ISCOs in Extremal Reissner Nordstrom Spacetimes

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Abstract

Circular geodesic orbits, both timelike and null, in extremal Reissner-Nordstrom spacetimes, are examined with regard to their stability, and compared with similar orbits in the non-extremal situation, focusing for simplicity on the near extremal case. Innermost Stable Circular Orbits (ISCOs), when they exist in the extremal case, are shown to lie infinitesimally close to the event horizon in coordinate distance, and correspond to zero energy trajectories. It is shown that this class of ISCOs are absent in the corresponding near-extremal spacetime.

1 Introduction

A number of features of near-extremal black hole spacetimes indicate the absence of a smooth limit to extremality [1]. One aspect related to black hole thermodynamics is the definition of the so-called Entropy Function [2] used widely nowadays to match the 'macroscopic' entropy of a class of extremal black holes emerging in the supergravity limit of string theories, to the 'microscopic' entropy obtained from counting of string states [3]. While the state counting is strictly restricted to BPS states, the use of the Entropy

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Function is stymied by the fact that its existence depends on that of a *bifurcation two-sphere* (in four spacetime dimensions). This bifurcation sphere only exists away from extremality, which forces one to begin with a *near*-extremal situation, and then proceed eventually to the extremal *limit*.

However, as has been suspected earlier [4] and succinctly pointed out recently [5], the existence of this limit cannot be taken for granted. In other words *the extremal limit of a near-extremal spacetime is not necessarily the extremal spacetime.* One manifestation of this concerns the near horizon limit: Consider for instance the near-horizon geometry of a non-extremal Reissner Nordstrom (RN) spacetime

$$ds^2 = -\left[\frac{(r-r_+)(r-r_-)}{r^2}\right]dt^2 + \left[\frac{(r-r_+)(r-r_-)}{r^2}\right]^{-1}dr^2 + r^2 d\Omega^2 \ . \tag{1}$$

Near extremality, defining $\delta \equiv (r_+ - r_-)/r_0 << 1$, $\epsilon \equiv (r - r_0)/r_0 << 1$ where r_0 is the radius of the horizon in the extremal case, this metric reduces, close to the event horizon, to

$$ds^2 = -\frac{\epsilon(\epsilon+\delta)}{(1+\epsilon)^2}dt^2 + \frac{(1+\epsilon)^2}{\epsilon(\epsilon+\delta)}dr^2 + r_0^2(1+\epsilon)^2 d\Omega^2 \tag{2}$$

which leads to two distinct outcomes depending on the order in which the limits $\delta \to 0$ and $\epsilon \to 0$ are taken. If the extremal limit $\delta \to 0$ is taken first and then the near-horizon limit is taken, the *local* geometry is that of an $AdS_2 \times S_2$

$$ds^2 \simeq -\epsilon^2 dt^2 + \frac{r_0^2}{\epsilon^2}d\epsilon^2 + r_0^2 d\Omega^2 \ . \tag{3}$$

On the other hand, if the near horizon limit is taken before the extremal limit, one gets

$$ds^2 \simeq -\epsilon\delta dt^2 + \frac{r_0^2}{\epsilon\delta}d\epsilon^2 + r_0^2 d\Omega^2 \ , \tag{4}$$

which certainly does not correspond to an $AdS_2 \times S_2$; in fact the extremal limit is now singular.

Indeed, it is known [1] that extremal spacetimes do not have any trapped surface inside the event horizon (itself usually a marginal outer trapped surface). This makes use of the fact, first pointed out in [6], that the proper distance between the event and Cauchy horizons in the extremal geometry (in the RN case, for instance) is actually infinite, even though the coordinate distance vanishes. This is certainly not the situation in the non-extremal situation where the proper distance between the inner and outer horizons is finite, as is the coordinate distance. However, despite such discontinuities, the extremal limit continues to be used as the *definition* of the extremal spacetime [3].

One aspect that has not been studied in detail is the behaviour of geodesics, both timelike and null, in the exterior of such spacetimes, both in the near-extremal and extremal cases. The interest here is in what happens to the geodesics near the horizon.

There is a class of geodesics in spherically symmetric spacetimes like Schwarzschild and RN, which circle around the event horizon in a stable orbit. Those in this class closest (in radial coordinate distance) to the event horizon are called Innermost Stable Circular Orbits (ISCOs). Since our interest is in the behaviour of such orbits near the event horizon, the use of the Kruskal-Szekeres extension of such spacetimes becomes crucial. Already at this level, there seems to be a discontinuity in the extremal limit of the Kruskal extension of a generic RN spacetime, vis-a-vis the extension of the extremal RN spacetime. It further ensues that ISCOs near the event horizon of an extremal RN spacetime exhibit a behaviour quite different from that in non-(or more precisely, near-)extremal situations. In what follows, a detailed investigation of such orbits is presented for near-extremal and extremal RN spacetimes.

A further motivation for the work comes from Hawking radiation, which is known to be absent for the extremal spacetime, as the surface gravity on the event horizon which measures the equilibrium temperature for the thermal distribution of the radiation vanishes in this case. Once again, this thermal state cannot be achieved in a continuous manner from a radiant black hole (however weakly) without violating energy conditions [7]. The behaviour of ISCOs is relevant to this in order to ascertain what really happens at extremality. This is an important issue for rotating black holes for which black hole radiance also includes *superradiance* in addition to Hawking radiation. While we do not consider rotating black holes in this paper, these issues serve as motivation for the present work.

