Late–time Kerr tails: generic and non–generic initial data sets, "up" modes, and superposition

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Three interrelated questions concerning Kerr spacetime late–time tails are considered, specifically the evolutions of generic and non–generic initial data sets, the excitation of "up" modes, and the resolution of an apparent paradox related to the superposition principle. We propose to generalize the Barack–Ori formula for the decay rate of any tail multipole given a generic initial data set, to the contribution of any initial multipole mode. We also show explicitly that the angular symmetry group of a multipole does not determine its late–time decay rate.

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I. INTRODUCTION

Perturbations of black holes decay first with complex frequencies, known as the quasinormal modes of of the black hole, taken over at late times by a superposition of modes —known as the tails of the perturbation field— decaying as power–laws of time. The decay rate of the late–time tails in the spacetime of spinning black holes has been the subject of much debate and some confusion. Much of the debate in the literature was focused on the late–time decay rate of an initially pure azimuthal hexadecapole ($\ell = 4$) scalar field perturbation. All authors agreed the decay rate would be according to an inverse power of time, but different claims were made as to the value of the power–law exponent. Specifically, there were claims for decay rate along fixed Boyer–Lindquist r according to t^{-3} [1], t^{-5} [2], and even $t^{-5.5}$ [3].

In the last couple of years the behavior of the power–law tails has been clarified in a number of papers [4–6] (and the reader is referred to the detail therein), yet there are some remaining interesting detail waiting to be unveiled. In particular, it was shown in [6] that the disagreement between the t^{-3} and t^{-5} behaviors can be attributed to the use of different initial data sets (see also [5]), but not to the use of different slicing conditions as was suggested in [4]. Specifically, [6] suggested that starting with an initial pure multipole ℓ', the decay rate of an (allowed) multipole ℓ (such that $\ell < \ell'$) is given by t^{-n} where $n = \ell' + \ell + 1$ if $\ell' - \ell - 2 > 0$ and $n = 2\ell + 3$ otherwise.

Barack and Ori [7] studied analytically the late–time tails for generic families of initial data (only assuming compactly supported outgoing initial pulses). Instead of carrying the analysis in the frequency domain —which turns out to be complicated on a Kerr background because the frequency dependence of the spheroidal harmonics implies that separation of the two angular variables depends on the frequency— Barack and Ori analyzed the evolution of perturbations in the time domain. Barack and Ori found the decay rate of the ℓ multipole given generic initial data, which they considered to be data in which all multipoles are present. In fact, because the smallest multipole contribution to the initial data turns out as we show below to dominate the decay rate of the ℓ multipole at late times, Barack and Ori found the decay rate of an even (odd) mode ℓ excites by the scalar field monopole (dipole) mode of the initial data. The Barack–Ori formula for the decay rate of the ℓ multipole has not been confirmed by numerical simulations, which we do below for the first time. We also propose to generalize it to any initial multipole, not just the monopole or dipole as in [7].

The organization of this Paper is as follows: In Section II we describe the numerical code and the type of initial data sets that we use. In this Paper we discuss three related questions: First, in Section III we address the difference in the time evolution of late–time tails between generic and non–generic initial data sets. Generic initial data sets were first considered by Barack and Ori [7], but their results have not been confronted with numerical simulations. In addition to verifying the results of [7], we also discuss the meaning of genericity of initial data sets. As we show below, not any mixture of all multipole modes is in fact generic. Then, in Section IV, we consider an aspect of late–time tails that has not been studied before, specifically the decay rate of excited higher multipole modes, which we call "up"–excited modes. The decay rate of "up" excited modes generalized the Barack–Ori result in [7] to some interesting cases: the case of non–generic initial data in which (only) the lowest dynamically allowed mode is not present, and the case in which only one multipole mode is present in the initial data. Lastly, in Section V, we discuss an apparent paradox and its resolution, specfically the apparent failure of the superposition principle for linear interactions, and the conclusion that the angular symmetry group of multipole modes does not determine their time evolution.

