Acoustic geometry for general relativistic barotropic irrotational fluid flow

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9 January 2010; LaTeX-ed January 8, 2010

Abstract

"Acoustic spacetimes", in which techniques of differential geometry are used to investigate sound propagation in moving fluids, have attracted considerable attention over the last few decades. Most of the models currently considered in the literature are based on non-relativistic barotropic irrotational fluids, defined in a flat Newtonian background. The extension, first to special relativistic barotropic fluid flow, and then to general relativistic barotropic fluid flow in an arbitrary background, is less straightforward than it might at first appear. In this article we provide a pedagogical and simple derivation of the general relativistic "acoustic spacetime" in an arbitrary $(d+1)$ dimensional curved-space background.

Keywords: acoustic spacetime; relativistic; fluid flow; barotropic; irrotational.

Contents

1 Introduction

In this article we shall present a simple and pedagogical derivation of the general relativistic version of the "acoustic metric" defined on an arbitrary curved $(d + 1)$ dimensional background spacetime. While there are related observations and more limited derivations extant in the literature, we feel that the current analysis has some definite advantages. For instance, the early 1980 analysis due to Moncrief is restricted to perturbations of a spherically symmetric fluid flow on a Schwarzschild background [1], and in the more recent 1999 derivation due to Bilic [2] it can be somewhat difficult to discern what is truly fundamental input from what is derived output. Since scientific interest in this field is both significant and ongoing [3, 4, 5], we feel it useful to carefully lay out the minimal set of assumptions and logic flow behind the derivation.

Our strategy is as follows:

- We shall first motivate the result (up to a conformal factor) by considering the acoustic version of the Gordon metric, which was introduced for geometrical optics in 1923 [6].
- We shall then carefully specify what is meant by "irrotational flow" in a general relativistic context, introducing the appropriate notion of velocity potential.
- From the relativistic Euler equation, using only the irrotational condition and the barotropic condition, we will derive the relativistic Bernoulli equation.
- From the relativistic energy equation, using only the barotropic condition, (that is, without using the irrotational condition), we shall derive a flux conservation law (continuity equation).
- We shall delay the introduction of thermodynamic arguments as long as practicable. (We would argue that thermodynamics is in fact a side issue not central to the derivation.)
- As usual, the acoustic metric follows from combining the linearized Bernoulli equation and linearized equation of state with the linearized continuity equation.
- We shall carefully explain the subtleties involved in taking the non-relativistic limit.
- We shall finish with some discussion, and relegate several thermodynamic observations to the appendices.

The key result that we shall be aiming for is this: the (contravariant) acoustic metric governing acoustic perturbations of an irrotational barotropic fluid flow in $(d+1)$ dimensions is

$$\mathcal{G}^{ab} = \left(\frac{n_0^2 \; c_s^{-1}}{\varrho_0 + p_0} \right)^{-2/(d-1)} \left\{ -\frac{c^2}{c_s^2} \; V_0^a \, V_0^b + h^{ab} \right\}. \tag{1}$$

In counterpoint, the (covariant) acoustic metric in $(d+1)$ dimensions is

$$\mathcal{G}_{ab} = \left(\frac{n_0^2 \; c_s^{-1}}{\varrho_0 + p_0} \right)^{2/(d-1)} \left\{ -\frac{c_s^2}{c^2} \; [V_0]_a \, [V_0]_b + h_{ab} \right\}. \tag{2}$$

Here:

- c_s is the speed of sound, defined as usual via $c_s^2 = c^2\, \partial p/\partial \varrho$.
- c is the speed of light, used for instance in defining $x^0 = c\,t$.
- V_0 is the dimensionless 4-velocity of the background fluid flow.
- $h_{ab} = g_{ab} + [V_0]_a [V_0]_b$ is the dimensionless orthogonal projection of the physical spacetime metric g_{ab} onto the 3-space perpendicular to the 4-velocity of the fluid.
- the indices on the background fluid flow $[V_0]^a$ are lowered and raised using the physical spacetime metric g_{ab} and its inverse g^{ab}.
- in contrast $\mathcal{G}_{ab}$ and $\mathcal{G}^{ab}$ are defined to be matrix inverses of each other; these indices are *not* to be raised and lowered with the physical spacetime metric.
- n_0 is the background number density of fluid particles.
- ϱ_0 is the background energy density of the fluid.
- p_0 is the background pressure of the fluid.