The plan of the paper is as follows: in section 2 we present computation of ISCOs for the Kruskal-Szekeres extension of Schwarzschild spacetime, demonstrating enroute that the 'effective potential' usually derived using Schwarzschild coordinates to discuss stability of circular orbits, is the same when derived using Kruskal-Szekeres coordinates. In section 3 we exhibit the Kruskal-Szekeres extension for non-extremal RN spacetime. We show explicitly how the extremal limit is singular, similar to what we have already seen above. The extremal RN geometry has to be thus extended directly, instead of by a limiting procedure. In section 4 we calculate the effective potential for non-extreme RN black hole using the Kruskal extension, and show that it is the same as that derived using Schwarzschild coordinates which are ill-behaved near the horizon. In section 5 we calculate ISCOs for extremal RN spacetime and compare their behaviour with ISCOs in near-extremal RN black holes. The concluding section includes a discussion of our results in the light of trapped surfaces and also presents our future outlook.

2 ISCOs in Kruskal-extended Schwarzschild spacetime

The standard discussion of ISCOs (see, e.g., [8]) is given in terms of Schwarzschild coordinates which are known to be ill-behaved on the event horizon. In this section, we begin with the Kruskal-Szekeres extension of the Schwarzschild spacetime and consider both timelike and null geodesics in it.

2.1 Timelike ISCOs

The metric of the Schwarzschild black hole in Kruskal coordinates is given by

$$ds^2 = -2f(U,V)dUdV + r^2(U,V)d\Omega^2 \tag{5}$$

where $d\Omega^2$ is the metric on the unit sphere and

$$\begin{aligned} f(U,V) &= \frac{16M^3}{r(U,V)}\exp(-r(U,V)/2M) && (6)\\ UV &= (1-\frac{r}{2M})\exp(r/2M) && (7)\end{aligned}$$

The Killing vector field ξ generating timelike isometries for the Schwarzschild spacetime have the Kruskal components

$$\begin{aligned} \xi^V &= \frac{\partial V}{\partial t}\xi^t + \frac{\partial V}{\partial r}\xi^r + \frac{\partial V}{\partial \theta}\xi^\theta + \frac{\partial V}{\partial \phi}\xi^\phi \\ \xi^U &= \frac{\partial U}{\partial t}\xi^t + \frac{\partial U}{\partial r}\xi^r + \frac{\partial U}{\partial \theta}\xi^\theta + \frac{\partial U}{\partial \phi}\xi^\phi \end{aligned} \tag{8}$$

In Schwarzschild coordinates the components of Killing vector field is given by $\xi^\mu = (1,0,0,0)$. Therefore, $\xi^V = \frac{\partial V}{\partial t}$ and $\xi^U = \frac{\partial U}{\partial t}$. We introduce null coordinates u and v as

$$\begin{aligned} du = dt - dr^* = dt - \frac{r^2dr}{\Delta} \\ dv = dt + dr^* = dt + \frac{r^2dr}{\Delta} \end{aligned} \tag{9}$$

since $dr^*(u,v) = \frac{r^2dr}{\Delta}$, $r = r(u,v)$ where $\Delta = r^2 - 2Mr$, v and u axes represent the outgoing and ingoing radial null geodesics respectively.

The Kruskal-Szekeres coordinates are defined by transformation

$$\begin{aligned} U &= -\exp(-\gamma u), V = \exp(\gamma v), \gamma = 1/4M \\ dU &= -\gamma U[dt - \frac{r^2dr}{\Delta}] \\ dV &= \gamma V[dt + \frac{r^2dr}{\Delta}] \end{aligned} \tag{10}$$

Therefore the Killing vector field using (10) in Kruskal-Szekeres coordinates are (for a circular orbit of radius $r(U,V) = const. \equiv R$)

$$\begin{aligned} \xi^U &= \frac{\partial U}{\partial t} = -\frac{U}{4M} \\ \xi^V &= \frac{\partial V}{\partial t} = \frac{V}{4M} \end{aligned} \tag{11}$$

Since the scalar $\xi.\mathbf{u}$ is conserved along geodesics, the energy function can be expressed as

$$\xi.\mathbf{u} = -E \tag{12}$$

where ξ is Killing vector field, $\mathbf{u}$ is the four velocity of the particle and E is the conserved energy per unit rest mass. In terms of the function f introduced in (5) this equation would be

$$f(U,V)\ [\xi^U \dot{V} + \xi^V \dot{U}] = E \tag{13}$$

Substituting the value of $\frac{\partial U}{\partial t}$, $\frac{\partial V}{\partial t}$ in (13) from (11), then this equation will be

$$V\dot{U} - U\dot{V} = 4ME/f(U,V) \tag{14}$$

There is also another Killing vector field associated with rotational symmetry. The particle angular momentum conserved along geodesics is represented as

$$L = \zeta.\mathbf{u} = r^2 \sin^2\theta \frac{d\phi}{d\tau} \tag{15}$$

where ζ is the Killing vector associated with rotational symmetry and L is the angular momentum per unit rest mass. The geodesic equation in Kruskal-Szekeres coordinates is expressed in terms of the normalization of the 4-velocity

$$\mathbf{u}.\mathbf{u} = g_{\mu\nu} u^\mu u^\nu = -1 \tag{16}$$

and the conserved quantities E and L. Since the conservation of angular momentum implies that the orbits lie in a plane as they do in Newtonian gravity, using (5), (15) and (16), the particle orbit for the equatorial plane $\theta = \pi/2$ is given by

$$\dot{U}\dot{V} = (1 + \frac{L^2}{r^2})/2f(U,V)\ . \tag{17}$$

Now we know for a particle to describe a circular orbit at radius $r(U,V) = R$, its radial velocity must vanish i.e. $\dot{r} = 0$. Then we have from equation (7) and (14)

$$V\dot{U} = -U\dot{V} = \frac{2ME}{f(U,V)} \tag{18}$$

Using (7),(17),(18) the energy equation at $r(U,V) = R$ would be

$$E^2 = V_{eff}(R) = (1 + \frac{L^2}{R^2})(1 - \frac{2M}{R}) \tag{19}$$

which is identical to the effective potential for a timelike circular geodesic in Schwarzschild coordinates.

