6}$$

where

$$s = (s_i) = (G_{ij} c^j / \sqrt{C}) \tag{3.7}$$

is a co-vector, obeying relations

$$(s, U) = 0, \tag{3.8}$$

$$\frac{1}{(U,U)} + (s, s) = 0, \tag{3.9}$$

following just from (2.19) and (2.20). In derivation of (3.6) we used the relation

$$c^0 = (s, U^{\Lambda})\sqrt{C}, \tag{3.10}$$

following from (2.21) and (3.7).

In what follows we will use the inequality

$$|(s, U^{\Lambda})| > \frac{|(U^{\Lambda}, U)|}{(U,U)}, \tag{3.11}$$

proved in Appendix C. The proof used relations (3.8), (3.9) and $(U, U) > 0$.

The parameter c^0 is a non-zero one (otherwise the relation (2.20) would be incompatible with the conditions $C > 0$, $K > 0$).

It follows from (3.11) that $b_{\pm}$ are also non-zero and

$$\mathrm{sign}(b_{\pm}) = \mathrm{sign}((s, U^{\Lambda})) = \mathrm{sign}(c^0). \tag{3.12}$$

It may be verified that due to (3.11) the lapse function $e^{\gamma_0(u)}$ is monotonically increasing from $+0$ to $+\infty$ for $c^0 > 0$ and monotonically decreasing from $+\infty$ to $+0$ for $c^0 < 0$.

We define a synchronous-like variable to be

$$\tau = \int_{-\infty}^{u} d\bar{u} e^{\gamma_0(\bar{u})} \tag{3.13}$$

for $c^0 > 0$ and

$$\tau = \int_{u}^{+\infty} d\bar{u} e^{\gamma_0(\bar{u})} \tag{3.14}$$

for $c^0 < 0$. Then, $\tau = \tau(u)$ is monotonically increasing from $+0$ to $+\infty$ for $c^0 > 0$ and monotonically decreasing from $+\infty$ to $+0$ for $c^0 < 0$.

We have the following asymptotical relations for $\tau = \tau(u)$

$$\tau \sim \mathrm{const}\ b_{\pm}^{-1} \exp(b_{\pm}\sqrt{C}u), \tag{3.15}$$

as $u \to \pm\infty$.

For $\beta = (\beta^i)$ from (2.22) we get (see (3.3))

$$\beta^i(u) \sim \mp \frac{U^i \sqrt{C} u}{(U,U)} + c^i u + \hat{c}^i \tag{3.16}$$

as $u \to \pm\infty$, where $\hat{c}^i$ are constants. Hence, due to (3.15), we are led to Kasner-like asymptotics

$$\beta^i \sim \alpha^i_{\pm} \ln \tau + \beta^i_{\pm} \tag{3.17}$$

for $u \to \pm\infty$, where $\beta^i_{\pm}$ are constants and

$$\alpha^i_{\pm} = [\mp \frac{U^i}{(U,U)} + s^i]/b_{\pm} \tag{3.18}$$

are Kasner-like parameters corresponding to $u \to \pm\infty$.

Asymptotical relations (3.17) could be also rewritten in the form of proper time asymptotics, i.e.

$$\beta^i \sim \alpha^i_0 \ln \tau + \beta^i_0, \text{ as } \tau \to +0, \tag{3.19}$$

$$\beta^i \sim \alpha^i_{\infty} \ln \tau + \beta^i_{\infty}, \text{ as } \tau \to +\infty. \tag{3.20}$$

Here

$$\alpha^i_0 = \alpha^i_-, \quad \alpha^i_{\infty} = \alpha^i_+ \tag{3.21}$$

for $c^0 > 0$ and

$$\alpha^i_0 = \alpha^i_+, \quad \alpha^i_{\infty} = \alpha^i_- \tag{3.22}$$

for $c^0 < 0$ and β^i_0, β^i_{∞} are constants.

It follows from definitions of Kasner parameters (3.18) that

$$G_{ij} \alpha^i_{\pm} \alpha^j_{\pm} = 0, \tag{3.23}$$

$$U(\alpha_{\pm}) = U_i \alpha^i_{\pm} = \mp \frac{1}{b_{\pm}}, \tag{3.24}$$

$$U^{\Lambda}(\alpha_{\pm}) = 1, \tag{3.25}$$

see (3.6), (3.8) and (3.9).