II. NUMERICAL CODE AND INITIAL DATA SETS

Linearized perturbations of Kerr black holes are described by Teukolsky's master equation, given in Boyer–Lindquist coordinates for a scalar field ($s = 0$) ψ, known as the Teukolsky function, by

$$\begin{aligned} -&\left[\frac{(r^2+a^2)^2}{\Delta} - a^2\,\sin^2\theta\right]\partial_{tt}\psi - \frac{4Mar}{\Delta}\,\partial_{t\varphi}\psi \\ +&\ \partial_r\left(\Delta\,\partial_r\psi\right) + \frac{1}{\sin\theta}\,\partial_\theta\left(\sin\theta\,\partial_\theta\psi\right) \\ +&\ \left(\frac{1}{\sin^2\theta} - \frac{a^2}{\Delta}\right)\partial_{\varphi\varphi}\psi = 0\,, \end{aligned} \tag{1}$$

where M, a are the black hole's mass and spin angular momentum per unit mass, respectively, and the horizon function $\Delta = r^2 - 2Mr + a^2$.

To solve Eq. (1) numerically, we define, following [8], $b(r,\theta) := (r^2+a^2)/\Sigma$ where $\Sigma^2 = (r^2+a^2)^2 - a^2\,\Delta\,\sin^2\theta$. We next define the 'momentum' Π of the field ϕ defined by $\psi = e^{im\varphi}\,\phi$, according to

$$\Pi := \frac{\partial\phi}{\partial t} + b(r,\theta)\,\frac{\partial\phi}{\partial r_*}$$

where r_* is the regular Kerr spacetime 'tortoise' coordinate defined by $dr_*/\,dr = (r^2+a^2)/\,\Delta$.

Our code is a 2+1D first–order code for the time evolution of modes m of ϕ, Π given two independent functions of two variables at $t = 0$ as initial data, based on [8]. The convergence order is taken to be second–order in the radial and temporal directions, and sixth order in the angular direction (although some of the higher–ℓ results were obtained with tenth–order angular operators). The standard grid resolution we use is $64,000$ grid points in the radial direction, and 64 angular grid points (which we denote as $64K \times 64$), unless stated otherwise, taking the temporal step to be the largest step consistent with the Courant condition.

In practice, we choose as initial data $\phi(r,\theta)|_{t=0} = 0$ and $\Pi(r,\theta)|_{t=0} = f(r)\,P_\ell(\cos\theta)$, where the radial function $f(r)$ is chosen to be a gaussian $f(r) = (1/\sqrt{2\pi\sigma^2})\,\exp[-(r_* - r_{*\,0})^2/(2\sigma^2)]$ with $r_{*\,0} = 25M$ and $\sigma = 6M$. The outer and inner boundaries are placed at $r_*^{\rm boundary} = \pm 800M$, which allows us to integrate to $t = 1,500M$ at $r_{*\,0}$ without seeing boundary reflection effects. We choose the Kerr parameter $a = 0.995M$. All our simulations below are for azimuthal ($m = 0$) modes. We use quadrupole precision floating–point arithmetic throughout.

III. NUMERICAL EVOLUTION OF GENERIC INITIAL DATA

We first show for the first time numerical tests of the Barack–Ori formula [7]. Then, in Section IV we show how the Barack–Ori formula is a particular case for "up" excitation of modes. Barack and Ori noted that in the generic case of black hole perturbations, all dynamically allowed modes are present. Specifically, in the case of scalar field perturbations, all modes, starting with the monopole ($\ell = 0$) evolve. The decay rate of a multipole ℓ with azimuthal number m is given by t^{-n} where

$$n_\ell = \ell + |m| + 3 + q\,, \tag{2}$$

where $q = 0, 1$ if $\ell + m$ is even or odd, respectively.

The meaning of "generic" initial data is that the relative amplitudes of the various multipoles and their radial profiles are not fine tuned. Consider, e.g., the following initial data set, $\phi(r,\theta)|_{t=0} = 0$ and $\Pi(r,\theta)_{t=0} = A\,\sin^2\theta\,f(r)$, where $f(r)$ is any localized radial function, say a gaussian. These initial data include non-vanishing monopole and quadrapole moments. The late–time tail includes all even multipoles, and is dominated by the monopole field that decays according to t^{-3} at late times. The quadrapole projection of the late–time field decays according to t^{-5}, in accordance with the Barack–Ori formula.

