# General class of vacuum Brans-Dicke wormholes

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Recently, traversable wormhole geometries were constructed in the context of $f(R)$ gravity. The latter is equivalent to a Brans-Dicke theory with a coupling parameter $\omega = 0$, which is apparently excluded from the narrow interval, $-3/2 < \omega < -4/3$, extensively considered in the literature of static wormhole solutions in vacuum Brans-Dicke theory. However, this latter interval is only valid for a specific choice of an integration constant of the field equations derived on the basis of a post-Newtonian weak field approximation, and there is no reason for it to hold in the presence of compact objects with strong gravitational fields. In this context, we construct a general class of vacuum Brans-Dicke wormholes that include the value of $\omega = 0$. Furthermore, we present the general condition for the existence of vacuum Brans-Dicke wormhole geometries, and show that the presence of effective negative energy densities is a generic feature of these vacuum solutions.

*Introduction*: Wormholes are hypothetical tunnels in spacetime and in classical general relativity are supported by exotic matter, which involves a stress energy tensor that violates the null energy condition (NEC) [1]. Several candidates have been proposed in the literature, such as solutions in Einstein-Gauss-Bonnet theory [2]; wormholes on the brane [3]; solutions in Brans-Dicke theory [4–7], which will be further explored in this brief report; wormhole solutions in semi-classical gravity [8]; exact wormhole solutions using conformal symmetries [9]; solutions supported by equations of state responsible for the cosmic acceleration [10]; and NEC respecting geometries were further explored in conformal Weyl gravity [11]; the possibility of distinguishing wormhole geometries by using astrophysical observations of the emission spectra from accretion disks was also explored [12], etc (see Refs. [13, 14] for more details and [14] for a recent review).

Recently, traversable wormhole geometries in the context of $f(R)$ modified theories of gravity were also constructed [15]. The matter threading the wormhole was imposed to satisfy the energy conditions, so that it is the effective stress-energy tensor containing higher order curvature derivatives that is responsible for the NEC violation. Thus, the higher order curvature terms, interpreted as a gravitational fluid, sustain these non-standard wormhole geometries, fundamentally different from their counterparts in general relativity. Furthermore, we note that $f(R)$ modified theories of gravity are equivalent to a Brans-Dicke theory with a coupling parameter $\omega = 0$, and a specific potential related to the function $f(R)$ and its derivative. However, the value $\omega = 0$ is apparently excluded from the interval, $-3/2 < \omega < -4/3$, of the coupling parameter, extensively considered in the literature of static wormhole solutions in vacuum Brans-Dicke theory.

In Brans-Dicke theory, analytical wormhole solutions were constructed [4–6]. It was shown that static wormhole solutions in vacuum Brans-Dicke theory only exist in a narrow interval of the coupling parameter [6], namely, $-3/2 < w < -4/3$. However, this result is only valid for vacuum solutions and for a specific choice of an integration constant of the field equations given by $C(w) = -1/(w + 2)$. The latter relationship was derived on the basis of a post-Newtonian weak field approximation, and it is important to emphasize that there is no reason for it to hold in the presence of compact objects with strong gravitational fields.

In this context, we construct a general class of vacuum Brans-Dicke wormholes that include the value of $\omega = 0$, and thus constructing a consistent bridge with the wormhole solutions in $f(R)$ gravity found in [15]. Furthermore, we present the general condition for the existence of Brans-Dicke wormhole geometries based on the NEC violation, and show that the presence of effective negative energy densities is a generic feature of these vacuum solutions.

*General class of Brans wormholes*: The matter-free action in Brans-Dicke theory is given by

$$S = \frac{1}{2} \int d^4x (-g)^{\frac{1}{2}} \left[ \varphi \mathbf{R} - \varphi^{-1} \omega(\varphi) g^{\mu\nu} \varphi_{,\mu} \varphi_{,\nu} \right] , \quad (1)$$

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where $\mathbf{R}$ is the curvature scalar, $\omega$ is a constant dimensionless coupling parameter, and $\varphi$ is the Brans-Dicke scalar. We adopt the convention $8\pi G = c = 1$ throughout this work.