In components relations (3.23) and (3.25) read as

$$\sum_{i=1}^{n} d_i \alpha^i_{\pm} = \sum_{i=1}^{n} d_i (\alpha^i_{\pm})^2 = 1. \tag{3.26}$$

Thus, we are led to Kasner-like relations (1.2) and (1.3) for $\alpha_{\pm} = (\alpha^i_{\pm})$. Hence, $\alpha_0 = (\alpha^i_0)$ and $\alpha_{\infty} = (\alpha^i_{\infty})$ also obey relations (1.2) and (1.3).

So, we obtained a Kasner-like asymptotical behaviour of our special solution (with $C > 0$, $K > 0$ and $A > 0$) for i) $\tau \to +0$ and for ii) $\tau \to +\infty$, as well. The Kasner-like behaviour in the case i) is in agreement with the general result of the billiard approach from [22]. The the case ii) was considered in [26].

Using (3.12) and (3.24) we get

$$U(\alpha_0) = U_i \alpha^i_0 > 0, \tag{3.27}$$

$$U(\alpha_{\infty}) < 0. \tag{3.28}$$

3.2 Scattering law

Now, we derive a relation between Kasner sets α_0 and α_{∞}.

We start with formulae:

$$b_+ \alpha_+ - b_- \alpha_- = -\frac{2\bar{U}}{(U,U)} \tag{3.29}$$

and

$$b_+ - b_- = -\frac{2(U^{\Lambda}, U)}{(U,U)}, \tag{3.30}$$

following from (3.18) and (3.6), respectively. (Recall that $\bar{U} = (U^i)$.) Using these relations and (3.24) we get

$$\alpha^i_{\pm} = \frac{\alpha^i_{\mp} - 2U^i U(\alpha_{\mp})(U,U)^{-1}}{1 - 2U(\alpha_{\mp})(U,U^{\Lambda})(U,U)^{-1}}. \tag{3.31}$$

This formula gives a scattering law formula for Kasner parameters in our case (see definitions (2.10), (3.21) and (3.22)) or

$$\alpha_{\infty} = \frac{\alpha_0 - 2\bar{U} U(\alpha_0)(U,U)^{-1}}{1 - 2U(\alpha_0)(U,U^{\Lambda})(U,U)^{-1}} = S(\alpha_0). \tag{3.32}$$

coinciding with the scattering law formula (1.4) derived in [1] for another S-brane solution when scalar fields are absent and U is coinciding with the brane U-vector.

Due to (3.31) the inverse function S^{-1} is given by just the same relation

$$\alpha_0 = \frac{\alpha_{\infty} - 2\bar{U} U(\alpha_{\infty})(U,U)^{-1}}{1 - 2U(\alpha_{\infty})(U,U^{\Lambda})(U,U)^{-1}} = S^{-1}(\alpha_{\infty}). \tag{3.33}$$

3.3 Geometric meaning of the scattering law

Here we analyze the geometric meaning of the scattering for $n > 2$ as it was done in [1] for the S-brane solution.

The Kasner-like relations (1.2) and (1.3) describe an ellipsoid isomorphic to a unit $(n-2)$-dimensional sphere S^{n-2} belonging to $\mathbb{R}^{n-1}$. The sets of Kasner parameters α may be parametrized by vectors $\vec{n} \in S^{n-2}$, i.e. $\alpha = \alpha(\vec{n})$.

For $(U, U^{\Lambda}) \neq 0$ (or, equivalently, when $\sum_{i=1}^{n} U_i \neq 0$, see (2.14)) the scattering law formula (1.4) in terms of $\vec{n}$-vectors reads as in [1]

$$\vec{n}_{\infty} = \frac{(\vec{v}^2 - 1)\vec{n}_0 + 2(1 - \vec{v}\vec{n}_0)\vec{v}}{(\vec{v} - \vec{n}_0)^2} \tag{3.34}$$

where $\vec{v}$ is a vector belonging to $\mathbb{R}^{n-1}$ with $|\vec{v}| > 1$.