Let us now consider the following scenario: one starts with initial data as above, and evolves the field to a certain value of time, say to $t = T$. The field at $t = T$ is a complicated mixture of modes, each with its own radial profile. Now consider a different initial value problem, in which at $t = 0$ only a pure multipole, say the quadrapole moment, is non-vanishing, and evolve these initial data to $t = T$. These pure mode initial data also evolve to a complicated mixture of multipole moments at $t = T$, and at first look one does not notice much qualitative difference between the fields at $t = T$. However, treating the fields at $t = T$ as new initial value problems, further time evolution leads to different decay rates at late times. Specifically, the generic data lead to quadrapole projection that drops off like t^{-5}, and the pure data lead to quadrapole projection that drops off like t^{-7}. Considering only the values at $t = T$ as the initial data sets, what is the fundamental difference between the two sets, that leads to very different late–time evolutions? Even though either set appears at $t = T$ to be a set of amplitudes with weighted ratios for the various multipole moments (as functions of the radial coordinate), the set describing the pure mode evolution is made at $t = T$ of a very carefully chosen set of (radius dependent) amplitudes. In fact, it is the outcome of fine–tuned evolution of a pure mode of the original initial value problem. Examination of the data set at $t = T$ does not reveal anything qualitative different about it: it is the specific amplitude ratios and radial profiles that make it non-generic.

One may test these ideas by injecting at $t = T$ another field, say that of a pure mode. Start at $t = 0$ with a pure quadrapolar field. At $t = T$ the projection of the field is a mixture of all even multipoles. Then, at $t = T$, inject either a pure monopole or a pure quadrapole with some radial profile. The multipolar content of the field after the injection is changed in that the relative

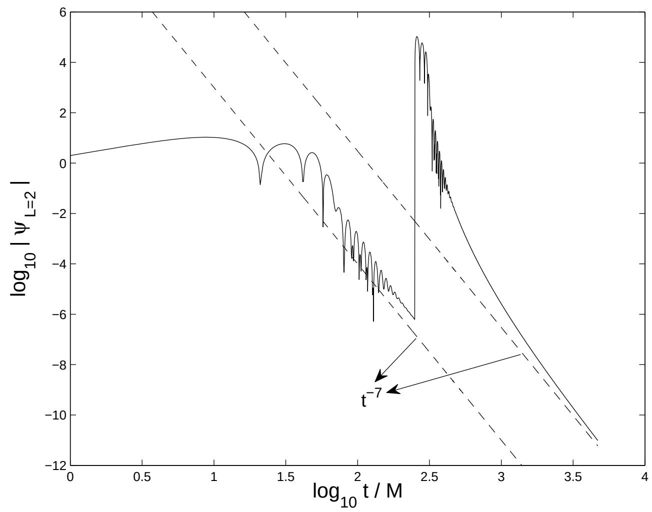

FIG. 1: The quadrapole projection of the full field as a function of time, for initial data of a pure quadrapole field, then injected at $T = 250M$ with a pure quadrapole field. The late–time field behavior is computationally found to be given by $\psi \sim t^{-7.05}$.

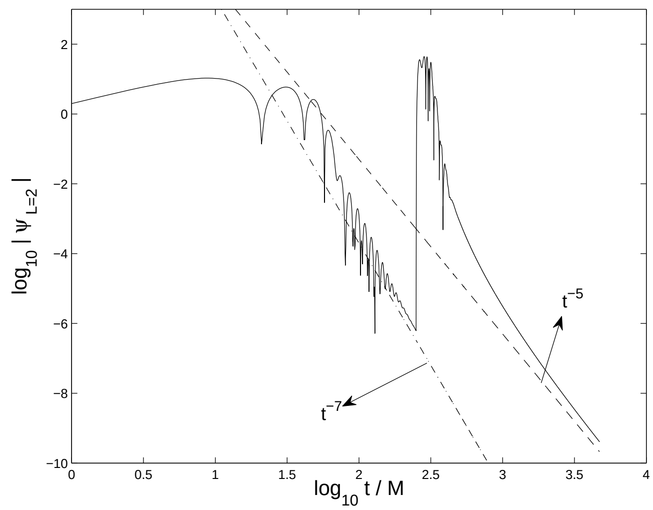

FIG. 2: The quadrapole projection of the full field as a function of time, for initial data of a pure quadrapole field, then injected at $T = 250M$ with a pure monopole field. The late–time field behavior is computationally found to be given by $\psi \sim t^{-5.03}$.

multipolar amplitudes changed (as functions of the radius). When the injected field is quadrapolar, the late–time decay rate of the quadrapole moment is t^{-7}, as should be expected from the fact that we have a superposition of two linear pure mode evolutions (which a certain time delay between them) (Fig. 1). When the injected field is a monopole, the late–time decay rate of the quadrapole moment is t^{-5} (Fig. 2). The addition of a monopole field (without fine tuning its amplitude and radial profile) makes the multipolar content of the total field generic. One may pose the following question: When the injected field is a quadrapole, its time evolution will excite a monopole. Why in the presence of this excited monopole the quadrapole projection of the field decays like t^{-7}, whereas in the presence of an injected monopole the decay rate is the slower t^{-5}? There is a fundamental difference between the two cases: the excited monopole is not an arbitrary field: it is carefully chosen by the dynamics of the problem, and may be viewed as a fine tuned field; the injected monopole is arbitrary, and is not fine tuned to lead to a different late–time decay rate.

