A null frame for spacetime positioning by means of pulsating sources

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We introduce an operational approach to the use of pulsating sources, located at spatial infinity, for defining a relativistic positioning and navigation system, based on the use of four-dimensional bases of null four-vectors, in flat spacetime. As a prototypical case, we show how pulsars can be used to define such a positioning system. The reception of the pulses for a set of different sources whose positions in the sky and periods are assumed to be known allows the determination of the user's coordinates and spacetime trajectory, in the reference frame where the sources are at rest. In doing so, the phases of the received pulses play the role of coordinates in the null frame. We describe our approach in flat Minkowski spacetime, and discuss the valididty of this and other approximations considered.

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I. INTRODUCTION

The current positioning systems, such as GPS and GLONASS [1, 2], are essentially conceived as *Ptolemaic*, since they are based on a reference frame centered in the Earth, and *Newtonian*, since positioning is defined in a classical (Euclidean) space and absolute time, over which relativistic (post-Newtonian) corrections are introduced [1]. Furthermore, these systems are effective for positioning on the Earth, but they can hardly be used for navigating in the space outside the Earth, as in the Solar System and beyond. In contrast, some authors [3–9] recently proposed to use the worldlines of electromagnetic signals, emitted by objects in geodesic motion, in order to build a relativistic positioning system, based on the use of the so called *light* (or *emission*) *coordinates*. These relativistic positioning systems are also autonomous or autolocated, since any user can determine its own position (and spacetime trajectory) by solely elaborating the received signals.

The simplest way of understanding what emission coordinates are, is to consider four emitting clocks, in motion in spacetime, broadcasting their proper times: the intersection of the past lightcone of an event with the world lines of the emitting clocks corresponds to the proper times of emission along the world lines of the emitters; these proper times are the emission coordinates of the given event. For example, one may think of a set of satellites orbiting around the Earth and equipped with onboard clocks, however, such a system would hardly be effective for the navigation in the Solar System; for that purpose, a set of *pulsars* could rather be used. In fact, known pulsars emit their signals at a highly regular rate (this is the case, in particular, of the *millisecond pulsars*, see e.g. [10]) , which makes them natural beacons for building an emission-coordinates-based positioning system. What can be measured with great accuracy is the arrival time of the N-th pulse, so that counting these pulses can in principle allow to define something similar to the emission coordinates. Actually, the idea of using pulsars as stellar beacons has been considered since the early years of pulsar discovery [11], and other proposals are actually under study [12, 13], some of which use the emission coordinates approach [14].

Here we introduce an operational approach to the use of periodic sources, such as pulsars (anticipated in [15]), for defining a relativistic positioning and navigation system, based on the use of a four-dimensional basis of null four-vectors. We assume that a user is equipped with a receiver that can count pulses from a set of sources whose periods and positions in the sky are known; then, reckoning the periodic electromagnetic signals coming from (at least) four sources and measuring the proper time intervals between successive arrivals of the signals allow to localize the user, within an accuracy controlled by the precision of the clock he is equipped with. This system can allow autopositioning with respect to an arbitrary event in spacetime and three directions in space, so that it could be used for space navigation and positioning in the Solar System and beyond. We describe our approach in flat Minkowski spacetime, and discuss the validity of this and other approximations considered, for actual physical situations.

II. THE BASIC NULL FRAME AND DEFINITION OF THE GRID

Let us consider a number of sources of periodic electromagnetic signals, at rest at spatial infinity, in a four-dimensional Minkowski spacetime. For our purposes, at least four sources are needed. Each of these sources is characterized by the frequency of its periodic signals and by their directions in space; since the sources are sup-