Here

$$\vec{v}\vec{n}_0 < 1 \qquad \vec{v}\vec{n}_{\infty} > 1, \tag{3.35}$$

for $(U, U^{\Lambda}) < 0$ (or, equivalently, when $\sum_{i=1}^{n} U_i > 0$) and

$$\vec{v}\vec{n}_0 > 1 \qquad \vec{v}\vec{n}_{\infty} < 1, \tag{3.36}$$

for $(U, U^{\Lambda}) > 0$ (or, equivalently, when $\sum_{i=1}^{n} U_i < 0$).

The vector $\vec{v} = (v_i) \in \mathbb{R}^{n-1}$ is defined by the formula

$$v_i = -\hat{U}_i / \hat{U}_0, \tag{3.37}$$

$i = 1, \dots, n-1$, where

$$\hat{U}_a = e_a^i U_i, \tag{3.38}$$

and the invertible matrix (e_i^a) satisfies the relations

$$\eta^{ab} = e_i^a G^{ij} e_j^b, \tag{3.39}$$

$a, b = 0, \dots, n-1$, with

$$e_i^0 = q^{-1} U_i^{\Lambda}, \tag{3.40}$$

and

$$q = [-(U^{\Lambda}, U^{\Lambda})]^{1/2} = [(D-1)/(D-2)]^{1/2}. \tag{3.41}$$

(Here $(\eta_{ab}) = (\eta^{ab}) = diag(-1, +1, \dots, +1)$.)

This implies

$$\hat{U}_0 = -q^{-1}(U, U^{\Lambda}) \tag{3.42}$$

and hence $\hat{U}_0 \neq 0$ when $(U, U^{\Lambda}) \neq 0$.

Relations (3.34), (3.35) and (3.36) could be readily proved from (3.31), (3.27) and (3.28) if the following "frame" Kasner-like parameters

$$\hat{\alpha}^a = e^a_i \alpha^i, \tag{3.43}$$

with

$$\hat{\alpha}^0 = q^{-1}, \qquad \hat{\alpha}^i = q^{-1} n^i, \tag{3.44}$$

$i = 1, \dots, n-1$, are used (see [1]). An important relation here is the following one

$$U(\alpha) = U_A \alpha^A = \hat{U}_a \hat{\alpha}^a = q^{-1} \hat{U}_0 (1 - \vec{v}\vec{n}). \tag{3.45}$$

Thus, for $(U, U^{\Lambda}) \neq 0$ we get just a modified inversion with respect to a point v located outside the Kasner sphere S^{n-2} (see Fig. 1). For $(U, U^{\Lambda}) < 0$ the function (3.34) maps a shadow part of the Kasner sphere S^{n-2} onto illuminated one, while for $(U, U^{\Lambda}) > 0$ this function maps an illuminated part of the Kasner sphere S^{n-2} onto shadow one. Here the shadow and illuminated parts of the Kasner sphere are defined w.r.t. a point-like source of light located at v.

Figure 1: *The graphical representation of the modified inversion S w.r.t. a point V for $n = 3$, and $(U, U^{\Lambda}) < 0$: $N' = S(N)$.*

For $(U, U^{\Lambda}) = 0$ (or, equivalently, when $\sum_{i=1}^{n} U_i = 0$) the main formula (1.4) in terms of $\vec{n}$-vectors reads

$$\vec{n}_{\infty} = \vec{n}_0 - 2(\vec{b}\vec{n}_0)\vec{b}, \tag{3.46}$$

where $\vec{b} = (b_i)$ is a unit vector belonging to $\mathbb{R}^{n-1}$ ($|\vec{b}| = 1$) with components

$$b_i = \hat{U}_i / (\sum_{j=1}^{n-1} \hat{U}_j^2)^{1/2}, \tag{3.47}$$

$i = 1, \dots, n-1$. The inequalities on Kasner-like parameters (3.27) and (3.28) in this case reads as follows

$$\vec{b}\vec{n}_0 > 0, \qquad \vec{b}\vec{n}_{\infty} < 0. \tag{3.48}$$

Thus, for $(U, U^{\Lambda}) = 0$ the function (3.34) is just a reflection with respect to a hyperplane $\{\vec{y} : \vec{b}\vec{y} = 0\}$, which contains a center of the Kasner sphere.