IV. EXCITATION OF "UP" MODES

Most interest has been devoted to finding the decay rate of the slowest decaying mode, for obvious reasons. Specifically, the slowest decaying mode determines the late–time behavior of the full field. As lower multipole modes typically have a slower decay rate, most interest has naturally focused on the excitation of "down" modes, specifically the excitation of lower multipole modes starting with higher multipole modes.

Here, we are interested in the converse, i.e., in the excitation and decay rate of "up" modes, i.e., the excitation and decay rate of a higher multipole mode than the one initially excited. Such modes are typically sub-dominant, and do not dominate the late–time behavior of the full field. Denoting the initially excited multipole by ℓ', and the multipole moment of the field whose excitation and decay are is of interest by ℓ (so that $\ell > \ell'$), we propose that the late–time decay rate for such "up" excited modes along a $r =$ const worldline, is given by t^n where

$$ {}_{\ell'}n_{\ell} = \ell' + \ell + |m| + 3 + q \,, \tag{3} $$

where m is the azimuthal number of the multipole, and $q = 0, 1$ depending on whether $\ell' + \ell + m$ is even or odd, correspondingly.

Letting the initial $\ell' = 2$, we projects the multipoles $\ell = 0, 2, 4$, and the local power indices are shown in Fig. 3. Of particular interest here is the excitation of the hexadecapole mode, $\ell = 4$. In this case, $\ell' = 2, \ell = 4, m = 0, q = 0$, so that our proposal is that $n = 2 + 4 + 0 + 3 + 0 = 9$. In our numerical simulations, we have found the value of ${}_2n_4 = 9.01$, with uncertainly in the last figure.

Starting with initial data for a pure $\ell' = 0$ monopole mode, we project the $\ell = 0, 2$ and 4 modes. The local power indices ${}_0n_0$ and ${}_0n_2$ are shown in Fig. 4. The hexadecapole mode $\ell = 4$, ${}_0\psi_4$, is shown in Fig. 5. Figure 5 suggests that our grid density is not sufficiently high to obtain accurately the very late time hexadecapole projection to the required accuracy. Indeed, as Fig. 5 shows, the tail of the field "curves up" at very late times. However, increasing the grid density appears to straighten up the field, from which we infer that this curving is a numerical artifact. We attribute the problem to our calculation of a mode that is a second–order excited "up" mode, i.e.,

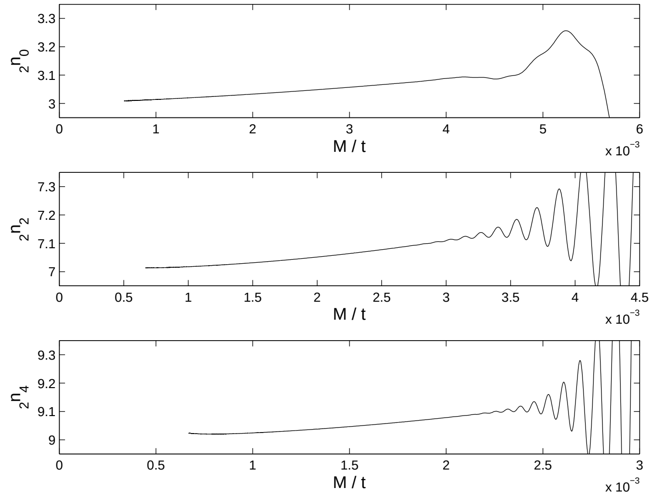

FIG. 3: The $\ell = 0, 2, 4$ multipole projections, for initial data of a pure quadrupole field as functions of time. The late–time field behavior is computationally found to be given by ${}_2\psi_0 \sim t^{-3.005}$, ${}_2\psi_2 \sim t^{-7.009}$, and ${}_2\psi_4 \sim t^{-9.01}$, respectively.

