Towards a more exact value of deflection of light due to static gravitational mass

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Abstract The deflection of a ray of light passing close to a gravitational mass, is generally calculated from the null geodesic which the light ray (photon) follows. However, there is an alternate approach, where the effect of gravitation on the ray of light is estimated by considering the ray to be passing through a *material medium*. Calculations have been done in this paper, following the later approach, to estimate the amount of deflection due to a static non-rotating mass. The refractive index of such a material medium, has been calculated in a more rigorous manner in the present work and the final expression for the amount of deflection calculated here is claimed to be more accurate than all other expressions derived so far using *material medium* approach. Based on this expression, the amount of deflection for a sun grazing ray has been also calculated.

1 Introduction

The gravitational deflection of light is one of the important predictions of the General Theory of Relativity (GTR) proposed by Einstein, which plays a key role in understanding problems related to Astronomy, Cosmology, Gravitational Physics and other related branches.

Newtons theory of universal gravitation had already predicted that the path of any material particle moving at a finite speed is affected by the pull of gravity. By the late 18th century, it was possible to apply Newtons law to compute the deflection of light by gravity. Cavendish commented briefly on the gravitational deflection of light in the late 1700s and Soldner gave a detailed derivation in 1801.

The idea of bending of light was revived by Einstein in 1911 and the quantitative prediction for the amount of deflection of light passing near a large mass (M) was identical to the old Newtonian prediction, $d = 2GM/(c^2 r_\odot)$, where $r_\odot$ is the closet distance of approach and in this case approximately the solar radius. It wasn't until late in 1915, as Einstein completed the general theory, he realized his earlier prediction was incorrect and the angular deflection should actually be twice the size he predicted in 1911. This was subsequently confirmed by Eddington in 1919 through an experiment performed during the solar eclipse.

The exact amount of deflection for a ray of light passing close to a gravitational mass can be worked out from the null geodesic, which a ray of light follows [1,2,3].

The deflection of a light ray passing close to a gravitational mass can be alternately calculated by following an approach, where the effect of gravitation on the light ray is estimated by considering the light ray to be passing through a material medium with a value of refractive index decided by the value of gravitational field [4].

The concept of this *equivalent material medium* was discussed by Balazs [5] as early as in 1958, to calculate the effect of a rotating body, on the polarization of an electromagnetic wave passing close to it. Plebanski [6] had also utilized this concept in 1960, to study the scattering of a plane electromagnetic wave by gravitational field, where the author mentioned that this concept of *equivalent material medium* was first pointed out by Tamm [7] in 1924. A general procedure for utilizing this concept, for deflection calculation has been worked out by Felice [8]. Later this concept was also used by Mashoon [9,10], to calculate the deflection and polarization due to the Schwarzschild and Kerr black holes. Fischbach and Freeman [11], derived the effective refractive index of the material medium and calculated the second order contribution to the gravitational deflection. In a similar way Sereno [12] has used this idea, for gravitational lensing calculation by drawing the trajectory of the ray by Fermat's principle. More recently Ye and Lin [13], emphasized the simplicity of this approach and calculated the gravitational time delay and the effect of lensing.

On the other hand, the calculation of higher order deflection terms, due to Schwarzschild Black hole, from the null geodesic, has been performed recently by Iyer and Petters ([14] and references their in). Using null geodesics, gravitational lensing calculations have been done by a number of authors in past [15,16].

With the above background, in the present work, we follow the *material medium* approach, to calculate a more accurate value for the deflection term due to a non-rotating sphere (Schwarzschild geometry). It is claimed that the present calculated value will be more accurate than all other values calculated in past, using *material medium* concept.

2 The effective refractive index and the trajectory of light ray

As discussed earlier, the gravitational field influences the propagation of electromagnetic radiation by imparting to the space an effective index of refraction $n(r)$[4].