Relations (3.47) and (3.48) may be obtained from (3.34), (3.35) and (3.36) by means of the limiting procedure: $\hat{U}_0 \to \pm 0$ ($|\vec{v}| \to +\infty$.).

It should be noted that all formulas presented above are also valid for $n = 2$. In this case the zero-dimensional Kasner sphere $S^0 = \{-1, 1\}$ should be considered.

4 Example: $n = 2$

Here we consider the simplest case of the solution with $C > 0$, $K > 0$, when $n = 2$. We put $U_1 \neq 0$ and $U_2 = 0$, i.e. $\hat{p}_1 = w_1 \hat{\rho}$ with $w_1 \neq 1$ and $\hat{p}_2 = \hat{\rho}$.

For Kasner set $\alpha = (\alpha^1, \alpha^2)$ we get from (1.2) and (1.3) [16, 30]

$$\alpha_{\pm} = (\alpha^1_{\pm}, \alpha^2_{\pm}) = \frac{1}{d_1 + d_2} \left(1 \pm \frac{r}{d_1}, 1 \mp \frac{r}{d_2} \right), \tag{4.1}$$

where $r = \sqrt{d_1 d_2 (d_1 + d_2 - 1)}$. (The number $r > 0$ is integer one when $d_1 = 1$ or $d_2 = 1$ and also for $(d_1, d_2) = (3,6), (5,5), (2,8), (13,13)$ etc [30].)

Let $d_2 > 1$. Then $\alpha^1_+ > 0$ and $\alpha^1_- < 0$. Due to $U_2 = 0$: $U(\alpha) = U_1 \alpha^1$ and hence $U(\alpha_+) > 0$ and $U(\alpha_-) < 0$ for $U_1 > 0$ ($w_1 < 1$) and $U(\alpha_+) < 0$ and $U(\alpha_-) > 0$ for $U_1 < 0$ ($w_1 < 1$).

It follows from (3.27) and (3.28) that

$$\alpha_0 = \alpha_+, \qquad \alpha_\infty = \alpha_-. \tag{4.2}$$

for $U_1 > 0$ and

$$\alpha_0 = \alpha_-, \qquad \alpha_\infty = \alpha_+. \tag{4.3}$$

for $U_1 < 0$.

Relation $U_i c^i = U_1 c^1 = 0$ implies $c^1 = 0$. Here $c^0 = d_2 c^2$. Due to (3.21) and (3.22) we should put $c^2 < 0$ for $U_1 > 0$ and $c^2 > 0$ for $U_1 < 0$. In this case the sets $\alpha_{\pm}$ given by (3.18) are coinciding with those given by (4.1). This may be also verified by straightforward calculations using the following relations

$$U^1 = \frac{(d_2 - 1) U_1}{d_1 (D - 2)}, \qquad U^2 = \frac{U_1}{(2 - D)} = (U, U^{\Lambda}), \tag{4.4}$$

$$K = U^1 U_1, \qquad C = K d_2 (d_2 - 1) c_2^2, \tag{4.5}$$

where $D = d_1 + d_2 + 1 > 3$.

Accelerated expansion of 3-dimensional factor-space. After replacing $\tau \to \tau_0 - 0$, where τ_0 is constant, we get for $w = -1$ two asymptotical Kasner type metrics

$$g_{as} = -d\tau \otimes d\tau + \sum_{i=1}^{2} A_i^2 (\tau_0 - \tau)^{2\alpha^i} g^{(i)}, \tag{4.6}$$

where either $\alpha^i = \alpha_0^i$ ($A_i = A_{i,0} > 0$) as $\tau \to \tau_0 - 0$, or $\alpha^i = \alpha_\infty^i$ ($A_i = A_{i,\infty} > 0$) as $\tau \to -\infty$.

Let M_1 be a flat 3-dimensional factor space ($d_1 = 3$), with the metric $g^{(1)} = dy^1 \otimes dy^1 + dy^2 \otimes dy^2 + dy^3 \otimes dy^3$. Then, due to relations (4.2), (4.3) and $\alpha_\infty < 0$ for $d_2 > 1$, we get an asymptotical accelerated expansion of our 3-dimensional factor space M_1 either as $\tau \to \tau_0 - 0$ for $U_1 < 0$, $c_2 > 0$ or as $\tau \to -\infty$ for $U_1 > 0$ and $c_2 < 0$.