The Innermost Stable Circular Orbit(ISCO) for the Schwarzschild black hole can now be determined by applying first the condition for an extremum i.e. $\frac{dV_{eff}}{dR} = 0$. Then we have

$$R_{extr} = \frac{L^2}{2M}[1 \pm \sqrt{1 - 12(\frac{M}{L})^2}] \ . \tag{20}$$

To further focus on the ISCO, one uses the condition of stability i.e. $\frac{d^2V}{dR^2} > 0$, which corresponds also to the true *minimum* of V_{eff}. Therefore the ISCO occurs at $R = R_{min}$ of the effective potential given by(19). When L/M decreases, the radius of such an orbit also decreases, although no such orbit is possible below a threshold L, since the geodesic falls through the event horizon in that case. For the Schwarzschild spacetime, the ISCO occurs, with $\frac{L}{M} = \sqrt{12}$, at the radius $R_{min} = 6M$.

2.2 Null ISCOs

The determination of the effective potential for circular null geodesics proceeds in parallel with the scheme for timelike geodesics with some changes. Since the scalar $\xi.\mathbf{u}$ does not change in any coordinate system, the energy function would be

$$\xi.\mathbf{u} = -E \tag{21}$$

where ξ is Killing vector field, $\mathbf{u}$ is the four velocity obeying $\mathbf{u}^2 = 0$ and $E =$ is the energy. Therefore in terms of metric this equation (21) would be

$$f(U,V)[\xi^U \frac{dV}{d\lambda} + \xi^V \frac{dU}{d\lambda}] = E \tag{22}$$

Substituting the value of ξ^U, ξ^V in (22) from (11), then this equation will be

$$V\frac{dU}{d\lambda} - U\frac{dV}{d\lambda} = 4ME/f(U,V) \ . \tag{23}$$

Similarly

$$L = \zeta.\mathbf{u} = r^2 \sin^2\theta \frac{d\phi}{d\lambda} \tag{24}$$

where λ is an affine connection, E and L are conserved along the null geodesics.

The null geodesic equation is essentially implied by normalization of the four velocity and using (5), (22) and (24); the orbit for massless particles in the equatorial plane $(\theta = \pi/2)$ is given by

$$2f(U,V)\frac{dU}{d\lambda}\frac{dV}{d\lambda} = \frac{L^2}{r^2} \tag{25}$$

With $dr/d\lambda = 0$ for circular orbits $(r(U,V) = R)$, one obtains, from equation (7) and (23)

$$V\frac{dU}{d\lambda} = -U\frac{dV}{d\lambda} = \frac{2ME}{f(U,V)} \tag{26}$$

Using (7),(25),(26) the energy equation at $r(U,V) = R$ would be

$$E^2 = V_{eff}(R) = \frac{L^2}{R^2}(1 - \frac{2M}{R}) \tag{27}$$

Therefore the effective potential for photons orbit is given by

$$V_{eff}(R) = \frac{L^2}{R^2}(1 - \frac{2M}{R}) \tag{28}$$

which is exactly similar to the effective potential as determined in Schwarzschild coordinates for circular null geodesics. The effective potential has an extremum for $\frac{dU_{eff}}{dR} = 0$ which yields a solution at $R = 3M$, but the orbit at this radius is *unstable*. Thus there are no null ISCOs in the Schwarzschild spacetime.

3 ISCOs in extremal RN spacetime: Schwarzschild coordinates

The well known metric of the RN black hole spacetime is given by eq. (1) with horizons appearing at $r = r_{\pm}$, where $r_{\pm} = M \pm \sqrt{M^2 - Q^2}$. The line $Q^2 = M^2$ in the parameter space of electrovac solutions is referred to as the *extremal* RN spacetime. For such spacetimes the metric is given by

$$ds^2 = -(1 - \frac{M}{r})^2 dt^2 + (1 - \frac{M}{r})^{-2} dr^2 + r^2(d\theta^2 + \sin\theta^2 d\phi^2) \tag{29}$$

Similar to the Schwarzschild spacetime, the RN spacetime is spherically symmetric and hence static, with the geodetically conserved quantities $E \equiv \xi \cdot \mathbf{u}$ and $L \equiv \zeta \cdot \mathbf{u}$ having the connotations of energy and angular momentum of the particle respectively, with $\xi \equiv \partial/\partial t$ and $\zeta \equiv \partial/\partial\phi$ being the timelike and azimuthal Killing vectors respectively. One thus has

$$E = -\xi.\mathbf{u} = (1 - \frac{M}{r})^2 \frac{dt}{d\tau} \tag{30}$$

$$L = \zeta.\mathbf{u} = r^2 \sin^2\theta \frac{d\phi}{d\tau} \ . \tag{31}$$

The equation for the particle orbit follows, for the timelike case, from normalization of the four velocity

$$\mathbf{u}.\mathbf{u} = g_{\mu\nu}u^{\mu}u^{\nu} = -1 \ , \tag{32}$$

and is given in the $\theta = \pi/2$ plane by

$$\frac{1}{2}(\frac{dr}{d\tau})^2 + V_{eff} = \epsilon \tag{33}$$

where

$$\begin{aligned} \epsilon &= (E^2-1)/2 \\ V_{eff} &= \frac{1}{2}[(1+\frac{L^2}{r^2})(1-\frac{M}{r})^2-1] \ . \end{aligned} \tag{34}$$

Circular orbits are determined by the extrema of the effective potential for $r=R$ and are found to be at

$$R=M \ , \ \frac{1}{2}[\frac{L^2}{M} \pm \frac{L^2}{M}\sqrt{1-8(\frac{M}{L})^2}] \ . \tag{35}$$

It is easy to show that the orbit $R=M$ corresponds to a minimum of V_{eff} and hence is the ISCO we are seeking. Of the other two, it is easy to see that for the threshold $L/M=\sqrt{8}$, there is one real positive solution, and that is at $R=4M$. This, it can be shown, is a point of inflection, and hence does not correspond to an ISCO.