This key result is easy to *motivate* (but not *derive*) in the limit of "ray acoustics", also known as "geometric acoustics", where we can safely ignore the wave properties of sound. In this limit we are interested only in the "sound cones". Let us pick a point in spacetime where the background fluid 4-velocity is V_0^a. Now adopt Gaussian normal coordinates, *and* go to the local rest frame of the fluid. Then taking $x^0 = ct$ we have

$$[V_0]^a \to (1; 0, 0, 0), \tag{3}$$

and

$$g_{ab} \to \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \qquad h_{ab} = g_{ab} + [V_0]_a\,[V_0]_b \to \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{4}$$

In the rest frame of the fluid the sound cones are (locally) given by

$$-\,c_s^2\;\mathrm{d}t^2 + ||\mathrm{d}\vec{x}||^2 = 0, \tag{5}$$

which we can rewrite as

$$-\frac{c_s^2}{c^2}\,(c\,\mathrm{d}t)^2 + ||\mathrm{d}\vec{x}||^2 = -\frac{c_s^2}{c^2}\,(\mathrm{d}x^0)^2 + ||\mathrm{d}\vec{x}||^2 = 0, \tag{6}$$

implying in these special coordinates the existence of an acoustic metric

$$\mathcal{G}_{ab} \propto \left[\begin{array}{cccc} -c_s^2/c^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]. \tag{7}$$

That is, transforming back to arbitrary coordinates:

$$\mathcal{G}_{ab} \propto -\frac{c_s^2}{c^2}\,[V_0]_a\,[V_0]_b + h_{ab}. \tag{8}$$

We now rewrite this as

$$\mathcal{G}_{ab} \propto -\frac{c_s^2}{c^2}[V_0]_a\,[V_0]_b + \{g_{ab} + [V_0]_a\,[V_0]_b\} \propto g_{ab} + \left\{1 - \frac{c_s^2}{c^2}\right\}[V_0]_a\,[V_0]_b. \tag{9}$$

Note that this is essentially a generalization of the derivation (dating back to 1923) of the so-called "Gordon metric" [6] used in ray optics to describe the "optical metric" appropriate to a (possibly) relativistic fluid with position-dependent refractive index $n(t, \vec{x})$. In Gaussian normal coordinates comoving with the fluid the Gordon metric is given by

$$\mathcal{G}_{ab} \propto \left[\begin{array}{cccc} -1/n^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]. \tag{10}$$

That is, transforming back to arbitrary coordinates:

$$\mathcal{G}_{ab} \propto -\frac{1}{n^2}[V_0]_a\,[V_0]_b + \{g_{ab} + [V_0]_a\,[V_0]_b\} \propto g_{ab} + \left\{1 - \frac{1}{n^2}\right\}[V_0]_a\,[V_0]_b. \tag{11}$$

Note that in either the ray acoustics or ray optics limits, because one only has the sound cones or light cones to work with, one can neither derive nor is it even meaningful to specify the overall conformal factor [3, 8, 9]. The calculation presented below is designed to go beyond the ray acoustics limit, to obtain a relativistic wave equation suitable for describing physical acoustics — all the "fuss" is simply over how to determine the overall conformal factor (and to verify that one truly does obtain a d'Alembertian equation using the conformally fixed acoustic metric).

2 Basic general relativistic fluid mechanics

In this Section we consider fully relativistic barotropic inviscid irrotational flow on an arbitrary general relativistic background, and derive the relevant wave equation for linearized fluctuations. We already know exactly what happens in the non-relativistic case [3, 7, 8, 9]. We go straight to curved spacetime relativistic fluid mechanics, where the geometry is described by a metric tensor $g_{ab}(x)$ of signature $-+++$, [or more generally $-(+)^d$], and the fluid is described by the energy density ϱ, pressure p, and 4-velocity V^a where

$$g_{ab} V^a V^b = -1. \tag{12}$$

The two relevant fluid dynamical equations are extremely well-known (see, for instance, Hawking and Ellis [10]).

