On the existence of cosmological event horizon

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Abstract

We show that, for general static or axisymmetric stationary spacetimes, a cosmological horizon exists only if $R_{ab}n^a n^b < 0$ for a hypersurface orthogonal timelike n^a, at least over some portion of the region of interest of the manifold. This implies violation of the strong energy condition by the matter fields, the simplest example of which is a positive cosmological constant.

Keywords: Cosmological horizon, strong energy condition, positive cosmological constant

It is generally accepted that a positive cosmological constant implies the existence of a cosmological horizon, i.e. an outer event horizon. If a positive cosmological constant Λ is added into the Einstein equation, we find de Sitter space in the absence of matter. If the spacetime is additionally assumed to be static, or axisymmetric and stationary, the solution to the vacuum Einstein equations is a Schwarzschild-de Sitter or Kerr-de Sitter black hole. What happens if there is matter? A sufficiently low matter density should produce a perturbation on the de Sitter black hole background. How does this perturbation affect the global properties of the spacetime? In particular, is there still an outer (cosmological) event horizon? More generally, what is the criterion for the existence of a cosmological event horizon? We were unable to find in the literature anything resembling an existence proof, so we decided to construct one. The motivation to look for horizons in spacetimes with a positive cosmological constant comes from recent observations that our universe is very likely endowed with one [1, 2].

The goal of this paper is to find the general conditions for which a stationary spacetime has an outer cosmological horizon. We consider two types of spacetimes, one static, and the other stationary and axisymmetric. An inner (black hole) event horizon is not assumed, although one may be present. We assume the existence of a null outer horizon and find the condition that the stress-energy tensor has to fulfil for the Einstein equations to hold. We find that the strong energy condition must be violated by the stress-energy tensor, at least over some part of the spacelike region inside the outer horizon. While a positive cosmological constant does this, we also find conditions on the stress-energy tensor due to ordinary matter so that $\Lambda > 0$ implies an outer horizon.

Let us then start with a spacetime which is static in some region. In this region the spacetime is endowed with a timelike Killing vector field ξ^a,

$$\nabla_a \xi_b + \nabla_b \xi_a = 0, \tag{1}$$

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with norm $\xi_a \xi^a = -\lambda^2$. Since the spacetime is static, ξ^a is orthogonal to a family of spacelike hypersurfaces Σ, and the Frobenius condition is satisfied,

$$\xi_{[a} \nabla_b \xi_{c]} = 0. \tag{2}$$

We shall define a horizon of this spacetime as a null hypersurface on which ξ^a becomes null, i.e., $\lambda^2 = 0$.

Let us consider the Killing identity

$$\nabla_a \nabla^a \xi_b = -R_{ab} \xi^a \,, \tag{3}$$

and contract both sides of Eq. (3) by ξ^b to obtain

$$\nabla_a \nabla^a \lambda^2 = 2R_{ab} \xi^a \xi^b - 2 \left(\nabla_a \xi_b\right) \left(\nabla^a \xi^b\right). \tag{4}$$

On the other hand, we can use Killing's equation (1) and the Frobenius condition (2) to get

$$\nabla_a \xi_b = \frac{1}{\lambda} \left(\xi_b \nabla_a \lambda - \xi_a \nabla_b \lambda\right), \tag{5}$$

which we substitute into Eq. (4) to obtain

$$\nabla_a \nabla^a \lambda^2 = 2R_{ab} \xi^a \xi^b + 4 \left(\nabla_a \lambda\right) \left(\nabla^a \lambda\right). \tag{6}$$

In order to project Eq. (6) onto Σ, we consider the usual projector or the induced metric on Σ

$$h_a{}^b = \delta_a{}^b + \lambda^{-2} \xi_a \xi^b. \tag{7}$$

Let us also write D_a for the induced connection on Σ. Then for any p-form Ω whose projection on Σ is ω, and which satisfies $\mathcal{L}_\xi \Omega = 0$ [3],

$$\nabla_a \Omega^{a \cdots} = \frac{1}{\lambda} D_a (\lambda \omega^{a \cdots}). \tag{8}$$