One shortcoming of the foregoing excercise in this section is this curious result of having an ISCO *right on* (the spatial foliation of) the event horizon of the extremal RN spacetime. Since this is also the surface on which the Schwarzschild coordinate system ceases to exist, the case for the location of this ISCO is open to question. Before we move into a discussion of a Kruskal-Szekeres extension of the RN spacetime, we continue with the Schwarzschild coordinate system, and probe if there is a stable photosphere for this spacetime. Proceeding as in the previous section, it is easy to see that the effective potential for the circular orbit of massless particles is given by

$$V_{eff}=\frac{1}{2}[\frac{L^2}{r^2}(1-\frac{M}{r})^2-1] \tag{36}$$

It is not difficult to see that $V_{eff}(r)$ has a minimum at $r=M$ exactly as in the case of a timelike particle. The prime issue to address then is whether this behaviour is indeed an artifact of the coordinate system we are using, which is of course ill-behaved precisely at the horizon $r=M$.

4 Kruskal-Szekeres extension of RN spacetime

The Kruskal-extended RN spacetime was first worked out by Carter [9] who gave the extended geometry only for the extremal case. Here we consider first the extension of the *non*-extremal or generic RN spacetime. The idea is to show that the *extremal limit* of the Kruskal extension is *not* the same as the Kruskal-extended extremal RN spacetime found by Carter.

4.1 Non-extremal case

We begin by defining 'tortoise' coordinates, and then use that to derive the Kruskal extension. The tortoise coordinate r^* is given by

$$dr^* = \frac{r^2 dr}{\Delta} = \frac{r^2 dr}{(r - r_-)(r - r_+)} \ . \tag{37}$$

Integrating this equation, we obtain

$$r^* = r + \frac{r_+^2}{(r_+ - r_-)} \ln|r - r_+| - \frac{r_-^2}{(r_+ - r_-)} \ln|r - r_-| + c \tag{38}$$

where as usual $r_\pm = M \pm \sqrt{M^2 - Q^2}$ and c is integration constant. The outer horizon r_+ is an event horizon and the inner horizon r_- is a Cauchy horizon. Now near the event horizon $r = r_+$, the tortoise coordinate is given by

$$r^* \approx \frac{r_+^2}{(r_+ - r_-)} \ln|r - r_+| \tag{39}$$

Here r^* has logarithmic dependence on $r - r_+$ and is singular at $r = r_+$. Introducing the radial null coordinates u and v, given by $u = t - r^*$, $v = t + r^*$, we observe that the surface $r = r_+$ appears at $v - u = -\infty$. This implies that so we need another transformation, which is the Kruskal-Szekeres extension. The null coordinates

$$du = dt - dr^* \ = \ dt - \frac{r^2 dr}{\Delta} \tag{40}$$

$$dv = dt + dr^* \ = \ dt + \frac{r^2 dr}{\Delta} \ , \tag{41}$$

since $dr^* = \frac{r^2 dr}{\Delta}$, where $u = \mathit{constant}$ are outgoing radial null geodesics and $v = \mathit{constant}$ ingoing radial null geodesics respectively. The RN metric now assumes the form

$$\begin{aligned} ds^2 &= -\frac{\Delta}{r^2}(dt^2 - dr^{*2}) + r^2 d\Omega^2 \\ &= -\frac{\Delta}{r^2} du dv + r^2 d\Omega^2 \end{aligned} \tag{42}$$

where $dudv = dt^2 - dr{*}^2$.

The Kruskal-Szekeres frame is now defined by the transformations

$$U^+ \ = \ -\exp(-\kappa_+ u) \tag{43}$$

$$V^+ \ = \ \exp(\kappa_+ v) \tag{44}$$

where $\kappa_\pm = \frac{r_\pm - r_\mp}{2r_\pm^2}$ is the surface gravity of the null hypersurfaces.

Therefore the metric, near $r = r_+$, becomes

$$ds^2 = -\frac{r_+ r_-}{\kappa_+^2}\frac{\exp(-2\kappa_+ r)}{r^2}(\frac{r_-}{r - r_-})^{\kappa_+/\kappa_- - 1} dU^+ dV^+ + r_+^2 d\Omega^2 \tag{45}$$

where

$$U^+ V^+ = -\exp(2\kappa_+ r)(\frac{r - r_+}{r_+})(\frac{r - r_-}{r_-})^{\frac{\kappa_+}{\kappa_-}} \tag{46}$$

This implies that, when in terms of the Kruskal coordinates (U^+, V^+), the metric is well behaved at the event horizon $(r = r_+)$ but is singular at the inner (Cauchy) horizon $(r = r_-)$. From this we can conclude that these coordinates (U^+, V^+) are valid for the region $(r_- < r < r_+)$ which is different from the original patch covering the region $r > r_+$. These coordinates do not cover $r \leq r_-$ because of the singularity at $r = r_-$, so another new coordinate patch is required to cover this region. In this region $g_{tt} > 0$ and $g_{rr} < 0$ such that t is spacelike and r is timelike. Note that in the above metric if we take the extremal limit $r_- \to r_+$ then the metric diverges, proving that the extremal limit is not continuous insofar as this extension is concerned. This follows from the fact that the coordinate chart U^+, V^+ considered here does not extend to the Cauchy horizon.