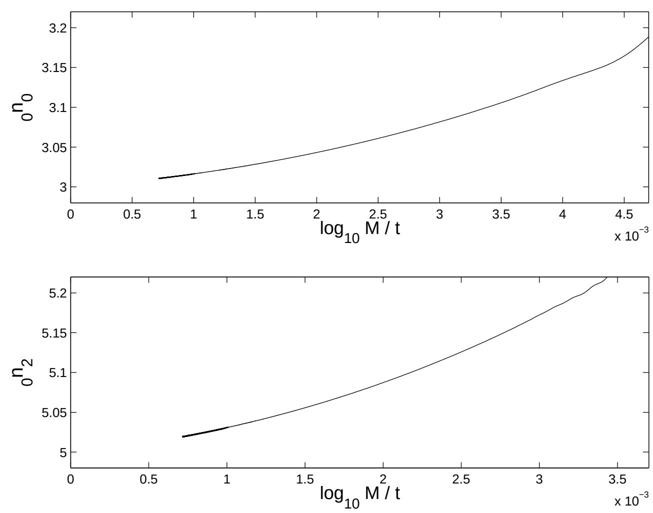

FIG. 4: The local power indices ${}_0n_0$ (upper panel) and ${}_0n_2$ (lower panel) as functions of the time t.

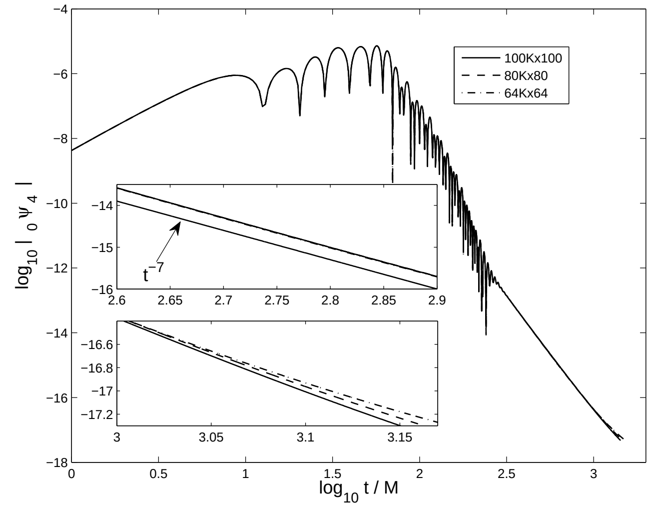

FIG. 5: The $\ell = 4$ multipole projections, for initial data of a pure monopole field as functions of time. Shown are the fields with three different grid densities, $64K \times 64$ (dash–dotted curve), $80K \times 80$ (dashed curve), and $100K \times 100$ (solid curve). The inserts show the same for segments of the full data, in addition to a reference curve $\sim t^{-7}$.

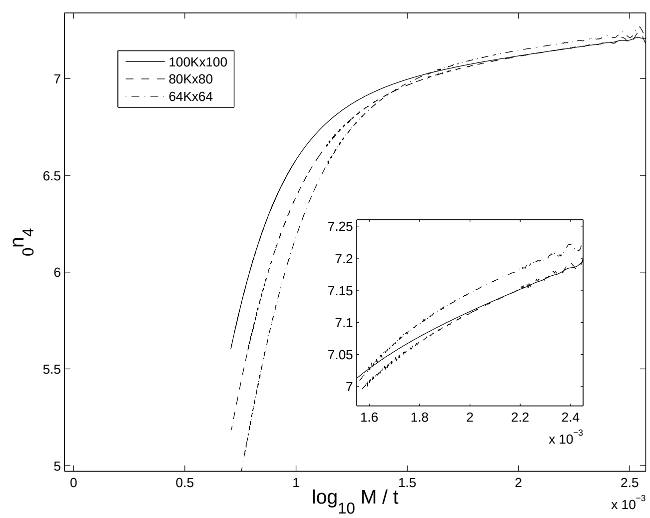

FIG. 6: The local power index ${}_0n_4$ as a function of the time t for three grid resolutions, $64K \times 64$ (dash–dotted curve), $80K \times 80$ (dashed curve), and $100K \times 100$ (solid curve).

the $\ell = 4$ excited mode from an initial $\ell' = 0$ mode. The practical problems in determining the decay rate accurately are accentuated by Fig. 6, which displays the local power indices for three grid densities. Figure 6 suggests that indeed a higher resolution simulation would be successful at determining the late–time decay rate with sufficient accuracy. Our results do suggest that ${}_0n_4 = 7$, as indicated by the reference curve in Fig. 5, but any determination of the decay rate at higher accuracy than that would require higher resolution simulations.