Relativistic energy equation:

$$\nabla_a(\varrho V^a) + p(\nabla_a V^a) = 0. \tag{13}$$

Relativistic Euler equation:

$$(\varrho + p) V^b \nabla_b V^a = -(g^{ab} + V^a V^b) \nabla_b p. \tag{14}$$

These equations can be combined into the single statement that the stress-energy tensor is covariantly conserved (see, for instance, Hawking and Ellis [10])

$$\nabla_a \left[(\varrho + p) V^a V^b + p g^{ab} \right] = 0. \tag{15}$$

Note the subtle (perhaps not so subtle) differences from the non-relativistic case: ϱ is now energy density, *not mass density* ρ, and the continuity equation looks different — at least at this stage of the calculation it has been replaced by an "energy conservation" equation, and the Euler equation now contains the combination $(\varrho + p)$. The 4-acceleration of the fluid is

$$A^a = V^b \nabla_b V^a, \qquad \text{such that} \qquad A^a \, V_a = 0. \tag{16}$$

2.1 Defining relativistic irrotational flow

What do we mean by irrotational flow in a relativistic setting? Construct the 1-form

$$v = V_a \, \mathrm{d}x^a, \tag{17}$$

and consider the 2-form ω_2 and 3-form t_3:

$$\omega_2 = \mathrm{d}v; \qquad t_3 = v \wedge \omega_2. \tag{18}$$

In a relativistic context, setting the 2-form $\omega_2 = 0$ is too strong a condition, setting the 3-form $t_3 = 0$ is just right. (See, for instance, Hawking and Ellis [10].) Indeed, adopting Gaussian normal coordinates and going to the local rest frame of the fluid, where $V^a \to (1; 0, 0, 0)$, setting $t_3 = 0$ implies $\partial_{[i} v_{j]} = 0$, so the *spatial* components of the flow velocity are locally irrotational (in the sense of being curl-free). But $t_3 = 0$ implies (via the Frobenius theorem) that locally there exist functions α and Θ such that

$$v = \alpha \; \mathrm{d}\Theta; \qquad V^a = \alpha \; g^{ab} \; \nabla_b \Theta. \tag{19}$$

But then, from the normalization condition for the 4-velocity

$$V^a = \frac{g^{ab} \; \nabla_b \Theta}{\sqrt{-g^{ab} \; \nabla_a \Theta \; \nabla_b \Theta}}. \tag{20}$$

We shall find it extremely useful to define

$$||\nabla\Theta||^2 = -g^{ab} \; \nabla_a \Theta \; \nabla_b \Theta \tag{21}$$

so that

$$V^a = \frac{g^{ab} \; \nabla_b \Theta}{||\nabla\Theta||}. \tag{22}$$

This is the *relativistic* condition for irrotational flow. The function Θ can now be interpreted as the general relativistic version of the velocity potential. Note that for any smooth function $F(\cdot)$ we can replace $\Theta \leftrightarrow F(\Theta)$ *without affecting the 4-velocity* V^a — both numerator and denominator above pick up factors of $F'(\Theta)$ which then cancel. This freedom in choosing the scalar potential, Θ, will be very useful when analyzing the Euler equation, and will allow us to obtain a relativistic Bernoulli equation.

For the discussion below define the projection operator

$$h^{ab} = g^{ab} + V^a \, V^b. \tag{23}$$

Using this projection operator, and the 4-orthogonality of 4-velocity with 4-acceleration, it is easy (for relativistic irrotational flow) to calculate

$$\nabla_b V_a = (\delta_a{}^c + V_a V^c) \frac{\nabla_c \nabla_b \Theta}{||\nabla\Theta||}. \tag{24}$$

Therefore, for relativistic irrotational flow, the 4-acceleration reduces to

$$A_a = V^b \nabla_b V_a = -\frac{1}{2} (\delta_a{}^c + V_a V^c) \frac{\nabla_c(-g_{de} \nabla_d \Theta \nabla_e \Theta)}{-g^{ab} \, \nabla_a \Theta \, \nabla_b \Theta}, \tag{25}$$

so that

$$A^a = -(g^{ab} + V^a V^b) \nabla_b \left(\log ||\nabla\Theta||\right). \tag{26}$$

Note that automatically $A^a V_a = 0$, as required. Furthermore, note that the 4-acceleration is the *projection* of the gradient of a scalar, and that Θ can be chosen to have the dimensions of a distance so that $\nabla\Theta$ is dimensionless.