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posed to be far away (i.e. at spatial infinity), their signals can be seen as corresponding to plane waves. In the inertial frame where the sources are at rest, once Cartesian coordinates are chosen, we associate to each source a null four-vector[1] $\boldsymbol{f}$ whose Cartesian contravariant components are given by

$$f^{\mu} \doteq \frac{1}{cT}(1, \vec{\mathbf{n}}), \tag{1}$$

T being the (proper) signal period, and $\vec{\mathbf{n}}$ is the unit vector describing the direction of propagation in the given frame. If in the same reference frame we consider the position four-vector

$$\boldsymbol{r} \doteq (ct, \vec{\mathbf{x}}), \tag{2}$$

with respect to an arbitrary and yet unspecified origin, then we can define the scalar function X at the spacetime event identified by the position four-vector $\boldsymbol{r}$

$$X(\boldsymbol{r}) \doteq \boldsymbol{f} \cdot \boldsymbol{r}, \tag{3}$$

where dot stands for Minkowski scalar product. The scalar X might be thought of as the phase difference of the wave described by $\boldsymbol{f}$ with respect to its value at the origin of coordinates. Four linearly independent four-vectors constitute a basis, or a frame: we may think of choosing four null four-vectors to serve as a basis (see e.g. [7]), so that the four wave four-vectors $\{\boldsymbol{f}_{(a)}, \boldsymbol{f}_{(b)}, \boldsymbol{f}_{(c)}, \boldsymbol{f}_{(d)}\}$ in the form (1) constitute our *null frame*, or *null tetrad*. Consequently, we define the corresponding phase differences (3)

$$X_{(N)} \doteq \boldsymbol{f}_{(N)} \cdot \boldsymbol{r}, \quad N = a, b, c, d \tag{4}$$

at any event $\boldsymbol{r}$ whose coordinates are defined by (2), where a, b, c, d label the sources.

According to the general tetrad formalism (see e.g. [16]), the null tetrad allows the definition of the symmetric matrix

$$\eta_{(M)(N)} = \boldsymbol{f}_{(M)} \cdot \boldsymbol{f}_{(N)}, \tag{5}$$

which, in this case, has constant components, and whose inverse is determined by the relation

$$\eta_{(M)(P)}\eta^{(P)(N)} = \delta^{(N)}_{(M)}. \tag{6}$$

Tetrad indices $N = a, b, c, d$ are lowered and raised by means of the matrices $\eta_{(M)(P)}$ and $\eta^{(M)(P)}$. In the null frame the position four-vector $\boldsymbol{r}$ is

$$\boldsymbol{r} = X^{(N)}\boldsymbol{f}_{(N)} = X_{(N)}\boldsymbol{f}^{(N)}. \tag{7}$$

[1] Arrowed bold face letters like $\vec{\mathbf{x}}$ refer to spatial vectors, while boldface letters like $\boldsymbol{f}$ refer to four-vectors; Greek indices refer to spacetime components, while Latin letters label the sources.

As a consequence we see that the phase differences $X_{(N)}$ are the components of the position four-vector with respect to the vectors

$$\boldsymbol{f}^{(N)} = \eta^{(N)(M)}\boldsymbol{f}_{(M)}, \tag{8}$$

or, differently speaking, the $X^{(N)}$ are the components with respect to the null tetrad vectors $\boldsymbol{f}_{(N)}$.

Considering the hyperplanes conjugated to the null frame $\{\boldsymbol{f}_{(a)}, \boldsymbol{f}_{(b)}, \boldsymbol{f}_{(c)}, \boldsymbol{f}_{(d)}\}$ vectors, we are able to define a spacetime grid[15], in which each event is identified by the relative phase of the electromagnetic signals with respect to an arbitrary origin: in other words these phases play the role of light coordinates in our frame.

III. LOCALIZATION WITHIN THE GRID

After having shown how to build a grid, we want to focus on how localization can be achieved within the grid. In particular, we suppose to deal with periodic signals, such as those coming from pulsars, and that these signals can be thought of as plane waves. Furthermore, we suppose that the user is equipped with a receiver able to recognize and count the pulses coming from the various sources, and a clock, that can be used to measure the proper time span between the arrivals from the various sources.