Choosing the 1-form $\mathrm{d}\lambda$ for Ω in this equation, and using Eq. (6), we find

$$\frac{1}{\lambda} D_a \left(\lambda D^a \lambda^2\right) = 2 \left[R_{ab} \xi^a \xi^b + 2 \left(D_a \lambda\right) \left(D^a \lambda\right)\right]. \tag{9}$$

Eq. (9) can be now integrated over the spacelike hypersurface Σ to produce

$$\oint_{\partial\Sigma} \lambda D_a \lambda^2 d\gamma^{(2)a} = 2 \int_\Sigma \lambda \left[R_{ab} \xi^a \xi^b + 2 \left(D_a \lambda\right) \left(D^a \lambda\right)\right], \tag{10}$$

where the surface integral on the left hand side is calculated over the boundary of Σ.

According to our assumption, the spacetime has a closed outer boundary or cosmological horizon, so that $\lambda = 0$ there. If we also have a black hole present in the spacetime, the inner boundary is the black hole event horizon, and we will also have $\lambda = 0$ there. We will also assume that the derivative of λ^2 is appropriately non-divergent on the horizons. Then the surface integral over the horizons in Eq. (10) vanishes, and we get

$$\int_\Sigma \lambda \left[R_{ab} \xi^a \xi^b + 2 \left(D_a \lambda\right) \left(D^a \lambda\right)\right] = 0. \tag{11}$$

The second term in Eq. (11) is a spacelike inner product and hence positive definite over Σ, so we must have a negative contribution from the first term $R_{ab} \xi^a \xi^b$. In other words, the outer horizon or the cosmological horizon will exist only if

$$R_{ab} \xi^a \xi^b < 0, \tag{12}$$

at least over some portion of Σ, so that the total integral in Eq. (11) vanishes. Using the Einstein equations

$$R_{ab} - \frac{1}{2} R g_{ab} = T_{ab}, \tag{13}$$

we see that the condition (12) implies that the strong energy condition (SEC) is violated by the energy-momentum tensor

$$\left(T_{ab} - \frac{1}{2} T g_{ab} \right) \xi^a \xi^b < 0, \tag{14}$$

at least over some portion of Σ. We know that a positive cosmological constant Λ, appearing on the right hand side of the Einstein equations as $-\Lambda g_{ab}$, violates the SEC. We now split the total stress-energy tensor T_{ab} as

$$T_{ab} = -\Lambda g_{ab} + T^{\mathrm{N}}_{ab}, \tag{15}$$

where the superscript 'N' denotes 'normal' matter fields satisfying SEC. Then Eq. (11) becomes

$$\int_\Sigma \lambda \left[X^{\mathrm{N}} + 2(D_a \lambda)(D^a \lambda) - \Lambda \lambda^2 \right] = 0. \tag{16}$$

X^{N} is a positive definite contribution from the normal matter satisfying SEC. So for the cosmological horizon to exist, we must have

$$\int_\Sigma \lambda \left[X^{\mathrm{N}} - \Lambda \lambda^2 \right] < 0. \tag{17}$$

In other words, the cosmological constant term (with $\Lambda > 0$) has to dominate the integral if there is to be an outer horizon. It is interesting to note that the observed values of Λ and matter densities in the universe satisfy this requirement. So would a universe with $\Lambda > 0$ in which all matter is restricted to a finite region in space. This has relevance in discussions of late time behavior of black holes formed by collapse.