2.2 From Euler equation to Bernoulli equation

The Euler equation for relativistic irrotational flow now reads

$$(g^{ab} + V^a V^b) \, \nabla_b \left(\log ||\nabla\Theta||\right) = (g^{ab} + V^a V^b) \, \frac{\nabla_b p}{\varrho + p}. \tag{27}$$

We now make use of the barotropic condition $\varrho = \varrho(p)$ to obtain

$$\frac{\nabla_b p}{\varrho + p} = \nabla_b \int_0^p \frac{\mathrm{d}p}{\varrho(p) + p}, \tag{28}$$

so the Euler equation becomes

$$(g^{ab} + V^a V^b) \nabla_b \left(-\log ||\nabla\Theta|| + \int_0^p \frac{\mathrm{d}p}{\varrho(p) + p} \right) = 0. \tag{29}$$

Note that for any arbitrary function $f(\Theta)$ we have

$$(g^{ab} + V^a V^b) \nabla_b f(\Theta) = 0, \tag{30}$$

(since the projection operator kills the gradient). Therefore (using both the irrotational and barotropic conditions) we can integrate the Euler equation to yield:

$$-\log ||\nabla\Theta|| + \int_0^p \frac{\mathrm{d}p}{\varrho(p) + p} + f(\Theta) = \text{constant}. \tag{31}$$

where $f(\Theta)$ is (at this stage) an arbitrary "function of integration". This is our *preliminary* version of the general relativistic Bernoulli equation. To see how we might further simplify this result, recall (see for example [3, 8, 9]) that for non-relativistic irrotational inviscid barotropic flow the non-relativistic Bernoulli equation is

$$\dot{\Theta} + \frac{1}{2}(\nabla\Theta)^2 + \int_0^p \frac{\mathrm{d}p}{\rho(p)} + f(t) = \text{constant}. \tag{32}$$

(In the non-relativistic case ρ is the mass density.) Now, in the non-relativistic case we can always redefine

$$\Theta \to \Theta + \int f(t)\mathrm{d}t, \tag{33}$$

and use this transformation to eliminate $f(t)$ — so we can without loss of generality write

$$\dot{\Theta} + \frac{1}{2}(\nabla\Theta)^2 + \int_0^p \frac{\mathrm{d}p}{\varrho(p)} = \text{constant}. \tag{34}$$

In the relativistic case we now note:

$$-\log||\nabla\Theta|| + f(\Theta) = -\log(e^{-f(\Theta)}\,||\nabla\Theta||) = -\log||\nabla F(\Theta)||. \tag{35}$$

That is, in the relativistic case, by making the transformation

$$\Theta \to F(\Theta) = \int_0^\Theta e^{-f(\bar{\Theta})}\,\mathrm{d}\bar{\Theta}, \tag{36}$$

we can (*without changing* V^a) absorb the arbitrary function $f(\cdot)$ into a redefinition of Θ, and so write

$$-\log||\nabla\Theta|| + \int_0^p \frac{\mathrm{d}p}{\varrho(p)+p} = \text{constant}. \tag{37}$$

Finally, without loss of generality we can rescale Θ to set the constant appearing above to zero and so obtain:

$$\log||\nabla\Theta|| = \int_0^p \frac{\mathrm{d}p}{\varrho(p)+p}. \tag{38}$$

This is our final form for the general relativistic Bernoulli equation. Note that there is no longer any freedom left in choosing Θ, we have now used it all up. It is relatively common to exponentiate the above and rewrite the general relativistic Bernoulli equation as

$$||\nabla\Theta|| = \exp\left(\int_0^p \frac{\mathrm{d}p}{\varrho(p)+p}\right). \tag{39}$$

For the sake of pedagogical development of the argument we have *not* yet rewritten the integral on the RHS in terms of other thermodymamic variables; we prefer to delay this for now.