Let us start with a toy model, where the emission from the sources is continuous and the phases of any pulse can be determined with an arbitrary precision, at any event. We choose a starting event, from which the phases of each pulse are measured, which is the origin of our coordinates (in other words, the event with $\boldsymbol{r} = \mathbf{0}$, according to what we have described above), and three directions in space, defining the Cartesian axes of the inertial frame of the sources. We point out that even though the starting event is arbitrary, in order to correctly define the null frame, the position of the sources in the sky has to be known: in other words, we have to know the unit vectors $\vec{\mathbf{n}}$ for each source (and their proper frequency ν too), which also enable us to calculate the matrices $\eta_{(N)(M)}, \eta^{(N)(M)}$ of the given frame.

To a subsequent event $\boldsymbol{r}$, we associate the measured phases

$$X_{(N)} = \boldsymbol{f}_{(N)} \cdot \boldsymbol{r}, \tag{9}$$

and, according to eq. (7), it is then possible to obtain the coordinates of the event $\boldsymbol{r}$, in terms of the measured phases:

$$\boldsymbol{r} = X_{(a)}\boldsymbol{f}^{(a)} + X_{(b)}\boldsymbol{f}^{(b)} + X_{(c)}\boldsymbol{f}^{(c)} + X_{(d)}\boldsymbol{f}^{(d)}. \tag{10}$$

and to reconstruct the user's worldline.

Coming to a more realistic situation, such as the one in which the emitters are pulsars, we should consider that the signals received consist in a series of pulses and are not continuos. In this case, we may proceed as follows.

First, we call "reception" the event corresponding to the arrival of a pulse from one of the sources. As a consequence, an arbitrary reception event can be written in the form

$$\boldsymbol{r} = X_{(a)}\boldsymbol{f}^{(a)} + X_{(b)}\boldsymbol{f}^{(b)} + X_{(c)}\boldsymbol{f}^{(c)} + X_{(d)}\boldsymbol{f}^{(d)}, \quad (11)$$

with

$$\begin{aligned} X_{(a)} &= n_{(a)} + p, & (12)\\ X_{(b)} &= n_{(b)} + q, & (13)\\ X_{(c)} &= n_{(c)} + s, & (14)\\ X_{(d)} &= n_{(d)} + w. & (15)\end{aligned}$$

where we have expressed the phases $X_{(N)}$ in terms of an integer $n_{(N)}$, describing the succession of signal's cycles, and a fractional value: e.g. p means a fractional value of the cycle in $X^{(a)}$, and the equivalent holds for q, s, w, where $0 < p, q, s, w$: in eqs. (12)-(15), only one of the p, q, s, w will in general be zero. Once we choose an arbitrary origin, we may count the pulses in order to measure the $n_{(N)}$, but we have no direct means to measure the fractional values p, q, s, w. However a procedure to determine these values can be obtained, based on geometric considerations: we suppose that the acceleration of the user is small during a limited series of reception events, so that we may identify the user's worldline with a straight line; furthermore, we also suppose that by means of his own clock the user can measure the proper time interval τ_{ij} between the i-th and j-th arrivals. With these assumptions we can proceed as follows to determine the fractional values p, q, s, w. Let us consider two sequences[2] of arrival times from the sources; we have eight events, each of them in the form

$$\boldsymbol{r}_j = X_{(a)j}\boldsymbol{f}^{(a)} + X_{(b)j}\boldsymbol{f}^{(b)} + X_{(c)j}\boldsymbol{f}^{(c)} + X_{(d)j}\boldsymbol{f}^{(d)}, j = 1, .., 8, \quad (16)$$

[2] They may be subsequent or not, provided the total time span does not spoil the hypothesis of linearity of the world-line.