This result can be generalized to stationary axisymmetric spacetimes, in general rotating, which satisfy some additional constraints. The basic scheme will be the same as before. For the spacetime we assume two commuting Killing fields $(\xi^a,\ \phi^a)$,

$$\nabla_{(a}\xi_{b)} = 0 = \nabla_{(a}\phi_{b)}\,, \tag{18}$$

$$[\xi, \phi]^a = 0\,. \tag{19}$$

ξ^a is locally timelike with norm $-\lambda^2$, whereas ϕ^a is a locally spacelike Killing field with closed orbits and norm f^2. We also assume that the vectors orthogonal to ξ^a and ϕ^a span an integral submanifold. In other words, local coordinates orthogonal to ξ^a and ϕ^a can be specified everywhere on the spacetime. This, and the last condition above, are the additional constraints mentioned above. We note that known stationary axisymmetric spacetimes obey these restrictions.

For a rotating spacetime, ξ^a is not orthogonal to ϕ^a, so in particular there is no spacelike hypersurface tangent to ϕ^a and orthogonal to ξ^a. Let us first construct a family of spacelike hypersurfaces. If we define χ_a as

$$\chi_a = \xi_a - \frac{1}{f^2} \left(\xi_b \phi^b \right) \phi_a \equiv \xi_a + \alpha \phi_a, \tag{20}$$

we will have $\chi_a \phi^a = 0$ everywhere. An orthogonal basis for the spacetime can be written as $\{\chi^a,\ \phi^a,\ \mu^a,\ \nu^a\}$. The norm of χ^a is calculated to be

$$\chi_a \chi^a = -\beta^2 = -\left(\lambda^2 + \alpha^2 f^2 \right), \tag{21}$$

i.e., χ_a is timelike when $\beta^2 > 0$. we can also calculate that

$$\nabla_{(a}\chi_{b)} = \phi_a \nabla_b \alpha + \phi_b \nabla_a \alpha. \tag{22}$$

Our assumption that $\{\mu^a, \nu^a\}$ span an integral 2-manifold implies that

$$\begin{aligned} \chi_{[a}\phi_b \nabla_c \phi_{d]} &= 0\,, && (23)\\ \phi_{[a}\chi_b \nabla_c \chi_{d]} &= 0\,. && (24) \end{aligned}$$

where we have also used Eq. (20). A straightforward calculation from here shows that

$$\nabla_a \chi_b - \nabla_b \chi_a = 2\beta^{-1}\left(\chi_b \nabla_a \beta - \chi_a \nabla_b \beta\right). \tag{25}$$

It follows that χ^a satisfies the Frobenius condition,

$$\chi_{[a}\nabla_b \chi_{c]} = 0\,. \tag{26}$$

So there is a family of spacelike hypersurfaces Σ orthogonal to χ^a, although we should note that χ^a is not a Killing vector field. In a rotating black hole spacetime, ξ^a becomes spacelike within the ergosphere [4], so for such spacetimes $\lambda^2 = 0$ does not define a horizon. The horizons are now at $\beta^2 = 0$.

Using the Killing identities $\nabla_a \nabla^a \xi_b = -R_b{}^a \xi_a$, and $\nabla_a \nabla^a \phi_b = -R_b{}^a \phi_a$, and also the orthogonality $\chi_a \phi^a = 0$, we obtain

$$\chi^b \nabla_a \nabla^a \chi_b = -R_{ab}\chi^a \chi^b + 2\chi^a \nabla_c \phi_a \nabla^c \alpha\,, \tag{27}$$

which is equivalent to

$$\nabla_a \nabla^a \beta^2 = 2R_{ab}\chi^a \chi^b - 2\nabla^c \chi^a \nabla_c \chi_a - 4\chi^a \nabla_c \phi_a \nabla^c \alpha. \tag{28}$$

Note that if we set $\alpha = 0$ in Eq. (28), we recover the static case of Eq. (4).