Milne-type asymptotics. Now we put $d_1 = 1$. We get

$$\alpha_+ = (1, 0), \qquad \alpha_- = \frac{1}{1 + d_2}(1 - d_2, 2). \tag{4.7}$$

For $M_1 = \mathbb{R}$, $g^{(1)} = -w dy^1 \otimes dy^1$, $-\infty < y^i < +\infty$, we get a Milne-type (flat) asymptotic:

i) as $\tau \to +0$ for $U_1 > 0$ and $c^2 < 0$;

ii) as $\tau \to +\infty$ for $U_1 < 0$ and $c^2 > 0$.

Both cases correspond to $u \to +\infty$.

For $M_1 = S^1$, $g^{(1)} = w dy^1 \otimes dy^1$, $0 < y^i < +2\pi$, we may get either non-singular (static) solution in the case i) ($\tau = \rho$) or asymptotically flat (static) solution in the case ii).

5 Conclusions and discussions

In this paper we have considered the exact cosmological type solution with 1-component anisotropic fluid. This solution is defined on the product manifold (2.2) containing n Ricci-flat factor spaces $M_1, ..., M_n$.

We have singled out a special solution governed by the *cosh* moduli function and shown that this solution has Kasner-like asymptotics in the limits $u \to \pm\infty$, where u is the harmonic variable, or, equivalently, in the limits $\tau \to +0$ and $\tau \to +\infty$, where τ is the synchronous type variable.

We have found a relation between two sets of Kasner parameters α_∞ and α_0. The relation between them $\alpha_\infty = S(\alpha_0)$ is coinciding with the "scattering law" formula obtained for the S-brane solution from [1] when scalar fields are absent and the fluid U-vector is equal to the brane one.

The function S (defined on the set of Kasner vectors obeying $U(\alpha) > 0$) is bijective. The inverse function S^{-1} (defined on the set of Kasner vectors obeying $U(\alpha) < 0$) is given by the same formula as the function S. The function S depends upon the co-vector $U = (U_i)$. It is invariant upon the replacement: $U \mapsto \lambda U$, where $\lambda > 0$ (see [26]). The transformation $U \mapsto -U$ implies the replacement $S \mapsto S^{-1}$.

We have also analyzed the geometric meaning of the scattering law formula in terms of transformation of the Kasner sphere S^{n-2}, $n \geq 2$. For $(U, U^\Lambda) \neq 0$ (or, equivalently, when $\sum_{i=1}^n U_i \neq 0$) we get just a modified inversion with respect to a point v located outside the Kasner sphere S^{n-2}, while for $(U, U^\Lambda) = 0$ (or, equivalently, when $\sum_{i=1}^n U_i \neq 0$) we are led to a reflection with respect to a hyperplane which contains a center of the Kasner sphere.

The scattering law formula may be applied for the solutions with Kasner-like asymptotical behaviours (written in a slightly different form)

$$g_{as} = -d\tau \otimes d\tau + \sum_{i=1}^{n} A_i^2 (\tau_0 - \tau)^{2\alpha^i} g^{(i)}, \tag{5.1}$$

where either $\tau \to \tau_0 - 0$, or $\tau \to -\infty$. In this case the metric (5.1) may describe an asymptotical accelerated expansion of flat 3-dimensional factor space M_1 if $d_1 = 3$, $g^{(1)} = dy^1 \otimes dy^1 + dy^2 \otimes dy^2 + dy^3 \otimes dy^3$ and $\alpha^1 < 0$.

Another application of the scattering law formula appears when $d_1 = 1$ and one of the asymptotical Kasner set of parameters in (1.1) is of Milne type: $\alpha = (1, 0, \dots, 0)$, e.g. when static non-singular solutions ($w = +1$, $M_1 = S^1$) or cosmological solutions ($w = -1$, $M_1 = \mathbb{R}$) with a horizon (for $\tau \to +0$) are considered. (Compare with flux-brane and S-brane solutions [31, 32]). These topics (mentioned above) may be a subject of separate publications.