where $X_{(N)j}$ are expressions like (12–15). The events are arranged in such a way that $\boldsymbol{r}_1$ is the arrival of the first signal from pulsar "a", $\boldsymbol{r}_2$ is the arrival of the first signal of pulsar "b" after $\boldsymbol{r}_1$, $\boldsymbol{r}_3$ is the arrival of the first signal of pulsar "c" after $\boldsymbol{r}_1$, and $\boldsymbol{r}_4$ is the arrival of the first signal of pulsar "d" after $\boldsymbol{r}_1$ (the pulsars are ordered from largest ("a") to shortest ("d") period); $\boldsymbol{r}_5$ is the arrival of the second signal from pulsar "a", and so on. The flatness hypothesis allows to write the displacement four-vector between two reception events in the form

$$\boldsymbol{r}_{ij} \doteq \boldsymbol{r}_i - \boldsymbol{r}_j = \left(X_{(N)i} - X_{(N)j}\right)\boldsymbol{f}^{(N)} \doteq \Delta X_{(N)ij}\boldsymbol{f}^{(N)}. \quad (17)$$

Indeed, the assumption that the worldline of the receiver is straight during a limited number of pitches of the signals can be used also to provide further information. In fact, let us consider three successive reception events i,j,k; we have

$$\boldsymbol{r}_{ji} = \Delta X_{(N)ji}\boldsymbol{f}^{(N)}, \qquad \boldsymbol{r}_{kj} = \Delta X_{(N)kj}\boldsymbol{f}^{(N)}. \quad (18)$$

The straight-line hypothesis allows us to write

$$\frac{\tau_{ji}}{\tau_{kj}} = \frac{\Delta X_{(a)ji}}{\Delta X_{(a)kj}} = \frac{\Delta X_{(b)ji}}{\Delta X_{(b)kj}} = \frac{\Delta X_{(c)ji}}{\Delta X_{(c)kj}} = \frac{\Delta X_{(d)ji}}{\Delta X_{(d)kj}}, \quad (19)$$

where τ_{ji}, τ_{kj} are the proper times elapsed between the i-th and j-th, and j-th and k-th reception events, respectively. These relations enable us to obtain the values we are interested in: in fact, we may write the coefficients of eqs. (16) in the form

$$X_{(N)i} = \left(\begin{array}{c|c|c|c} n_1^{(a)} & n_1^{(b)} + q_1 & n_1^{(c)} + s_1 & n_1^{(d)} + w_1 \\ \hline n_2^{(a)} + p_2 & n_2^{(b)} & n_2^{(c)} + s_2 & n_2^{(d)} + w_2 \\ \hline n_3^{(a)} + p_3 & n_3^{(b)} + q_3 & n_3^{(c)} & n_3^{(d)} + w_3 \\ \hline n_4^{(a)} + p_4 & n_4^{(b)} + q_4 & n_4^{(c)} + s_4 & n_4^{(d)} \\ \hline n_5^{(a)} & n_5^{(b)} + q_5 & n_5^{(c)} + s_5 & n_5^{(d)} + w_5 \\ \hline n_6^{(a)} + p_6 & n_6^{(b)} & n_6^{(c)} + s_6 & n_6^{(d)} + w_6 \\ \hline n_7^{(a)} + p_7 & n_7^{(b)} + q_7 & n_7^{(c)} & n_7^{(d)} + w_7 \\ \hline n_8^{(a)} + p_8 & n_8^{(b)} + q_8 & n_8^{(c)} + s_8 & n_8^{(d)} \end{array}\right) \quad (20)$$

because some of the values of the p, q, s, w are zero. Then, on using relations like (19) we obtain the fractional values in terms of observed quantities, i.e. proper times. For instance, we have

$$p_1 = 0, \quad q_1 = n_2^{(b)} - n_1^{(b)} - \left(n_6^{(b)} - n_2^{(b)}\right)\frac{\tau_{21}}{\tau_{62}}, \quad s_1 = n_3^{(c)} - n_1^{(c)} - \left(n_7^{(c)} - n_3^{(c)}\right)\frac{\tau_{31}}{\tau_{73}}, \quad w_1 = n_4^{(d)} - n_1^{(d)} - \left(n_8^{(d)} - n_4^{(d)}\right)\frac{\tau_{41}}{\tau_{84}}, \quad (21)$$

and so on. Moving the pair of sequences we consider one step further, we can reconstruct the whole worldline of the receiver, in terms of measured quantities, i.e. proper times.