We have also tested our proposal with odd modes. Specifically, Let $\ell' = 3$ and study the late time tail for the modes $\ell = 1, 3, 5$. Figure 7 shows the projections $\ell = 1, 3, 5$ as functions of time, and figure 8 displays the local power indices for these modes as functions of time.

Starting with $\ell' = 1$, we project the dipole and octupole modes and their respective local power indices in Fig. 9. We encounter the same problem finding the decay rate for $\ell = 5$ — i.e., find ${}_1\psi_5$ and ${}_1n_5$ — as we did when we we calculated ${}_0\psi_4$ and ${}_0n_4$ (Figs. 10 and 11). Again, the second–order "up"–excited mode requires high resolution simulations.

Our proposal (3) naturally extends the Barack–Ori formula (2) to cases where the initial data do not include the monopole ($\ell' = 0$) mode. Recall that the Barack–Ori

TABLE I: Late–time power law indices n for the $\ell = \ell' - 4, \cdots, \ell' + 4$ projections of the fields for pure mode initial data sets, $\ell' = 0, 1, 2, 3, 4$. This table includes both "down" and "up" excitations. Dashes relate to decay rate we have not computed. All figures are significant.

Initial ℓ' mode	Projected $\ell = \ell' - 4$	Projected $\ell = \ell' - 2$	Projected $\ell = \ell'$	Projected $\ell = \ell' + 2$	projected $\ell = \ell' + 4$
0	*	*	3.007	5.01	7
1	*	*	5.002	7.003	9
2	*	3.005	7.009	9.01	–
3	*	5.002	9.009	11.008	–
4	5.01	7.002	11.008	–	–

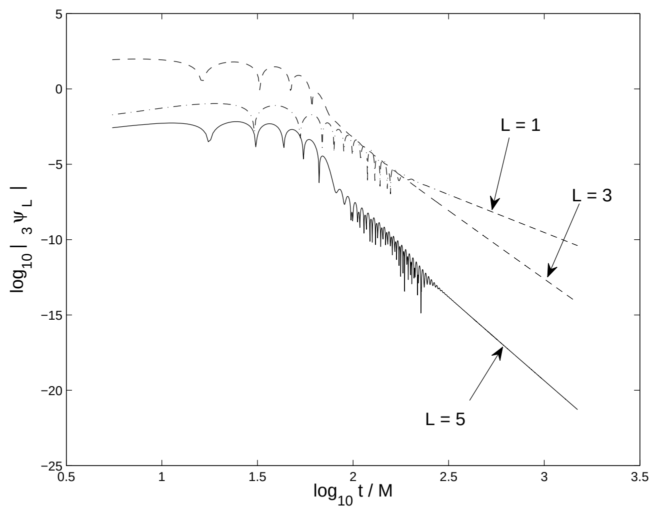

FIG. 7: The $\ell = 1, 3, 5$ multipole projections, for initial data of a pure octupole field as functions of time. The late–time field behavior is computationally found to be given by $\psi_1 \sim t^{-5.002}$, $\psi_3 \sim t^{-9.009}$, and $\psi_5 \sim t^{-11.008}$, respectively.

formula considers the case of generic initial data, which means a mixture of all multipole modes with arbitrary relative amplitudes and radial profiles. In particular, the monopole mode is generically present, so that the slowest decaying contribution of any "up"–excited ℓ mode is generated by the initial monopole. Our proposal degenerates to the Barack–Ori formula when $\ell' = 0$. When the monopole is not present in the generic initial data set, our proposal will produce the decay rate of a mixture of modes when ℓ' is taken to be the smallest multipole present in the generic data set, and as a particular case will also produce the decay rate of the ℓ multipole when the initial data is a pure $\ell' (< \ell)$ mode, the non–generic case, in the language of Barack and Ori. In summary, the Barack–Ori formula Eq. (2) is simply the particular case ${}_0n_\ell$ of Eq. (3).