The above action provides the following field equations:

$$\mathbf{G}_{\mu\nu} = -\frac{\omega}{\varphi^2}\left(\varphi_{,\mu}\varphi_{,\nu} - \frac{1}{2}g_{\mu\nu}\varphi_{,\sigma}\varphi^{,\sigma}\right)$$
$$-\frac{1}{\varphi}\left(\varphi_{;\mu}\varphi_{;\nu} - g_{\mu\nu}\Box^2\varphi\right), \tag{2}$$

$$\Box^2\varphi = 0, \tag{3}$$

where $\mathbf{G}_{\mu\nu}$ is the Einstein tensor and $\Box^2 \equiv \varphi^{;\rho}_{;\rho}$.

It is useful to work in isotropic coordinates, with the metric given by

$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + e^{2\nu(r)}r^2(d\theta^2 + \sin^2\theta d\psi^2). \tag{4}$$

Throughout this work, we consider the Brans class I solution, which corresponds to setting the gauge $\beta - \nu = 0$. Thus, the field equations yield the following solutions

$$e^{\alpha(r)} = e^{\alpha_0}\left(\frac{1-B/r}{1+B/r}\right)^{\frac{1}{\lambda}}, \tag{5}$$

$$e^{\beta(r)} = e^{\beta_0}(1+B/r)^2\left(\frac{1-B/r}{1+B/r}\right)^{\frac{\lambda-C-1}{\lambda}}, \tag{6}$$

$$\varphi(r) = \varphi_0\left(\frac{1-B/r}{1+B/r}\right)^{\frac{C}{\lambda}}, \tag{7}$$

$$\lambda^2 \equiv (C+1)^2 - C\left(1 - \frac{\omega C}{2}\right) > 0, \tag{8}$$

where $\alpha_0$, $\beta_0$, $B$, $C$, and $\varphi_0$ are constants. Note that the asymptotic flatness condition imposes that $\alpha_0 = \beta_0 = 0$, as can be readily verified from Eqs. (5) and (6).

In order to analyze traversable wormholes in vacuum Brans-Dicke theory, it is convenient to express the spacetime metric in the original Morris-Thorne canonical form [1]:

$$ds^2 = -e^{2\Phi(R)}dt^2 + \frac{dR^2}{1-b(R)/R} + R^2(d\theta^2 + \sin^2\theta d\psi^2) \tag{9}$$

where $\Phi(R)$ and $b(R)$ are the redshift and shape functions, respectively. To be a wormhole solution, several properties are imposed [1], namely: The throat is located at $R = R_0$ and $b(R_0) = R_0$. A flaring out condition of the throat is imposed, i.e., $[b(R) - Rb'(R)]/b^2(R) > 0$, which reduces to $b'(R_0) < 1$ at the throat, where the prime here denotes a derivative with respect to $R$. The condition $1 - b(R)/R \geq 0$ is also imposed. To be traversable, one must demand the absence of event horizons, so that $\Phi(R)$ must be finite everywhere.

Confronting the Morris-Thorne metric with the isotropic metric (4), the radial coordinate $r \to R$ is redefined as

$$R = re^{\beta_0}(1+B/r)^2\left(\frac{1-B/r}{1+B/r}\right)^{\Omega}, \qquad \Omega = 1 - \frac{C+1}{\lambda}, \tag{10}$$

so that $\Phi(R)$ and $b(R)$ are given by

$$\Phi(R) = \alpha_0 + \frac{1}{\lambda}\left\{\ln\left[1 - \frac{B}{r(R)}\right] - \ln\left[1 + \frac{B}{r(R)}\right]\right\}, \tag{11}$$

$$b(R) = R\left\{1 - \left[\frac{\lambda[r^2(R) + B^2] - 2r(R)B(C+1)}{\lambda[r^2(R) + B^2]}\right]^2\right\}, \tag{12}$$

respectively. The wormhole throat condition $b(R_0) = R_0$ imposes the minimum allowed $r$-coordinate radii $r_0^{\pm}$ given by

$$r_0^{\pm} = \alpha^{\pm}B, \qquad \alpha^{\pm} = (1-\Omega) \pm \sqrt{\Omega(\Omega-2)}. \tag{13}$$