Now consider what happens for the Cauchy horizon $r = r_-$. Near the Cauchy horizon $r = r_-$, the tortoise coordinates is given by

$$r^* \approx \frac{r_-^2}{(r_+ - r_-)} \ln|r - r_-| \tag{47}$$

Here r^* has a logarithmic singularity at $r = r_-$. The radial null coordinate's u and v are $u = t - r^*$, $v = t + r^*$, then the surface $r = r_-$ appears at $v - u = +\infty$. Therefore the Kruskal-Szekeres transformations are

$$U^- = -\exp(-\kappa_- u) \tag{48}$$

$$V^- = \exp(\kappa_- v) \tag{49}$$

where κ_- has been previously defined. Therefore near $r = r_-$, the metric becomes

$$ds^2 = -\frac{r_+ r_-}{\kappa_-^2}\frac{\exp(-2\kappa_- r)}{r^2}(\frac{r_+}{r - r_+})^{\kappa_-/\kappa_+ - 1} dU^- dV^- + r_-^2 d\Omega^2 \tag{50}$$

where

$$U^- V^- = -\exp(2\kappa_- r)(\frac{r - r_-}{r_-})(\frac{r - r_+}{r_+})^{\frac{\kappa_-}{\kappa_+}} \tag{51}$$

This implies that, when it express in terms of Kruskal coordinates (U^-, V^-), the metric is regular at the Cauchy (inner) horizon $(r = r_-)$ but singular at the event horizon $(r = r_+)$. From this we can conclude that these coordinates (U^-, V^-) are valid for the region $(0 < r < r_-)$. In this region $g_{tt} < 0$ and $g_{rr} > 0$ such that r is spacelike coordinates and t is timelike coordinates. Note that the singularity $r = 0$ appears in the RN geometry as a timelike hypersurface whereas for Schwarzschild geometry it is a spacelike hypersurface. Once again, the extended metric is singular in the extremal limit.

4.2 Extremal case

Now we want to see what happens for the extremal case. The tortoise coordinate is given by

$$r^* = \int \frac{r^2 dr}{(r-M)^2} = r + 2M[\ln|r-M| - \frac{M}{2(r-M)}] \ . \tag{52}$$

Near the horizon this becomes

$$r^* \approx \frac{M^2}{(r-M)} \tag{53}$$

Here r^* has no logarithmic dependence, but an extra pole : $M^2/(r-M)$; it is singular at $r = M$. Introducing the double null coordinates u and v as $u = t - r^*$, $v = t + r^*$, the surface $r = r_+$ appears at $v - u = \infty$, hence these coordinates are inappropriate there. We need another transformation as in the previous generic cases.

The metric in terms of double null coordinates u and v is given by

$$ds^2 = -(1 - M/r)^2 du dv + r^2(u,v) d\Omega^2 \tag{54}$$

Now from eqn. (52)

$$\exp(\alpha r^*) = \exp(\alpha r)(r-M)^2 \exp(-M/(r-M)) \ , \ \alpha = 1/M \tag{55}$$

The maximally extended spacetime can be obtained by substituting

$$\begin{aligned} \tan U &= -\exp(-\alpha u) = -\exp(-\alpha t)[\exp(\alpha r)(r-M)^2 \exp(-M/(r-M))] \\ \tan V &= +\exp(+\alpha v) = \exp(\alpha t)[\exp(\alpha r)(r-M)^2 \exp(-M/(r-M))] \end{aligned} \tag{56}$$

Therefore the complete extremal RN metric in Kruskal-Szekeres coordinate system is given by (after substituting the value of $\alpha = 1/M$)

$$\begin{aligned} ds^2 = -4M^2(1 - \frac{M}{r})^2 \csc 2U \csc 2V dU dV + r^2 d\Omega^2 \\ \tan U \tan V = -\exp(2r/M)(r-M)^4 \exp(-2M/(r-M)) \end{aligned} \tag{57}$$

This is the result of Carter [9].

5 ISCOs in extremal RN spacetime

5.1 Effective potential for circular orbits

5.1.1 Timelike geodesics

In this section we investigate the effective potential for circular timelike and null geodesic orbits in extremal Reisner Nordstrom spacetime in the Kruskal-Szekers coordinates introduced in the last section. The metric given by

$$ds^2 = -2f(U,V) dU dV + r^2(U,V) d\Omega^2$$

$$= \quad -4M^2(1-\frac{M}{r})^2 \csc 2U \csc 2V dUdV + r^2(U,V)d\Omega^2 \tag{58}$$

where

$$\tan U \tan V = -\exp(2r/M)(r-M)^4 \exp(-2M/r-M) \tag{59}$$

The components of the timelike Killing vector field in Kruskal-Szekeres coordinates is given by

$$\begin{aligned}
\xi^V &= \frac{\partial V}{\partial t}\xi^t + \frac{\partial V}{\partial r}\xi^r + \frac{\partial V}{\partial \theta}\xi^\theta + \frac{\partial V}{\partial \phi}\xi^\phi \\
\xi^U &= \frac{\partial U}{\partial t}\xi^t + \frac{\partial U}{\partial r}\xi^r + \frac{\partial U}{\partial \theta}\xi^\theta + \frac{\partial U}{\partial \phi}\xi^\phi
\end{aligned} \tag{60}$$