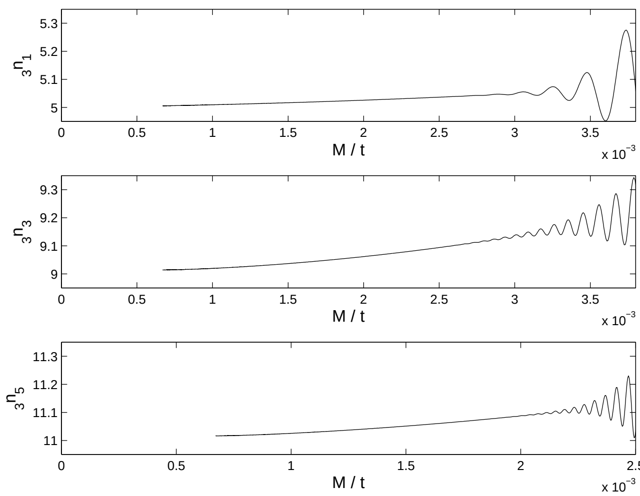

FIG. 8: The local power indices of the $\ell = 1, 3, 5$ multipole projections as functions of time, for initial data of a pure octupole field (same data as in Fig. 7).

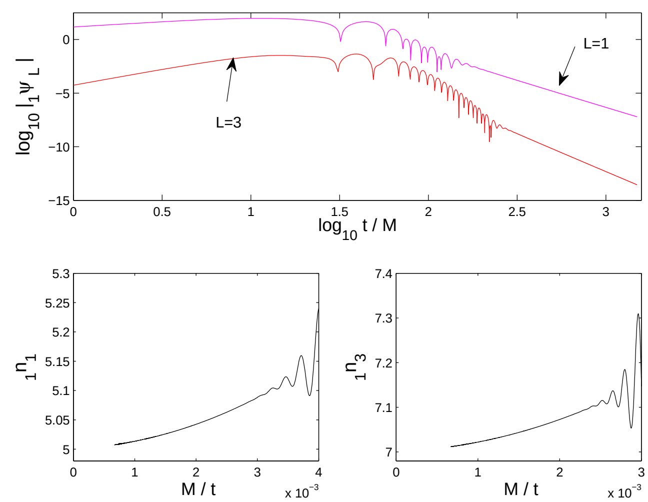

FIG. 9: The dipole ($\ell = 1$) and octupole ($\ell = 3$) projections, ${}_1\psi_3$ and ${}_1\psi_3$, correspondingly (upper panel), and their local power indices, ${}_1n_1$ and ${}_1n_3$, respectively (left and right lower panels), as functions of time, for initial data of a pure dipole ($\ell' = 1$) field.

V. APPARENT VIOLATION OF THE PRINCIPLE OF SUPERPOSITION

Consider the question of the decay rate of a certain multipole moment ℓ. In the Schwarzschild spacetime the decay rate depends only the the value of ℓ, specifically it is t^{-n} where $n = 2\ell + 3$. Most notably, the decay rate is independent of the history of the mode. This is no longer the case in the Kerr spacetime: the decay rate of a multipole ℓ does depend on how that mode came into existence. Therefore, in the Schwarzschild case one may say that "a multipole is a multipole," and consequently

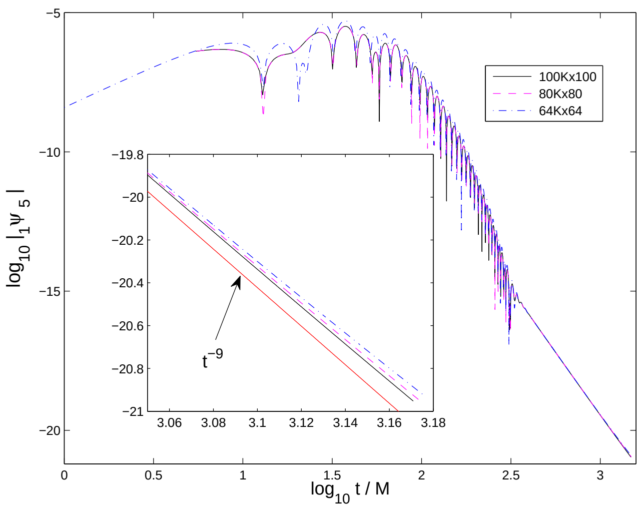

FIG. 10: The $\ell = 5$ multipole projections, for initial data of a pure monopole field as functions of time. Shown are the fields with three different grid densities, $64K \times 64$ (dash–dotted curve), $80K \times 80$ (dashed curve), and $100K \times 100$ (solid curve). The inserts show the same for segments of the full data, in addition to a reference curve $\sim t^{-9}$.