The values $R_0^{\pm}$ can be obtained from Eq. (10) using Eq. (13). Note that $R \to \infty$ as $r \to \infty$, so that $b(R)/R \to 0$ as $R \to \infty$. The condition $b(R)/R \leq 1$ is also verified for all $R \geq R_0^{\pm}$. The redshift function $\Phi(R)$ has a singularity at $r = r_S = B$, so that the minimum allowed values of $r_0^{\pm}$ must necessarily exceed $r_S = B$. It can also be verified from Eq. (10) that $r_0^{\pm} \geq B$ which implies $R_0^{\pm} \geq 0$.

The energy density and the radial pressure of the wormhole material are given by [6]

$$\rho = -\frac{4B^2r^4Z^2[(C+1)^2 - \lambda^2]}{\lambda^2(r^2 - B^2)^4}, \tag{14}$$

$$p_r = -\frac{4Br^3Z^2}{\lambda^2(r^2 - B^2)^4}[\lambda C(r^2 + B^2) - Br(C^2 - 1 + \lambda^2)], \tag{15}$$

respectively, where $Z$ is defined as

$$Z \equiv \left(\frac{r-B}{r+B}\right)^{(C+1)/\lambda}. \tag{16}$$

Adding Eqs. (14) and (15), one arrives at

$$\rho + p_r = -\frac{4Br^3Z^2}{\lambda^2(r^2 - B^2)^4}[\lambda C(r^2 + B^2) + 2Br(C+1-\lambda^2)], \tag{17}$$

which will be analyzed in the NEC violation below.

In [6], the authors considered negative energy densities, which consequently violates the weak energy condition (WEC). Now, Eq. (14) imposes the following condition:

$$[C(\omega) + 1]^2 > \lambda^2(\omega), \tag{18}$$

which can be rephrased as

$$C(\omega)\left[1 - \frac{\omega C(\omega)}{2}\right] > 0, \tag{19}$$

by taking into account Eq. (8). Note that the function $C(\omega)$ is still unspecified.

However, it is important to emphasize that negative energy densities are not a necessary condition in wormhole physics. The fundamental ingredient is the violation of the NEC, $\rho + p_r < 0$, which is imposed by the

flaring out condition [1]. To find the general restriction for $\rho + p_r < 0$ at the throat $r_0$, amounts to analyzing the factor in square brackets in Eq. (17), namely, the condition $\lambda C(r_0^2 + B^2) + 2Br_0(C + 1 - \lambda^2) > 0$. Using Eqs. (8) and (13), the latter condition is expressed as:

$$(-1)^{s+t+1} \left[ (-1)^s(C+1) + (-1)^t \sqrt{C\left(1 - \frac{\omega C}{2}\right)} \right] \times$$

$$\times \frac{C(1 - \omega C/2)}{\sqrt{(4+2\omega)C^2 + 4(C+1)}} > 0, \qquad (20)$$

where $s, t = 0, 1$. Note that a necessary condition imposed by the term in the square root, in square brackets, is precisely condition (19). Thus, a necessary condition for vacuum Brans-Dicke wormholes is the existence of negative effective energy densities. However, we emphasize that it is condition (20), i.e., the violation of the NEC at the throat, that generic vacuum Brans-Dicke wormholes should obey.

