The entropy and mean separation between energy levels of black hole

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Abstract

According to the widely accepted statistical interpretation of black hole entropy the mean separation between energy levels of black hole should be exponentially small. But this sharply disagrees with the value obtained from the quantization of black hole area. It is shown that the new statistical interpretation of black hole entropy proposed in my paper arXiv:0911.5635 gives the correct value.

According to Bekenstein [1], quantization of the black hole area means that the area spectrum of black hole is of the form

$$A_n = \Delta A \cdot n, \quad n = 0, 1, 2, ..., \tag{1}$$

where ΔA is the quantum of black hole area. Despite this, there is still no general agreement on the precise value of ΔA; in the literature there are several alternative proposals for ΔA (see, for example, [2] and references therein). From (1) the mass spectrum follows; for example, in the case where $\Delta A = 8\pi l_P^2$ (here the precise value of ΔA is not crucial for the form of the spectrum) the mass (energy) of a black hole is

$$M_n = m_P \sqrt{\frac{n}{2}} \tag{2}$$

which for large n ($n \gg 1$) gives the energy spacing

$$\Delta M_n = \frac{M_n}{2n} = \frac{m_P^2}{4M_n}. \tag{3}$$

This value however does not agree with an estimation obtained from the definition of entropy. As is well known, the entropy of an ordinary system S, by definition, is the logarithm of the number of states W with energy between E and $E + \delta E$

$$S = \ln W. \tag{4}$$

The width δE is some energy interval characteristic of the limitation in our ability to specify absolutely precisely the energy of a macroscopic system. Dividing δE by the number of states $\exp(S)$ we obtain the mean separation between energy levels of the system [3]:

$$\langle \Delta M \rangle \sim \delta E \exp(-S). \tag{5}$$

The interval δE is equal in order of magnitude to the mean canonical-ensemble fluctuation of energy of a system. However, there exist problems with the description of a black hole in a canonical ensemble. For example, because a black hole has negative specific heat C_v, $C_v = -8\pi G M^2$, energy fluctuations calculated in the canonical ensemble have formally negative variance: $\langle (\delta E)^2 \rangle = C_v T_H^2 \sim -m_P^2$, where T_H is the Hawking temperature. The situation is quite different if a black hole is part of a thermodynamical system which has a finite size. For example, if a black hole is placed in a reservoir of radiation and the total energy of the system is fixed, a stable equilibrium configuration can exist. It appears that the equilibrium is stable if the radiation and black hole temperatures coincide, $T_{rad} = T_H \equiv T$, and $E_{rad} < M/4$, where E_{rad} is the energy of radiation. The latter condition can be reformulated as the restriction on the volume of reservoir V, $4aVT^5 < 1$, where a is the radiation constant. According to this condition Pavon and Rubï found [4] that the mean square fluctuations of the black hole energy (mass) is given by

$$\langle (\delta E)^2 \rangle = (1/8\pi) T^2 Z, \tag{6}$$

Z being the quantity $4aVT^3/(1-4aVT^5)$, $G = c = \hbar = 1$ and the Boltzmann constant $k_B = (8\pi)^{-1}$. It is clear that (restoring G, c, $\hbar$, and k_B)

$$\langle (\delta E)^2 \rangle \sim \frac{m_P^4}{M^2} \tag{7}$$

and

$$\delta E \sim \frac{m_P^2}{M}. \tag{8}$$

So we obtain the mean separation between energy levels for a black hole [5]

$$\langle \Delta M \rangle \sim \frac{m_P^2}{M} \exp(-S). \tag{9}$$

This value is however exponentially smaller than (3). Thus a problem arises. It has not yet received attention in the literature.

In this note I propose a solution of the problem. The point is as follows. In the quantum mechanical description, the accuracy with which the mass of a black hole can be defined by a distant observer is limited by the time-energy uncertainty relation as well as by the decrease of the mass of the black hole due to transition from a higher energy level to a lower one. The lifetime of a state M_n is proportional to the inverse of the imaginary part of the effective action [6]; less formally, it is the time needed to emit a single Hawking quantum and this is proportional to the gravitational radius R_g. So $\delta E_q \sim 1/R_g$, where I have added the subscript "q" to refer to the quantum uncertainty. On the other hand, $\delta E_q \sim T_H$ for the transition from state n to state $n-1$. It is obvious that the energy interval δE_q contains only a single state. As is well known, in this case statistics is not applicable; by definition the statistical treatment is possible only if δE contains many quantum states. It is clear that δE_q is of the same order of magnitude as (8). This means that the formula (9) is not applicable.

Nevertheless, since the mean separation between energy levels is nothing but the energy per unit state, we can define it as follows. In [7] it was noticed that in contrast to alternative values, the quantum of area $\Delta A = 8\pi l_P^2$ does not follow from the accepted statistical interpretation of black hole entropy of the form (4); on the contrary, a new statistical interpretation follows from it. Namely, in [7] it was shown that if the number of microstates accessible to a black hole is to be an integer and the entropy of the black hole is to satisfy the generalized second law of thermodynamics, the black hole entropy should be of the form

$$S = 2\pi W, \tag{10}$$

that is, in contrast to (4), without logarithm, and the number of microstates is $W \equiv n = A/8\pi l_P^2$. So we can define

$$\langle \Delta M \rangle \sim \frac{\Delta M}{\Delta n} \sim \frac{dM}{dS} = T_H \sim \frac{m_P^2}{M}; \tag{11}$$

this agrees with (3), as it should. Thus the first law of thermodynamics as well as entropy defines the energy spacing of a black hole.

References

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