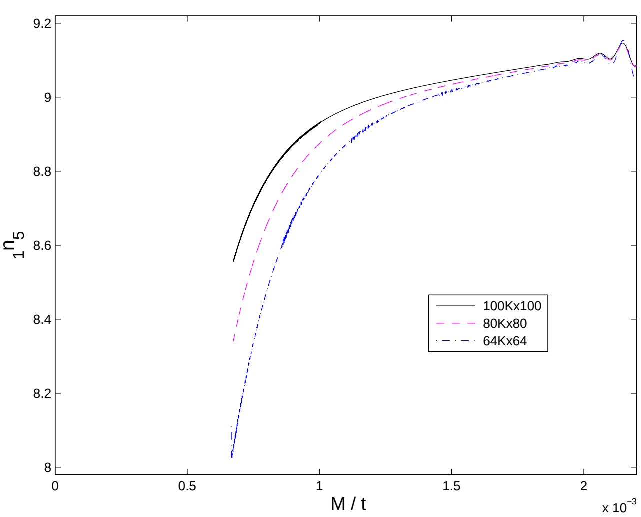

FIG. 11: The local power index ${}_1n_5$ as a function of the time t for three grid resolutions, $64K \times 64$ (dash–dotted curve), $80K \times 80$ (dashed curve), and $100K \times 100$ (solid curve).

its decay rate is intimately linked to its angular distribution and the associated symmetry group. Specifically, in Schwarzschild there is a one–to–one relation of the symmetry group of the mode in question and the mode's late–time decay rate, such that the symmetry determines the time evolution of the mode. We are next posing the question of whether the angular symmetry of a multipole uniquely determines the decay rate also in the Kerr spacetime. Put differently, is it still correct to say that "a multipole is a multipole?"

Consider the quadrapole mode in Kerr, i.e., $\ell = 2$. We first excite a pure $\ell = 2$ mode, and obtain the well known late–time decay rate of t^{-7} for the quadrapole projection of the full field. Notice, it is not the full field in whose decay rate we are interested here, but rather that of a subdominant projection. We then do a different simulation, and add to the initial data also a monopole field, $\ell = 0$. The monopole field excites higher multipoles, including the quadrapole. We then study the quadrapole projection of the late time field. This quadrapole field has two dominant contributions: the initial quadrapole field, and the excited quadrapole field from the initial monopole. (Notice, that there are also higher-order quadrapole components: the initial quadrapole, say, excites a hexadecapole field, which at later times contributes to the quadrapole moment through backreaction excitations; in what follows we ignore higher-order effects such as that.) If it is correct to say that "a multipole is a multipole," the quadrapole projection of the field would decay as t^{-7}, with a different amplitude than in the first case of pure quadrapole initial data, because of the contribution of the excited quadrapole. That is, one may expect direct application of the superposition principle: there would be two quadrapole fields present —one from the initial data, and one excited— and the full quadrapole projection is a linear superposition of the two, so that one may naively expect the total decay rate of the quadrapole projection of the full field at late time to decay as t^{-7}. This is found to be incorrect. The decay rate of the field is t^{-5}.

Why does the superposition of two quadrapole fields decay according to a different power law than the decay rate of pure quadrapole initial data? We find that the principle of superposition in fact does not fail, as indeed must be the case in a linear problem such as this one. Instead, we find that it is incorrect to say that "a multipole is a multipole," and the decay rate of the field does not depend only on the angular symmetry of the mode. Instead, there is also intricate dependence on the radial profile of the mode, as we argue in what follows. The monopole field excites an "up" quadrapole, and according to our formula the decay rate of this excited quadrapole field is t^{-5}. This excited quadrapole dominates over the evolved quadrapole from the initial data, that drops off faster at late times, specifically decays as t^{-7}. The full quadrapole projection of the field decays therefore at the slower rate of t^{-5}.

The decay rate of the excited quadrapole is understood from two complementary viewpoints: first, as an excited "up" mode, it has $\ell' = 0$, $\ell = 2$, $m = 0$ and $q = 0$, so that the proposed formula predicts $n = 5$; second, as a mode of "generic" multipolar evolution (i.e., a non–fine tuned mixture of multipoles) it has $\ell = 2$, $m = 0$ and $q = 0$, such that the Barack–Ori result also predicts decay rate of t^{-5}. We therefore have two kinds of quadapole fields: one decaying with $n = 7$ (the evolved initial data of a quadrapole field) and the other decaying with $n = 5$ (the evolved excited quadrapole of the initial monopole). Because fields with the same symmetry decay according to different power law depending on how they came into existence, it is not correct to say that "a multipole is a multipole," and symmetry does not determine the time evolution.

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