A specific choice of $C(\omega)$ considered extensively in the literature, is the Agnese-La Camera function [4] given by

$$C(\omega) = -\frac{1}{\omega + 2}. \qquad (21)$$

Using this function, it was shown that static wormhole solutions in vacuum Brans-Dicke theory only exist in a narrow interval of the coupling parameter [6], namely, $-3/2 < \omega < -4/3$. However, we point out that this result is only valid for vacuum solutions and for the specific choice of $C(\omega)$ considered by Agnese and La Camera [4]. As mentioned in the Introduction, relationship (21) was derived on the basis of a post-Newtonian weak field approximation, and it is important to emphasize that there is no reason for it to hold in the presence of compact objects with strong gravitational fields. The choice given by (21) is a tentative example and reflects how crucially the wormhole range for $\omega$ depends on the form of $C(\omega)$. Evidently, different forms for $C(\omega)$ different from Eq. (21) would lead to different intervals for $\omega$.

Another issue that needs to be mentioned is that the above-mentioned interval imposed on $\omega$ was also obtained by considering negative energy densities. In principle, the violation of the WEC combined with an adequate choice of $C(\omega)$ could provide a different viability and less restrictive interval (including the value $\omega = 0$) from the case of $-3/2 < \omega < -4/3$ considered in [6]. In this context, we consider below different forms of $C(\omega)$ that allow the value $\omega = 0$ in the permitted range. Thus, to satisfy the constraint (19), both factors $C(\omega)$ and $[1 - \omega C(\omega)/2]$ should both be positive, or both negative.

Consider the following specific choice

$$C(\omega) = \frac{1}{\omega^2 + a^2}, \qquad (22)$$

where $a$ is a real constant. The requirement that $\lambda^2 > 0$, i.e., Eq. (8), is satisfied. The function $C(\omega)$ is positive for all real $\omega$, and the second term, in square brackets, of Eq.

(19), is positive everywhere for $a^2 > 1/16$. Therefore, for this case, condition (19) is satisfied for all $\omega$. For $a^2 < 1/16$, $[1 - \omega C(\omega)/2]$ has two real roots, namely, $\omega_\pm^0 = (1 \pm \sqrt{1 - 16a})/4$; the lesser value is positive and thus both the second term and condition (19) will be positive at $\omega = 0$. Thus, if $a^2 < 1/16$, the condition (19) is satisfied for $\omega \in \mathbb{R} - [\omega_-^0; \omega_+^0]$. Figure 1 depicts condition (19) (depicted as a solid curve), i.e., negative energy densities, and condition (20) (depicted as the dashed curves), i.e., the violation of the NEC, for $a = 1$. For the latter, only the cases of $(s, t) = (0, 1)$ and $(s, t) = (1, 1)$ of condition (20) are allowed; and are depicted in Fig. 1 by the small and large peaks, respectively.
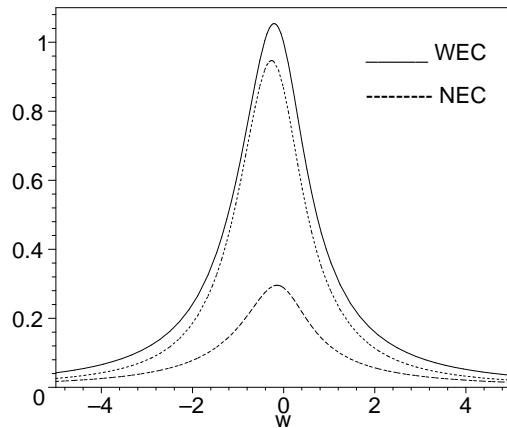


FIG. 1: Plot of the energy conditions for $C(\omega) = (\omega^2 + a^2)^{-1}$ for $a = 1$. In particular, the WEC expressed by condition (19) is given by the solid line; and the NEC, expressed by the condition (20), is given by the dashed curves. For the latter, only the cases of $(s, t) = (0, 1)$ and $(s, t) = (1, 1)$ of condition (20) are allowed; and are depicted by the small and large peaks, respectively.

In the limiting case, $C(\omega) \to 0$, $\lambda(\omega) \to 1$ as $\omega \to \infty$, one simply recovers the Schwarzschild exterior metric in standard coordinates. This can be verified from Eqs. (11) and (12), which impose $b(R) = 2M$ and $b'|_{r_0} = 0$. However, in this limit, the inequality (20) is violated, and there are no traversable wormholes.