The radial null coordinates, on the other hand, are given by

$$\begin{aligned}
du = dt - dr^* = dt - \frac{r^2 dr}{\Delta} \\
dv = dt + dr^* = dt + \frac{r^2 dr}{\Delta}
\end{aligned} \tag{61}$$

since $dr^* = \frac{r^2 dr}{\Delta}$, where $\Delta = (r-M)^2$, $u = const.$ is the outgoing radial null coordinate and $v = const.$ is the ingoing radial null coordinate respectively. Introducing the transformation to Kruskal-Szekeres coordinates, given by

$$\begin{aligned}
\tan U = -\exp(-u/M), \tan V = \exp(v/M) \\
dU = -\frac{\sin 2U}{2M}[dt - \frac{r^2 dr}{\Delta}] \\
dV = \frac{\sin 2V}{2M}[dt + \frac{r^2 dr}{\Delta}]
\end{aligned} \tag{62}$$

the components of ξ in Kruskal-Szekeres coordinates are (for a circular orbit at radius $r(U,V) = R$)

$$\begin{aligned}
\xi^U &= \frac{\partial U}{\partial t} = -\frac{\sin 2U}{2M} \\
\xi^V &= \frac{\partial V}{\partial t} = \frac{\sin 2V}{2M} \ .
\end{aligned} \tag{63}$$

Since the scalar $\xi.\mathbf{u}$ does not change in any coordinate system, the energy function is given by

$$\xi.\mathbf{u} = -E \ . \tag{64}$$

Using the form for $f(U,V)$ in eq. (58), one gets

$$f(U,V)[\xi^U \dot{V} + \xi^V \dot{U}] = E \ , \tag{65}$$

where, the dot represents $\partial/\partial t$. Substituting the values of $\dot{U}$, $\dot{V}$ in (65) from (63), this equation becomes

$$[\sin 2V\dot{U} - \sin 2U\dot{V}] = \frac{2ME}{f(U,V)} \ . \tag{66}$$

Now, the angular momentum of the geodesics is related to the rotational Killing vector of the spacetime:

$$L = \zeta.\mathbf{u} = r^2 \sin^2\theta \frac{d\phi}{d\tau} \tag{67}$$

where ζ is the Killing vector associated with rotational symmetry and L is the angular momentum per unit rest mass.

Using the normalization of the 4-velocity of the particle

$$\mathbf{u}^2 = g_{\mu\nu}u^\mu u^\nu = -1 \tag{68}$$

and assuming the geodesics lie in the equatorial plane $\theta = \pi/2$, one obtains

$$\dot{U}\dot{V} = (1 + \frac{L^2}{r^2})/2f(U,V) \ . \tag{69}$$

Now we know for a particle to describe a circular orbit at radius $r(U,V) = const. = R$, its radial velocity must vanish i.e. $\dot{r} = 0$. Then we have from equation (59)and(66)

$$\sin 2V\dot{U} = -\sin 2U\dot{V} = \frac{ME}{f(U,V)} \tag{70}$$

Using (59),(69),(70) the energy equation at $r(U,V) = R$ would be

$$E^2 = \mathcal{V}_{eff}(R) = (1 + \frac{L^2}{R^2})(1 - \frac{M}{R})^2 \tag{71}$$

which leads to an effective potential identical to that obtained in Schwarzschild coordinates. The problem thus reduces to the simple excercise of extremization of this effective potential, which we attempt in the next section, along with the case of circular orbits for near-extremal spacetime for comparison. Before that however, we briefly record the result for null geodesic circular orbits ('photon' orbits).

5.1.2 Null geodesics

$$\xi.u = -E \tag{72}$$

where ξ is killing vector field, u is the four velocity of the particle and E = Conserved energy per unit rest mass.

The most important difference is of course the vanishing of the squared norm of the 4-velocity:

$$\mathbf{u}^2 = g_{\mu\nu} u^\mu u^\nu = 0 \tag{73}$$

where $u^\mu = \frac{dx^\mu}{d\lambda}$. Proceeding as for case of timelike geodesics, the photon orbit for the equatorial plane ($\theta = \pi/2$) is given by

$$2f(U,V)\frac{dU}{d\lambda}\frac{dV}{d\lambda} = \frac{L^2}{r^2} \tag{74}$$

It is straightforward to deduce the energy equation

$$E^2 = \mathcal{U}_{eff}(R) = \frac{L^2}{R^2}(1 - \frac{M}{R})^2 \tag{75}$$

which leads us to the effective potential for circular photon orbits

$$\mathcal{U}_{eff}(R) = \frac{L^2}{R^2}(1 - \frac{M}{R})^2 \ . \tag{76}$$

5.2 Computation of ISCOs

It is convenient to define dimensionless quantities $\mathbf{r} \equiv R/M$, $\ell \equiv L/M$ and $q \equiv Q/M$. Thus, the extremal case corresponds to $q = 1$.