Next we note that the subspace spanned by $\{\chi^a,\ \mu^a,\ \nu^a\}$ do not form a hypersurface. This is because the necessary and sufficient condition that an arbitrary subspace of a manifold forms an integral submanifold or a hypersurface is the existence of a Lie algebra of the basis vectors of that subspace (see e.g. [4] and references therein). The condition Eq. (26) follows from this. On the other hand, Lie brackets among $\{\chi^a,\ \mu^a,\ \nu^a\}$ do not close. For example,

$$[\chi,\ \mu]^a = [\xi,\ \mu]^a + \alpha[\phi,\ \mu]^a + \phi^a \mu^b \nabla_b \alpha. \tag{29}$$

Since μ^a is not a Killing field, the last term on the right hand side of Eq. (29) is not zero. A similar argument holds for ν^a . Therefore the vectors spanned by $\{\chi^a,\ \mu^a,\ \nu^a\}$ do not form a Lie algebra. This implies that we cannot write a condition like $\phi_{[a}\nabla_b \phi_{c]} = 0$.

However, according to our assumptions, there are integral spacelike 2-manifolds orthogonal to both χ^a and ϕ^a. These are spanned by $\{\mu^a, \nu^a\}$. Then we must have

$$\phi_{[a} D_b \phi_{c]} = 0, \tag{30}$$

where D_b is the connection induced on Σ defined via the projector $h_a{}^b = \delta_a{}^b + \beta^{-2}\chi_a \chi^b$, exactly in the same manner as in the static case. Then we can write

$$D_a \phi_b = h_a{}^c h_b{}^d \nabla_c \phi_d = \nabla_a \phi_b + \beta^{-2}\left(\chi_a \phi^c \nabla_b \chi_c - \chi_b \phi^c \nabla_a \chi_c\right). \tag{31}$$

Using the expression of $\nabla_a \chi_b$ from Eq. (22) and Eq. (25), we can rewrite this as

$$D_a \phi_b = \nabla_a \phi_b + \frac{f^2}{2\beta^2}\left[\chi_a \nabla_b \alpha - \chi_b \nabla_a \alpha\right]. \tag{32}$$

It follows from this equation that we can write the Killing equation for ϕ_a on Σ as

$$D_a\phi_b + D_b\phi_a = 0. \tag{33}$$

Using this equation and the Frobenius condition of Eq. (30), we derive the expression

$$\nabla_a\phi_b = \frac{1}{f}\left[\phi_b D_a f - \phi_a D_b f\right] + \frac{f^2}{2\beta^2}\left[\chi_b\nabla_a\alpha - \chi_a\nabla_b\alpha\right]. \tag{34}$$

These are all that is needed to simplify Eq. (28). Substituting the expressions for $\nabla_a\chi_b$ and $\nabla_a\phi_b$ into Eq. (28) we get

$$\nabla_a\nabla^a\beta^2 = 2R_{ab}\chi^a\chi^b + 4\left(\nabla_a\beta\right)\left(\nabla^a\beta\right) + f^2\left(\nabla_a\alpha\right)\left(\nabla^a\alpha\right). \tag{35}$$

Now we know that $\chi^a\nabla_a\beta = 0$, and from the fact that ξ^a commutes with ϕ^a, it follows that $\chi^a\nabla_a\alpha = 0$. With this, using the same line of argument as for Eq. (16), we get

$$\int_\Sigma \beta\left[X^{\mathrm{N}} + 2\left(D_a\beta\right)\left(D^a\beta\right) + \frac{f^2}{2}\left(D_a\alpha\right)\left(D^a\alpha\right) - \Lambda\beta^2\right] = 0, \tag{36}$$

if the spacetime has an outer or cosmological event horizon. For $T_{ab} = 0$ in Eq. (36), we get Kerr-de Sitter solution [5]. We note that the assumption of integral 2-manifolds orthogonal to both the Killing fields ξ^a and ϕ^a was crucial to the proof. For a completely general stationary axisymmetric spacetime, the existence of such submanifolds is not guaranteed, and thus an outer horizon may not exist in such cases, even for $\Lambda > 0$.

To summarize, we have found that in both the static and the stationary axisymmetric cases, existence of an outer horizon requires a violation of the strong energy condition. This can be through a positive cosmological constant, for which there is strong observational evidence, or through exotic matter.

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