2.3 From energy equation to flux conservation

The energy conservation equation is

$$\nabla_a(\varrho V^a) + p(\nabla_a V^a) = 0, \tag{40}$$

which becomes

$$V^a \nabla_a \varrho + (\varrho + p)(\nabla_a V^a) = 0, \tag{41}$$

or

$$V^a \frac{\nabla_a \varrho}{\varrho + p} + (\nabla_a V^a) = 0. \tag{42}$$

Using the barotropic condition $\varrho = \varrho(p)$, (but note, now *without* using the irrotational condition), this can be written as

$$V^a \nabla_a \left[\int_{\varrho(p=0)}^{\varrho} \frac{\mathrm{d}\varrho}{\varrho + p(\varrho)} \right] + (\nabla_a V^a) = 0, \tag{43}$$

which implies

$$V^a \nabla_a \left\{ \exp\left[\int_{\varrho(p=0)}^{\varrho} \frac{\mathrm{d}\varrho}{\varrho + p(\varrho)} \right] \right\} + \exp\left[\int_{\varrho(p=0)}^{\varrho} \frac{\mathrm{d}\varrho}{\varrho + p(\varrho)} \right] (\nabla_a V^a) = 0. \tag{44}$$

This can now be rewritten as a continuity equation (flux conservation equation):

$$\nabla_a \left\{ \exp\left[\int_{\varrho(p=0)}^{\varrho} \frac{\mathrm{d}\varrho}{\varrho + p(\varrho)} \right] V^a \right\} = 0. \tag{45}$$

This is now a "standard form" zero-divergence continuity equation — note this the ability to derive this transformed form of the energy equation depends only on the barotropic assumption. Also note that we have not yet needed to even introduce, let alone discuss in any detail, the particle number density n, nor introduce any thermodynamic arguments.

2.4 Introducing the number density

Subject only to the barotropic condition, we have just derived the flux conservation equation given immediately above. We now suppose the barotropic fluid contains some type of conserved "tracker" particles. For example, one might be interested in counting baryons, Fe^{56} nuclei, or leptons. We now explicitly assume translation invariant and time invariant *composition* of the fluid. (That is, we assert that the equation of state is the same throughout the fluid; in other words there is no explicit time or position dependence in the equation of state.) Therefore the *ratios* of these tracker particle densities must be translation and time invariant constants. Furthermore since these "tracker particles" are all assumed to be conserved

$$\nabla_a \{n_i \, V^a\} = 0, \tag{46}$$

while, since the fluid is assumed to be barotropic, there must be functions of pressure p and energy density ϱ such that

$$n_i = n_i(p) = n_i(\varrho). \tag{47}$$

But the only way to satisfy *all* these constraints is if

$$n_i(p) = n_{i\,(p=0)} \; \exp\left[\int_{\varrho_{(p=0)}}^{\varrho(p)} \frac{\mathrm{d}\varrho}{\varrho + p(\varrho)} \right]. \tag{48}$$

In particular for the total particle density we have

$$n(p) = n_{(p=0)} \; \exp\left[\int_{\varrho_{(p=0)}}^{\varrho(p)} \frac{\mathrm{d}\varrho}{\varrho + p(\varrho)} \right]. \tag{49}$$

This observation is extremely convenient in that allows us to physically interpret the quantity

$$\exp\left[\int_{\varrho(p=0)}^{\varrho(p)} \frac{\mathrm{d}\varrho}{\varrho + p(\varrho)} \right] = \frac{n(p)}{n_{(p=0)}} = \frac{n_i(p)}{n_{i(p=0)}} \tag{50}$$

as being proportional to the number density of constituents making up the fluid. (If one wishes to take an extreme point of view and eschew all thermodyamic arguments completely, one could simply take this equation as the definition of a "shorthand symbol" $n(p)$, and ignore the physical interpretation of $n(p)$ as particle number density.) In terms of the number density the conservation equation now simply reads

$$\nabla_a \{n \, V^a\} = 0. \tag{51}$$

Furthermore, note that using these results we can rewrite the relativistic Bernoulli equation (39) as

$$\log ||\nabla\Theta|| = \int_0^p \frac{\mathrm{d}p}{\varrho(p) + p} = \int_0^p \frac{\mathrm{d}[\varrho(p) + p]}{\varrho(p) + p} - \int_{\varrho(p=0)}^{\varrho} \frac{\mathrm{d}\varrho}{\varrho + p(\varrho)} \tag{52}$$

$$= \log\left[\frac{\varrho(p) + p}{\varrho_{(p=0)}}\right] - \log\left[\frac{n(p)}{n_{(p=0)}}\right] = \log\left[\frac{[\varrho(p) + p]\, n_{(p=0)}}{n(p)\, \varrho_{(p=0)}}\right]. \tag{53}$$