5.2.1 Timelike orbits

In this notation, the effective potential becomes

$$\mathcal{V}_{eff}(\mathbf{r}) = \left(1 + \frac{\ell^2}{\mathbf{r}^2}\right)\left(1 - \frac{1}{\mathbf{r}}\right)^2 \ . \tag{77}$$

Setting $\mathcal{V}'_{eff}(\mathbf{r}) = 0$ one obtains

$$\mathbf{r}^2(\mathbf{r}-1) - \ell^2(\mathbf{r}-2)(\mathbf{r}-1) = 0 \ , \tag{78}$$

which has the solutions

$$\begin{aligned} \mathbf{r}_0 &= 1 \\ \mathbf{r}_\pm &= \frac{\ell^2}{2}\left[1 \pm \left(1 - \frac{8}{\ell^2}\right)^{1/2}\right] \end{aligned} \tag{79}$$

It is easy to check that the circular orbit with $\mathbf{r} = \mathbf{r}_0 = 1$ *indeed corresponds to the ISCO*, being a stable minimum of $\mathcal{V}_{eff}$ given in eq. (71), since $\mathcal{V}''_{eff}(1) > 0$. It is also quite independent of the angular momentum ℓ and corresponds to $E = 0$. In other words, this is an orbit where the particle hovers over the event horizon with vanishingly small energy, and *yet, does not fall in* ! As we shall show below, there is no such orbit even for *near* extremal RN spacetime, i.e., where q differs from unity infinitesimally. In the next section we argue that this somewhat strange orbit in the extremal case is an artifact of the unusual nature of the spacetime geometry in this case, having to do with the nonexistence of outer trapped surfaces inside the event horizon, and also with the infinite difference between coordinate and proper distance.

Of the two other orbits at $\mathbf{r} = \mathbf{r}_{\pm}$, clearly one must choose $\ell^2 \geq 8$ for them to be real; we choose $\ell^2 = 9$ for simplicity and get $\mathbf{r}_+ = 6$, $\mathbf{r}_- = 3$. One obtains $\mathcal{V}''_{eff}(6) > 0$, showing that the orbit with radius $\mathbf{r}_+$ is stable, while $\mathcal{V}''_{eff}(3) < 0$, so that the orbit with radius $\mathbf{r}_-$ is unstable. Once again this is slightly counterintuitive, in view of the ISCO at $\mathbf{r}_0$.

5.2.2 Null orbits

With the effective potential given by eq. (76), extremization leads to

$$\frac{2\ell^2}{\mathbf{r}^3}\left(1-\frac{1}{\mathbf{r}}\right)\left(\frac{2}{\mathbf{r}}-1\right) = 0 \tag{80}$$

so that one has circular orbits at $\mathbf{r} = 1, 2$. It is easy to check that $\mathcal{U}''_{eff}(1) > 0$ which corresponds to the photon ISCO, while $\mathcal{U}''_{eff}(2) < 0$ implying that there is no other stable circular photon orbit.

6 ISCOs in near extremal RN spacetime

6.1 Effective potential for circular orbits and ISCOs

6.1.1 Timelike orbits

The effective potential for timelike circular geodesics in generic non-extremal RN spacetime is determined exactly in the last section. In terms of the dimensionless quantities introduced in that section, this effective potential can be expressed as

$$\mathcal{V}_{eff}(\mathbf{r}) = \left(1+\frac{\ell^2}{\mathbf{r}^2}\right)\ \Delta(\mathbf{r}) \tag{81}$$

where,

$$\Delta(\mathbf{r}) \equiv 1 - \frac{2}{\mathbf{r}} + \frac{q^2}{\mathbf{r}^2}\ . \tag{82}$$

The extremization condition $\mathcal{V}'_{eff}(\mathbf{r}) = 0$ leads to the cubic equation

$$\mathbf{r}^3 - (q^2+\ell^2)\mathbf{r}^2 + 3\ell^2\mathbf{r} - 2q^2\ell^2 = 0\ . \tag{83}$$

It is sufficient for us to solve eq. (83) in the *near extremal* approximation, to compare the results with those in the extremal situation. To this end, we define $\chi \equiv 1 - q > 0$, $\rho \equiv \mathbf{r} - 1$, where, recall that for the extremal situation $q = 1$, the ISCO is at $\mathbf{r} = \mathbf{r}_0 = 1$. The cubic equation (83) is to be solved perturbatively around this extremal solution to yield ρ to *linear* order in the perturbation χ. Likewise, terms of $O(\rho^2)$ or higher in (83) are ignored. One gets,

$$\rho = -2\chi \left(\frac{1 + 2\ell^2}{\ell^2 - 1} \right) + O(\chi^2) \ , \tag{84}$$

leading to a circular orbit with radius

$$\mathbf{r}_0^{nex} \simeq 1 - 2\chi \left(\frac{1 + 2\ell^2}{\ell^2 - 1} \right) < \mathbf{r}_0 \ ! \tag{85}$$

Thus, the timelike ISCO at $\mathbf{r}_0 = 1$ found for the extremal spacetime has no analogue in the near-extremal situation, since the circular orbit found in the latter case is an *unstable* one, corresponding to an orbit *inside* the horizon. There is no ISCO near the horizon with vanishingly small energy hovering over the horizon, for a near-extremal RN spacetime. Thus, the extremal spacetime admits this unique timelike ISCO infinitesimally close to the horizon, which is absent in the near extremal situation, pointing to another aspect in which the extremal spacetime differs from the extremal limit of the near extremal situation.

To find the timelike ISCO in this case, we proceed as before. Perturbing around the unstable orbit at $\mathbf{r}_- = (1/2)\ell^2[1 - (1 - 8/\ell^2)^{1/2}]$ we obtain the perturbation

$$\rho_- = -2\chi \left[\frac{\ell^2 \mathbf{r}_-}{(\ell^2 - 2)\mathbf{r}_- - 3\ell^2} \right] \ . \tag{86}$$

It is obvious that with $\ell^2 > 8$ one gets $\rho_- < 0$, this orbit does indeed correspond to an unstable circular orbit in the near extremal spacetime. In fact, for $\ell^2 = 9$, $\rho_- = -(36/5)\chi$. On the other hand, perturbation around the stable circular orbit at $\mathbf{r}_+ = (1/2)\ell^2[1 + (1 - 8/\ell^2)^{1/2}]$ in the extremal case, leads to a perturbed orbit with

$$\rho_+ = -2\chi \left[\frac{\ell^2 \mathbf{r}_+}{(\ell^2 - 2)\mathbf{r}_+ - 3\ell^2} \right] \ . \tag{87}$$

It is not difficult to see that for $\ell^2 > 8$ we have $\rho_+ > 0$. In fact, for $\ell^2 = 9$, $\rho_+ = 9\chi$. This does indeed correspond to a stable circular orbit in the near extremal spacetime, indeed it *is the ISCO* in this case.