Appendix

A Solution for Liouville system

Let

$$L=\frac{1}{2}<\dot{x},\dot{x}>-A\exp[2<b,x>] \tag{A.1}$$

be a Lagrangian, defined on $V\times V$, where $V=\mathbb{R}^n$, $A\neq 0$, and $<\cdot,\cdot>$ is non-degenerate real-valued quadratic form on V . (Here $\dot{x}=\frac{dx}{dt}$ etc.)

Let $<b,b>\neq 0$. Then, the Euler-Lagrange equations for the Lagrangian (A.1)

$$\ddot{x}+2Ab\exp[2<b,x>]=0 \tag{A.2}$$

have the following solution [13]

$$x(t)=-\frac{b}{<b,b>}\ln|f(t-t_0)|+t\alpha+\beta, \tag{A.3}$$

where $\alpha,\beta\in V$,

$$<\alpha,b>=<\beta,b>=0, \tag{A.4}$$

and

$$f(\tau)=\begin{array}{lll} R\sinh(\sqrt{C}\tau), & C>0, & <b,b>A<0, \\ R\sin(\sqrt{|C|}\tau), & C<0, & <b,b>A<0, \\ R\cosh(\sqrt{C}\tau), & C>0, & <b,b>A>0, \\ |2A<b,b>|^{1/2}\tau, & C=0, & <b,b>A<0, \end{array} \tag{A.5}$$

where $R=|\frac{2A<b,b>}{C}|^{1/2}$ and C , t_0 are constants.

The energy

$$E=\frac{1}{2}<\dot{x},\dot{x}>+A\exp[2<b,x>] \tag{A.6}$$

calculated for the solution (A.2) reads

$$E=\frac{C}{2<b,b>}+\frac{1}{2}<\alpha,\alpha>. \tag{A.7}$$

B Lagrange representation

The Einstein equations (2.1) imply the conservation law

$$\nabla_M T^M_N = 0. \tag{B.8}$$

that due to relations (2.3) and (2.4) may be written in the following form

$$\dot{\hat{\rho}} + \sum_{i=1}^{n} d_i \dot{\beta}^i (\hat{\rho} + \hat{p}_i) = 0. \tag{B.9}$$

Using the equation of state (2.5) we get

$$\kappa^2 \hat{\rho} = -wAe^{2U_i\beta^i - 2\gamma_0}, \tag{B.10}$$

where $\gamma_0(\beta) = \sum_{i=1}^{n} d_i\beta^i$, and A is constant.

The Einstein equations (2.1) with the relations (2.5) and (B.10) imposed are equivalent to the Lagrange equations for the Lagrangian (for $w = -1$ see [14])

$$L = \frac{1}{2} e^{-\gamma+\gamma_0(\beta)} G_{ij}\dot{\beta}^i\dot{\beta}^j - e^{\gamma-\gamma_0(\beta)} V, \tag{B.11}$$

where

$$V = Ae^{2U_i\beta^i}, \tag{B.12}$$

is the potential and the components of the minisupermetric G_{ij} are defined in (2.8).

For $\gamma = \gamma_0(\beta)$, i.e. when the harmonic time gauge is considered, we get the set of Lagrange equations for the Lagrangian

$$L = \frac{1}{2} G_{ij}\dot{\beta}^i\dot{\beta}^j - V, \tag{B.13}$$

with the zero-energy constraint imposed

$$E = \frac{1}{2} G_{ij}\dot{\beta}^i\beta^j + V = 0. \tag{B.14}$$

C The solution

The exact solutions for the Lagrangian (B.13) with the potential (B.12) could be readily obtained using the relations from Appendices **A** and **B**.

The solutions read:

$$\beta^i(u) = -\frac{U^i}{(U,U)} \ln |f(u)| + c^i u + \bar{c}^i, \tag{C.15}$$

where u_0 is constant. Function $f(u)$ in (C.15) is the following:
function reads

$$f(u) = R \sinh(\sqrt{C}(u-u_0)),\ C>0,\ KA<0; \tag{C.16}$$
$$R \sin(\sqrt{|C|}(u-u_0)),\ C<0,\ KA<0; \tag{C.17}$$
$$R \cosh(\sqrt{C}(u-u_0)),\ C>0,\ KA>0; \tag{C.18}$$
$$|2AK|^{1/2}(u-u_0),\ C=0,\ KA<0, \tag{C.19}$$

where $K=(U,U)$, $R=|2AK/C|^{1/2}$ and C, u_0 are constants.