That is

$$||\nabla\Theta|| = \frac{[\varrho(p)+p]\, n_{(p=0)}}{n(p)\, \varrho_{(p=0)}}. \tag{54}$$

3 Linearization

Let us now write

$$\Theta = \Theta_0 + \epsilon\, \Theta_1 + \dots, \tag{55}$$

which in particular implies that

$$V = V_0 + \epsilon\, V_1 + \dots, \tag{56}$$

and further assert

$$\varrho = \varrho_0 + \epsilon\, \varrho_1 + \dots, \tag{57}$$

$$p = p_0 + \epsilon\, p_1 + \dots. \tag{58}$$

Using these relations we now linearize the fluid equations around some assumed background flow. (Note that both the background fluid flow (V_0, ϱ_0, p_0), and the linearized fluctuations, satisfy the Bernoulli and energy conservation equations.) In a wider context, extending far beyond fluid dynamics, we mention that it is quite common for linearized fluctuations around an appropriately defined background to exhibit an "effective spacetime" behaviour [3, 11, 12].

3.1 Linearized continuity equation

From the continuity equation we see

$$\nabla_a \left\{ \exp\left[\int_{\varrho_{(p=0)}}^{\varrho_0+\epsilon\varrho_1+\dots} \frac{\mathrm{d}\varrho}{\varrho+p(\varrho)} \right] [V_0^a + \epsilon V_1^a + \dots] \right\} = 0. \tag{59}$$

Then to first order in ϵ

$$\nabla_a \left\{ \exp\left[\int_{\varrho_{(p=0)}}^{\varrho_0} \frac{\mathrm{d}\varrho}{\varrho+p(\varrho)} \right] \left(\frac{\varrho_1}{\varrho_0+p_0}\, V_0^a + V_1^a \right) \right\} = 0. \tag{60}$$

Using the number density we can rewrite this as

$$\nabla_a \left\{ n(p_0) \left(\frac{\varrho_1}{\varrho_0+p_0}\, V_0^a + V_1^a \right) \right\} = 0. \tag{61}$$

3.2 Linearized irrotational flow

Perturbing the 4-velocity for relativistic irrotational flow yields

$$V_0 + \epsilon V_1 + \dots = \frac{g^{ab}\, \nabla_b(\Theta_0 + \epsilon\Theta_1 + \dots)}{\sqrt{-g^{ab}\, \nabla_a(\Theta_0 + \epsilon\Theta_1 + \dots)\, \nabla_b(\Theta_0 + \epsilon\Theta_1 + \dots)}}. \tag{62}$$

Then expanding to first order in ϵ we see

$$V_1^a = \frac{(g^{ab} + V_0^a V_0^b)\, \nabla_b \Theta_1}{||\nabla\Theta_0||}. \tag{63}$$

Now use the relativistic Bernoulli equation to write

$$V_1^a = (g^{ab} + V_0^a V_0^b) \;\; \nabla_b \Theta_1 \;\; \exp\left(-\int_0^{p_0} \frac{\mathrm{d}p}{\varrho(p) + p} \right). \tag{64}$$

Alternatively

$$V_1^a = \frac{n_0 \;\; \varrho_{(p=0)}}{[\varrho_0 + p_0]\; n_{(p=0)}} \;\; (g^{ab} + V_0^a V_0^b) \;\; \nabla_b \Theta_1. \tag{65}$$

Note that we automatically have

$$g_{ab} V_1^a V_0^b = 0, \tag{66}$$

as required by the normalization condition, $g_{ab} V^a V^b = -1$, for V^a.