Consider a second specific choice given by

$$C(\omega) = A \exp\left(-\frac{\omega^2}{2}\right). \qquad (23)$$

The requirement that $\lambda^2 > 0$, i.e., Eq. (8), is also satisfied. This function, for $A > 0$, is positive for all $\omega$. Therefore, in order to satisfy condition (19), the restriction $(1 - \omega C(\omega)/2) > 0$ is imposed. We verify that if $0 < A < 2\exp(1/2)$, then $(1 - \omega C(\omega)/2) > 0$ for all $\omega$, so that conditions (19) and (20) are both satisfied. If $A > 2\exp(1/2)$, then the second term $(1 - \omega C(\omega)/2)$ will have two real positive roots, i.e., $\omega_{0,1} > 0$. For this choice of $A$, we have the following range of allowed $\omega$: $\mathbb{R} - ]\omega_0, \omega_1[$. Moreover, since $\omega_0 > 0$, the value $\omega = 0$ will always be in the set of allowed values.

Figure 2 depicts condition (19) (depicted as a solid curve), i.e., negative energy densities, and condition (20) (depicted as dashed curves), i.e., the violation of the NEC for $A = 3\exp(1/2)$. For the latter, only the cases of $(s,t) = (0,1)$ and $(s,t) = (1,1)$ of condition (20) are allowed; and are depicted in Fig. 2 by the smaller and larger peaks, respectively.
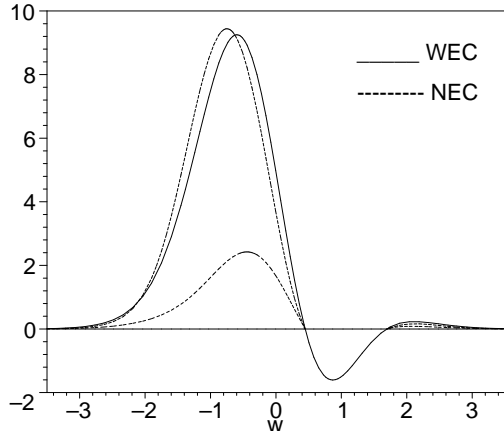


FIG. 2: Plot of the energy conditions for $C(\omega) = A\exp(-\omega^2/2)$, with $A = 3\exp(1/2)$. In particular, the WEC expressed by condition (19) is given by the solid line; and the NEC, expressed by the condition (20), is given by the dashed curve. For the latter, only the cases of $(s,t) = (0,1)$ and $(s,t) = (1,1)$ of condition (20) are allowed; and are depicted by the smaller and larger peaks, respectively.

*Conclusion*: Recently, in the context of $f(R)$ modified theories of gravity, traversable wormhole geometries were constructed. As $f(R)$ gravity is equivalent to a Brans-Dicke theory with a coupling parameter $\omega = 0$, one may be tempted to find these solutions inconsistent with the permitted interval, $-3/2 < \omega < -4/3$, extensively considered in the literature of static wormhole solutions in vacuum Brans-Dicke theory. Thus the choice provided by Eq. (21), in addition to the WEC and NEC violation, reflects how crucially the range of $\omega$ depends on the form of $C(\omega)$, and we have shown that adequate choices of $C(\omega)$ provide different viability regions and less restrictive intervals, that include $\omega = 0$. In this context, we have constructed a more general class of vacuum Brans-Dicke wormholes that include the value of $\omega = 0$, proving the consistency of the solutions constructed in $f(R)$ gravity. Furthermore, we deduced the general condition for the existence of vacuum Brans-Dicke wormhole geometries, and have shown that the presence of effective negative energy densities is a generic feature of these vacuum solutions. It will also be interesting to generalize this analysis in Brans-Dicke theory in the presence of matter. Work along these lines is presently underway.

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