Vectors $c=(c^i)$ and $\bar{c}=(\bar{c}^i)$ satisfy the linear constraint relations (see (A.4) in Appendix **A**)

$$U(c) = U_i c^i = 0, \tag{C.20}$$
$$U(\bar{c}) = U_i \bar{c}^i = 0. \tag{C.21}$$

The zero-energy constraint reads (see (A.6) in Appendix **A**)

$$E = \frac{C}{2(U,U)} + \frac{1}{2} G_{ij} c^i c^j = 0. \tag{C.22}$$

D Proof of the inequality (3.11)

Let us prove the inequality (3.11)

$$|(s,U^\Lambda)| > \frac{|(U^\Lambda,U)|}{(U,U)} > 0,$$

for a vector $s=(s^A) \in \mathbb{R}^n$ obeying relations $(s,U)=0$, $(s,s) = -1/(U,U)$. Here the scalar-product $(U,U') = G^{ij}U_i U'_j$, where $G^{ij} = \delta^{ij} d_i^{-1} + (2-D)^{-1}$. We also use here the following relations $(U,U)>0$, $U^\Lambda = (d_i)$ and $(U^\Lambda, U^\Lambda) < 0$.

Proof. Let us define the vector

$$U_1 = U - \frac{(U, U^\Lambda)}{(U^\Lambda, U^\Lambda)} U^\Lambda. \qquad \text{(D.23)}$$

It is clear that $(U_1, U^\Lambda) = 0$ and

$$(U_1, U_1) = (U, U) - \frac{(U, U^\Lambda)^2}{(U^\Lambda, U^\Lambda)} > 0. \qquad \text{(D.24)}$$

since $(U, U) > 0$ and $(U^\Lambda, U^\Lambda) < 0$. Let us define vectors:

$$s_0 = \frac{(s, U^\Lambda)}{(U^\Lambda, U^\Lambda)} U^\Lambda, \qquad \text{(D.25)}$$

$$s_1 = \frac{(s, U_1)}{(U_1, U_1)} U_1, \qquad \text{(D.26)}$$

$$s = s - s_0 - s_1. \qquad \text{(D.27)}$$

s_0, s_1 and s_2 are mutually orthogonal and hence

$$(s, s) = (s_0, s_0) + (s_1, s_1) + (s_2, s_2). \qquad \text{(D.28)}$$

For the first two terms in r.h.s. of (D.28) we get

$$(s_0, s_0) = \frac{(s, U^\Lambda)^2}{(U^\Lambda, U^\Lambda)}, \qquad \text{(D.29)}$$

$$(s_1, s_1) = \frac{(s, U_1)^2}{(U_1, U_1)} = \frac{(s, U^\Lambda)^2}{(U^\Lambda, U^\Lambda)} \frac{(U, U^\Lambda)^2}{[(U, U)(U^\Lambda, U^\Lambda) - (U, U^\Lambda)^2]} \qquad \text{(D.30)}$$

that implies

$$(s, s) = \frac{(s, U^\Lambda)^2 (U, U)}{(U, U)(U^\Lambda, U^\Lambda) - (U, U^\Lambda)^2} + (s_2, s_2). \qquad \text{(D.31)}$$

For the third term in r.h.s. of (D.28) the following inequality is valid

$$(s_2, s_2) \geq 0, \qquad \text{(D.32)}$$

Indeed, due to $(s_2, U^\Lambda) = 0$, or, equivalently, $\sum_{i=1}^n s_2^i d_i = 0$, we obtain

$$(s_2, s_2) = G_{ij} s_2^i s_2^j = \sum_{i=1}^n (s_2^i)^2 d_i \geq 0. \qquad \text{(D.33)}$$

Using this inequality, (D.31), $(U^\Lambda, U^\Lambda) < 0$ and $(s,s) = -1/(U,U)$ we get

$$(s, U^\Lambda)^2 = [\frac{(U,U^\Lambda)^2}{(U,U)} - (U^\Lambda, U^\Lambda)][(U,U)^{-1} + (s_2, s_2)] > \frac{(U,U^\Lambda)^2}{(U,U)^2} > 0, \quad (D.34)$$

that is equivalent to the inequality (3.11). Thus, (3.11) is proved.

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