Before we proceed to probe the extremal situation in order to understand why the ISCO in the extremal case disappears beyond the horizon in the near extremal situation, we record our results for the null ISCO in the near extremal situation.

6.1.2 Null Orbits

In terms of the linear perturbation χ, the effective potential for null circular orbits in near extremal RN spacetime is given by

$$\mathcal{V}_{eff}(\mathbf{r}) = \frac{\ell^2}{\mathbf{r}^2}\left[\left(1-\frac{1}{\mathbf{r}}\right)^2 - \frac{2\chi}{\mathbf{r}^2}\right] \tag{88}$$

leading to circular orbits with radii $\mathbf{r} = 1-(1/2)\chi\ ,\ 2+(1/4)\chi$. It is easy to show that $\mathcal{V}''_{eff}(\mathbf{r}) < 0$ for both these radii, demonstrating that the near extremal spacetime does not admit any null ISCO at all! Once again the difference from the precisely extremal situation is clear.

7 Discussion

In this concluding section, we first attribute the existence of timelike and null ISCOs in extremal RN spacetime to the absence of trapped surfaces in such a spacetime. We then present our future outlook on the genre of problems at hand in spacetimes with rotation.

7.1 Absence of trapped surfaces in extremal RN spacetime

In a general spacetime $(M, g_{\mu\nu})$ with the metric $g_{\mu\nu}$ having signature $(-+++)$, one defines two future directed null vectors l^μ and n^μ whose expansion scalars are given by

$$\theta_{(l)} = q^{\mu\nu}\nabla_\mu l_\nu \ \ \theta_{(n)} = q^{\mu\nu}\nabla_\mu n_\nu\ . \tag{89}$$

where $q_{\mu\nu} = g_{\mu\nu} + l_\mu n_\nu + n_\mu l_\nu$ is the metric induced by $g_{\mu\nu}$ on the two dimensional spacelike surface formed by spatial foliation of the null hypersurface generated by l^μ and n^μ.

Then (i) a two dimensional spacelike surface S is said to be a *trapped* surface if both $\theta_{(l)} < 0$ and $\theta_{(n)} < 0$; (ii) S is to be *marginally trapped* surface if one of two null expansions vanish i.e. $\theta_{(l)} = 0$ or $\theta_{(n)} = 0$. The null vectors for non-extremal RN black hole are given by

$$l^\mu = \frac{1}{\Delta}(r^2, -\Delta, 0, 0)\ \ n^\mu = \frac{1}{2r^2}(r^2, \Delta, 0, 0) \tag{90}$$

$$l_\mu = \frac{1}{\Delta}(-\Delta, -r^2, 0, 0)\ \ n_\mu = \frac{1}{2r^2}(-\Delta, r^2, 0, 0) \tag{91}$$

where $\Delta = (r-r_+)(r-r_-)$ and $r_\pm = M \pm \sqrt{M^2-Q^2}$. The null vectors satisfy the following conditions :

$$l^\mu n_\mu = -1\ \ l^\mu l_\mu = 0\ \ n^\mu n_\mu = 0 \tag{92}$$

Using (89), one obtains

$$\theta_{(l)} = -\frac{2}{r}\,\theta_{(n)} = \frac{(r-r_+)(r-r_-)}{r^3} \tag{93}$$

In the region ($r_- < r < r_+$), $\theta_{(l)} < 0$ and $\theta_{(n)} < 0$. This implies that trapped surfaces exist for non extreme RN black hole in this region. In contrast, for the extreme RN black hole

$$\theta_{(l)} = -\frac{2}{r}\,\theta_{(n)} = \frac{(r-M)^2}{r^3} \tag{94}$$

Here inside or outside extremal horizon $r < M$ or $r > M$, $\theta_{(l)} < 0$ and $\theta_{(n)} > 0$. This implies that there are no trapped surfaces for extremal RN black hole beyond the event horizon .

7.2 Outlook

Our analysis reinforces earlier assertions in the literature [4], [5] that the extremal limit of a generic charged black hole is *not necessarily* the extremal black hole spacetime. However, recent work [10] has conclusively shown that extremal black holes can be modelled by *isolated horizons* on par with generic non-extremal black holes. One expects this to lead to a well-defined microcanonical entropy obtained for extremal macroscopic black holes as an infinite series in horizon area, with the leading Bekenstein-Hawking area term receiving precise subleading logarithmic and power law corrections [11], for spherically symmetric horizons, just like more generic black holes. Note that this is genuine *gravitational* entropy of extremal black holes, and has little to do with entanglement or such non-gravitational phenomena. This understanding of extremal black hole entropy, based on Loop Quantum Gravity, ought to find extensive applications in superstring theoretic black holes in four dimensional spacetime [12]. The object now is to discern whether this approach works for rotating black holes with a similar degree of precision. Apart from questions pertaining to geodesics and ISCOs in extremal Kerr and Kerr-Newman spacetimes, there is the issue of stability of such spacetimes with respect to *superradiance*. Isolated horizons are not expected to be very useful in this respect, since the properties of the ergosphere play a crucial role for superradiance. The aim in future then ought to be a study of perturbations of extremal Kerr black holes with regard to superradiance.

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