For a static and spherically symmetric gravitational field, the solution of Einstein's Field Equation was given by K. Schwarzschild in 1961, which is as follows[4]:

$$ds^2 = (1 - \frac{r_g}{r})c^2dt^2 - r^2(sin^2\theta d\phi^2 + d\theta^2) - \frac{dr^2}{(1 - \frac{r_g}{r})} \tag{1}$$

where $r_g = \frac{2km}{c^2}$ called Schwarzschild Radius, which completely defines the gravitational field in vacuum produced by any centrally-symmetric distribution of masses. The above equation can be expressed in an isotropic form by introducing a new radius co-ordinate (ρ) with the following transformation equation [4]

$$\rho = \frac{1}{2}[(r - \frac{r_g}{2}) + r^{1/2}(r - r_g)^{1/2}] \tag{2}$$

OR

$$r = \rho(1 + \frac{r_g}{4\rho})^2 \tag{3}$$

The resulting isotropic form of Schwarzschild equation will be now:

$$ds^2 = (\frac{1 - r_g/(4\rho)}{1 + r_g/(4\rho)})^2c^2dt^2 - (1 + \frac{r_g}{4\rho})^4(d\rho^2 + \rho^2(sin^2\theta d\phi^2 + d\theta^2)) \tag{4}$$

Now in spherical co-ordinate system the quantity $(d\rho^2 + \rho^2(sin^2\theta d\phi^2 + d\theta^2))$ has the dimension of square of infinitesimal length vector $d\overrightarrow{\rho}$.

By setting $ds = 0$, the velocity of light can be identified from the expression of the form $ds^2 = f(\rho)dt^2 - d\overrightarrow{\rho}^2$, as $v(\rho) = \sqrt{f(\rho)}$. Therefore the velocity of light in the present case (characterized by Schwarzschild radius r_g) can be expressed as :

$$v(\rho) = \frac{(1 - \frac{r_g}{4\rho})c}{(1 + \frac{r_g}{4\rho})^3} \tag{5}$$

But this above expression of velocity of light is in the unit of length ρ per unit time. We therefore write

$$\begin{aligned} v(r) = \quad & v(\rho)\frac{dr}{d\rho} \\ & = v(\rho)[(1 + \frac{r_g}{4\rho})^2 - \frac{r_g}{2\rho}(1 + \frac{r_g}{4\rho})] \\ & = (\frac{r_g - 4\rho}{r_g + 4\rho})^2 c \end{aligned} \tag{6}$$

Substituting the value of ρ from Eqn (2) in Eqn.(6), we get:

$$\begin{aligned} v(r) = \quad & (\frac{r_g/2 - 2\rho}{r_g/2 + 2\rho})^2 c \\ = \quad & (\frac{r_g/2 - ((r - \frac{r_g}{2}) + r^{1/2}(r - r_g)^{1/2})}{r_g/2 + ((r - \frac{r_g}{2}) + r^{1/2}(r - r_g)^{1/2})})^2 c \\ = \quad & (\frac{r_g - r - r^{1/2}(r - r_g)^{1/2}}{r + r^{1/2}(r - r_g)^{1/2}})^2 c \\ = \quad & \frac{c(r - r_g)}{r} \end{aligned} \tag{7}$$

Therefore the refractive index $n(r)$ at a point with spherical polar coordinate (r), can be expressed by the relation:

$$n(r) = \frac{c}{v(r)} = \frac{r}{r - r_g} \tag{8}$$

At this stage the entire problem, can become a problem of geometrical optics, where we have to find the trajectory of a light ray travelling in a medium, whose refractive index has spherical symmetry. The trajectory of the light ray and the center of mass (source of gravitational potential) will together define a plane. The equation of such a ray in plane polar co-ordinate system (r, θ) can be written as [17]:

$$\theta = A. \int_{r_{\odot}}^{\infty} \frac{dr}{r\sqrt{n^2 r^2 - A^2}} \tag{9}$$

The trajectory is such that $n(r).d$ always remains a *constant*, where d is the perpendicular distance between the trajectory of the light ray from the origin and the *constant* is taken here as A [17]. In our present problem the light is coming from infinity ($r = -\infty$) and it is approaching the gravitational mass, which is placed at the origin and characterized by Schwarzschild radius r_g. The closest distance of approach, for the approaching ray is b and the ray goes to $r = \infty$, after undergoing certain amount of deflection $(\triangle\phi)$.