3.3 Linearized equation of state

Linearizing the equation of state we see

$$p_0 + \epsilon p_1 + \dots = p(\varrho_0 + \epsilon\varrho_1 + \dots), \tag{67}$$

that is

$$p_1 = \left.\frac{\mathrm{d}p}{\mathrm{d}\varrho}\right|_{\varrho_0} \;\; \varrho_1, \tag{68}$$

which we use to define what we shall soon enough see is the speed of sound

$$p_1 = \frac{c_s^2}{c^2}\, \varrho_1. \tag{69}$$

3.4 Linearized Euler (Bernoulli) equation

Linearizing the Bernoulli equation requires us to consider

$$\frac{1}{2}\log\left[-g^{cd}\,\nabla_c(\Theta_0+\epsilon\Theta_1+\dots)\,\nabla_d(\Theta_0+\epsilon\Theta_1+\dots)\right] = \int_0^{(p_0+\epsilon p_1+\dots)} \frac{\mathrm{d}p}{\varrho(p)+p}. \tag{70}$$

Then to first order in ϵ

$$\frac{-g^{cd}\nabla_c\Theta_1\nabla_d\Theta_0}{-g^{cd}\nabla_c\Theta_0\nabla_d\Theta_0} = \frac{p_1}{\varrho_0+p_0}. \tag{71}$$

That is, using the linearized equation of state,

$$-\frac{V_0^a\nabla_a\Theta_1}{||\nabla\Theta_0||} = \frac{c_s^2}{c^2}\,\frac{\varrho_1}{\varrho_0+p_0}, \tag{72}$$

which we can rearrange to

$$\varrho_1 = -(\varrho_0+p_0)\frac{c^2}{c_s^2}\frac{V_0^a\nabla_a\Theta_1}{||\nabla\Theta_0||}. \tag{73}$$

That is, now using the general relativistic Bernoulli equation,

$$\varrho_1 = -(\varrho_0+p_0)\frac{c^2}{c_s^2}\,\exp\left(-\int_0^{p_0}\frac{\mathrm{d}p}{\varrho(p)+p}\right)\,V_0^a\nabla_a\Theta_1. \tag{74}$$

Finally, using the number density we can rewrite this as

$$\varrho_1 = -\frac{n_0\;c^2}{n_{(p=0)}\;c_s^2}\;\varrho_{(p=0)}\;\;V_0^a\;\nabla_a\Theta_1. \tag{75}$$

3.5 Deriving the d'Alembertian equation

Now combine these results: insert the linearized Euler (Bernoulli) equation (75), and the linearized irrotational condition (65), into the linearized continuity equation (61). We obtain

$$\nabla_a\left\{-\frac{n_0^2\;c^2\;\varrho_{(p=0)}}{n_{(p=0)}\;c_s^2\;(\varrho_0+p_0)}V_0^aV_0^b\nabla_b\Theta_1 + \frac{n_0^2\;\varrho_{(p=0)}}{n_{(p=0)}\;(\varrho_0+p_0)}(g^{ab}+V_0^aV_0^b)\;\nabla_b\Theta_1\right\} = 0. \tag{76}$$

But of course $\varrho_{(p=0)}$ and $n_{(p=0)}$ are irrelevant position-independent constants, so we can just as easily write

$$\nabla_a\left\{-\frac{c^2\;n_0^2}{c_s^2\;(\varrho_0+p_0)}V_0^aV_0^b\nabla_b\Theta_1 + \frac{n_0^2}{(\varrho_0+p_0)}(g^{ab}+V_0^aV_0^b)\;\nabla_b\Theta_1\right\} = 0. \tag{77}$$

Introducing the projection tensor $h^{ab} = g^{ab} + V_0^a V_0^b$, and factorizing, this becomes

$$\nabla_a \left\{ \frac{n_0^2}{\varrho_0 + p_0} \; \left[-\frac{c^2}{c_s^2} \; V_0^a V_0^b + h^{ab} \right] \; \nabla_b \Theta_1 \right\} = 0. \tag{78}$$

This is in fact *exactly* the result we want — a d'Alembertian equation for the perturbation in the velocity potential Θ.