IV. DISCUSSION AND CONCLUSIONS

The procedure that we have described allows to use pulsating signals for positioning purposes; in particular, pulsars signals can be used. It is based on the definition of a null frame by means of the four-vectors associated to the signals in the inertial reference frame where the sources are at rest (so that the emission directions and the frequencies of the pulsating signals have to be known) and far away (so that their signals can be dealt with as plane waves). The procedure is fully relativistic and allows position determination with respect to an arbitrary event. Once a null frame has been defined, it turns out that the phases of the electromagnetic signals play the role of emission coordinates with respect to the null basis, so that they can be used to label an arbitrary event in spacetime. If the sources emit continuously and the phases can be determined with arbitrary precision at any event, it is straightforward to obtain the coordinates of the user and his worldline. On the other hand, when the signals consist of a series of pulses, we have developed a simple method that can be used to determine the user's worldline by assuming that the worldline is a straight line during a proper time interval corresponding to the reception of a limited number of pulses, which means that the acceleration is negligibly small. Given the user's clock accuracy $\delta\tau$, this method allows to reach an accuracy in position determining which is of the order

$$\delta x \simeq c\frac{\delta\tau}{\delta t}T, \tag{22}$$

where δt is the maximum allowed time interval between reception events that does not break the straight-line hypothesis, and T is the order of magnitude of signals period. On setting $\delta t \simeq 10^{-1}$ s, $\delta\tau \simeq 10^{-10}$ s, $T = 10^{-3}$ s, we get $\delta x \simeq 10^{-4}$ m. Actually, this is a theoretical accuracy, which depends on the accuracies involved in position determining only: it does not deal with the actual measurement process, which involves the technical details of signals detection; moreover, this accuracy does not consider the sistematic error in determining the signals periods and the position of the sources in the sky.

We can define the maximum proper time interval $\Delta\tau_{\max}$ that can be considered in order to be self consistent with the straight-line hypothesis. Developing the worldline of the receiver in powers of its proper time up to the second order we see that, if a is the order of magnitude of the user's acceleration, and v his velocity, the following condition should be satisfied:

$$\Delta\tau_{\max} = \sqrt{2\frac{v}{a}\delta\tau}. \tag{23}$$

For instance, if the user is moving in flat spacetime with $\delta\tau \simeq 10^{-10}$ s, $v = 5\times 10^5$ m/s and an acceleration $a = 1$ m/s^2, we have $\Delta\tau_{\max} = 10^{-2}$ s, which corresponds to several periods of millisecond pulsars. In our approach, we have considered the case of positioning in flat spacetime; actually, the deviation from the linearity of the user's worldline can also be due to the curvature of spacetime, i.e. to the presence of the gravitational field. We can give a similar estimate of the corresponding maximum proper time interval $\Delta\tau_{\max}$ by setting $a = \nabla\Phi$ in (23) where Φ is the gravitational potential. For instance, for $a = 10^{-3}$ m/s^2, which is the order of magnitude of the Sun gravitational field at 1 A.U., we get for $v = 10^3$ m/s, $\Delta\tau_{\max} \simeq 10^{-2}$ s. In summary, the effects of acceleration, both of gravitational and non-gravitational origin, can be evaluated and their impact can be neglected in some actual physical events: in these cases, millisecond pulsars can be used for positioning in the scheme that we have depicted. In principle our approach can be applied also to satellites orbiting the Earth, much like in the GPS, acting as sources, by taking into account the variations of the null four-vectors of the null basis. The theoretical accuracy in position determining is controlled by the user's clock accuracy, but in order to deal with an actual measurement process, the technical details of the signal detection have to be considered. In fact, in order to build a positioning device based on this approach, several technological issues have to be considered: for instance, the extreme weakness of pulsars signals and the required design and sensitivity of the detectors. These and other issues require further investigations and perhaps technological improvements, however we believe that this approach could be useful for defining an autonomous and relativistic positioning system in space, which ultimately transfers the basic positioning frame from the Earth to spacetime, according to a truly relativistic viewpoint.

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