Here, the parameter b can be replaced by solar radius $r_{\odot}$. When the light ray passes through the closest distance of approach (ie $r = b$ or $r_{\odot}$), the tangent to the trajectory becomes perpendicular to the vector $\overrightarrow{r}$ (which is $\overrightarrow{r}_{\odot}$). Therefore, we can write $A = n(r_{\odot})r_{\odot}$. The trajectory of the light ray had been already constructed before like this, by Ye and Lin [13] and the value of deflection $(\triangle\phi)$, can be written as :

$$\triangle\phi = 2 \int_{r_{\odot}}^{\infty} \frac{dr}{r\sqrt{(\frac{n(r).r}{n(r_{\odot}).r_{\odot}})^2 - 1}} - \pi \tag{10}$$

However, Ye and Lin [13], had in our opinion used a value of refractive index $n(r)$ which was approximated and somewhat ad hoc. Fischbach and Freeman [11] also in their attempt to calculate a more accurate value of deflection, considered terms only up to second order in the expression for refractive index. However, in our attempt to do so we shall avoid making any such approximation in the following. We denote the above integral in Eqn. (10) by I and write

$$
\begin{aligned}
I = \quad & \int_{r_\odot}^{\infty} \frac{dr}{r\sqrt{(\frac{n(r).r}{n(r_\odot).r_\odot})^2 - 1}} \\
&= n(r_\odot)r_\odot \int_{r_\odot}^{\infty} \frac{dr}{r\sqrt{(n(r).r)^2 - (n(r_\odot).r_\odot)^2}} \\
&= n(r_\odot)r_\odot \int_{r_\odot}^{\infty} \frac{dr}{r\sqrt{\frac{r^4}{(r-r_g)^2} - \frac{r_\odot^4}{(r_\odot - r_g)^2}}} \\
&= n(r_\odot)r_\odot \int_{r_\odot}^{\infty} \frac{dr}{r^2\sqrt{\frac{1}{(1-\frac{r_g}{r})^2} - \frac{r_\odot^2 r^{-2}}{(1-\frac{r_g}{r_\odot})^2}}} \qquad (11)
\end{aligned}
$$

Now we change the variable to $x = \frac{r_g}{r}$ and introduce a quantity $a = \frac{r_g}{r_\odot}$. We also denote $n(r_\odot)$ by $n_\odot$. Accordingly we write:

$$
\begin{aligned}
I = \quad & n_\odot r_\odot \int_a^0 \frac{-x^{-2} r_g dx}{r^2\sqrt{\frac{1}{(1-x)^2} - \frac{x^2}{(a(1-a))^2}}} \\
&= n_\odot r_\odot \int_a^0 \frac{-x^{-2} r_g dx}{x r^2\sqrt{\frac{1}{(x(1-x))^2} - \frac{1}{(a(1-a))^2}}} \\
&= \frac{n_\odot r_\odot}{r_g} \int_0^a \frac{dx}{x\sqrt{\frac{1}{(x(1-x))^2} - \frac{1}{(a(1-a))^2}}} \\
&= \frac{n_\odot r_\odot}{r_g} \int_0^a \frac{(1-x)dx}{\sqrt{1 - \frac{(x(1-x))^2}{(a(1-a))^2}}} \qquad (12)
\end{aligned}
$$

For our convenience we can denote the quantity $1/(a(1-a))$ by D. This also implies

$$
D = \frac{r_\odot^2}{r_g(r_\odot - r_g)} \qquad (13)
$$

However, the Integral I can not be solved as a standard integral at this stage. We split the above Integral, as a sum of two Integrals and proceed as follows:

$$
\begin{aligned}
I = \quad & (\frac{n_{\odot} r_{\odot}}{r_g})[\int_0^a \frac{(1-2x)dx}{\sqrt{1-D^2x^2(1-x)^2}} + \int_0^a \frac{xdx}{\sqrt{1-D^2x^2(1-x)^2}}] \\
& = (\frac{n_{\odot} r_{\odot}}{r_g}) \int_0^a \frac{(1-2x)dx}{\sqrt{1-D^2x^2(1-x)^2}} + (\frac{n_{\odot} r_{\odot}}{r_g}) \int_0^a \frac{xdx}{\sqrt{1-D^2x^2(1-x)^2}} \\
& = (\frac{n_{\odot} r_{\odot}}{r_g}) I_1 + (\frac{n_{\odot} r_{\odot}}{r_g}) I_2 \qquad (14)
\end{aligned}
$$

where I_1 and I_2 are used to denote the above two integrals. Now we can identify

$$\frac{n_{\odot} r_{\odot}}{r_g} = \frac{1}{1-a}.\frac{1}{a} = \frac{1}{a(1-a)} = D$$

Changing the variable from x to $y = Dx(1-x)$, we can write $D(1-2x)dx = dy$. Accordingly the upper and lower limits $x = 0$ and $x = a$ change to $y = 0$ and $y = Da(1 - \frac{r_g}{r_{\odot}}) = \frac{1}{a(1-a)} a(1-a) = 1$. Therefore for the first part in Eqn (14) we can write :

$$
\begin{aligned}
(\frac{n_{\odot} r_{\odot}}{r_g}) I_1 = \quad & \int_0^a \frac{D(1-2x)dx}{\sqrt{1-D^2x^2(1-x)^2}} \\
& = \int_0^1 \frac{dy}{\sqrt{1-y^2}} \\
& = [sin^{-1} y]_0^1 \\
& = \pi/2 \qquad (15)
\end{aligned}
$$

Therefore, from Eqn (10), one may write the amount of deflection as:

$$\triangle\phi = \quad 2\int_{r_{\odot}}^{\infty} \frac{dr}{r\sqrt{(\frac{n(r).r}{n(r_{\odot}).r_{\odot}})^2 - 1}} - \pi$$

$$= 2(\frac{n_{\odot} r_{\odot}}{r_g}) I_1 + 2(\frac{n_{\odot} r_{\odot}}{r_g}) I_2 - \pi$$

$$= \pi + 2(\frac{n_{\odot} r_{\odot}}{r_g}) I_2 - \pi$$

$$= (\frac{2 n_{\odot} r_{\odot}}{r_g}) \int_0^a \frac{x dx}{\sqrt{1 - D^2 x^2 (1-x)^2}} \tag{16}$$

Thus the gravitational bending for a ray of light grazing the static gravitational mass (with Schwarzschild radius r_g) with the closest distance of approach $r_{\odot}$ can be expressed as:

$$\triangle\phi = 2D \int_0^a \frac{x dx}{\sqrt{1 - D^2 x^2 (1-x)^2}} \tag{17}$$

The above expression for gravitational deflection has been obtained from the Schwarzschild Equation (Eqn(1)), with out applying any approximation at any stage. Owing to this, it is claimed that this expression of bending is more exact as compared to all other expressions derived till today, using *equivalent material medium* concept. However, the integration of the quantity in Eqn (17)), involves some complicated algebraic expressions containing Elliptical functions. Using Mathematica, we obtain the following expression after integration :

$$\int \frac{x dx}{\sqrt{1 - D^2 x^2 (1-x)^2}} = 2 \frac{(\sqrt{D} + \sqrt{D-4})E - (2\sqrt{D-4})F}{D(\sqrt{D+4} - \sqrt{D-4})} \tag{18}$$

where $E \equiv E(p, q^2)$ is the Elliptic Integral of first kind and $F \equiv F(-q, p, q^2)$ is Incomplete Elliptic Integral of Third kind. The arguments p,q^2,-q,p,q^2 are expressed by the following mathematical relations:

$$p = \arcsin \sqrt{\frac{(\sqrt{D-4} - \sqrt{D+4})(\sqrt{D-4} + (2x-1)\sqrt{D})}{(\sqrt{D-4} + \sqrt{D+4})(\sqrt{D-4} - (2x-1)\sqrt{D})}} \tag{19}$$

$$q = \frac{(\sqrt{D-4}+\sqrt{D+4})}{(\sqrt{D-4}-\sqrt{D+4})} \tag{20}$$

Finally we can write the expression for gravitational deflection ($\triangle\phi$) of the light ray, due to a static mass (r_g) with the closest distance of approach $r_\odot$ as :

$$\triangle\phi = 4\left\{\frac{(\sqrt{D}+\sqrt{D-4})E-(2\sqrt{D-4})F}{(\sqrt{D+4}-\sqrt{D-4})}\right\}_{x=0}^{x=a} \tag{21}$$

where the value of D is given by Eqn.(13) as $D = \frac{r_\odot^2}{r_g(r_\odot - r_g)}$ and $a = r_g/r_\odot$. Eqn. (21) is a general expression for bending of light, where $r_\odot$ can be replaced by the closest distance of approach of the light ray. This mathematical expression for deflection, derived here is claimed to be more accurate than all other expressions derived so far using *material medium* approach and it is equally valid for strong field. For a Sun grazing ray, we can take the closest distance of approach as equal to solar radius which is $r_\odot = 695,500$ km and Schwarzschild radius corresponding to the mass of Sun as $r_g = 3$ km. We, therefore, get $a = (r_g/r_\odot) = 1/231,833$ and $D = 231,834$. Finally, we get a value of $\triangle\phi = 8.62690E10^{-6}$ radians or 1.77943 *arc sec*. This value of gravitational deflection suffered by a Sun grazing ray, is claimed to be more accurate than all other values obtained in past.

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