- After some trivial notational changes, this agrees (where the formalisms overlap), with both the observations of Moncrief [1], and with the presentation of Bilic [2], but the present exposition gives much more attention to the underlying details and makes only a bare minimum of technical assumptions.

- Note that everything so far is really dimension-independent, and that we can now read off the acoustic metric simply by setting

$$\sqrt{-\mathcal{G}} \; \mathcal{G}^{ab} = \frac{n_0^2}{\varrho_0 + p_0} \; \left[-\frac{c^2}{c_s^2} \; V_0^a V_0^b + h^{ab} \right]. \tag{79}$$

  The dimension-dependence now comes from solving this equation for $\mathcal{G}^{ab}$.

4 The general relativistic acoustic metric

From the dimension-independent result above we have, in (d+1) dimensions,

$$(-1)^{(d+1)/2} \; \mathcal{G}^{(d+1)/2-1} = -\frac{c^2}{c_s^2} \; \left(\frac{n_0^2}{\varrho_0 + p_0} \right)^{(d+1)}, \tag{80}$$

whence

$$(-\mathcal{G})^{(d+1)/2-1} = \frac{c^2}{c_s^2} \; \left(\frac{n_0^2}{\varrho_0 + p_0} \right)^{(d+1)}, \tag{81}$$

Dropping an irrelevant overall constant factor of $c^{-2/(d-1)}$, we finally have the (contravariant) acoustic metric

$$\mathcal{G}^{ab} = \left(\frac{n_0^2 \; c_s^{-1}}{\varrho_0 + p_0} \right)^{-2/(d-1)} \; \left\{ -\frac{c^2}{c_s^2} \; V_0^a V_0^b + h^{ab} \right\}, \tag{82}$$

and (covariant) acoustic metric

$$\mathcal{G}_{ab} = \; \left(\frac{n_0^2 \; c_s^{-1}}{\varrho_0 + p_0} \right)^{2/(d-1)} \; \left\{ -\frac{c_s^2}{c^2} \; [V_0]_a [V_0]_b + h_{ab} \right\}. \tag{83}$$

If one wishes to go to the extra trouble of making the acoustic metric dimensionless it is easy to re-insert appropriate position-independent constants in the conformal factor and obtain

$$\mathcal{G}^{ab} = \left(\frac{n_0^2 \; \varrho_{(p=0)} \; c}{n_{(p=0)}^2 \; (\varrho_0 + p_0) \; c_s} \right)^{-2/(d-1)} \left\{ -\frac{c^2}{c_s^2} \; V_0^a V_0^b + h^{ab} \right\}, \tag{84}$$

and

$$\mathcal{G}_{ab} = \left(\frac{n_0^2 \; \varrho_{(p=0)} \; c}{n_{(p=0)}^2 \; (\varrho_0 + p_0) \; c_s} \right)^{2/(d-1)} \left\{ -\frac{c_s^2}{c^2} \; [V_0]_a [V_0]_b + h_{ab} \right\}. \tag{85}$$

Of course these extra position-independent constants in the conformal factor carry no useful information and are commonly suppressed.

5 The non-relativistic limit

Compare this general relativistic acoustic metric with the non-relativistic limit, where the coordinates are most conveniently taken to be $x^a = (t; x^i)$ and where the d'Alembertian equation reduces to [3, 8, 9]

$$\nabla_a \left\{ \rho_0 \quad \left[-\frac{1}{c_s^2} \; V_0^a V_0^b + h^{ab} \right] \; \nabla_b \Theta_1 \right\} = 0, \tag{86}$$

with $\rho_0 \neq \varrho_0$ now being the *non-relativistic mass density*, (not the relativistic energy density), and the meanings of V_0 and h are suitably adjusted. In the non-relativistic case

$$V_0^a = (1; v^i); \qquad\qquad h^{ab} = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & \delta^{ij} \end{array} \right], \tag{87}$$

and independent of dimensionality we have

$$\sqrt{-\mathcal{G}} \; \mathcal{G}^{ab} = \rho_0 \quad \left[-\frac{1}{c_s^2} V_0^a V_0^b + h^{ab} \right], \tag{88}$$

implying

$$\mathcal{G}^{ab} = \left(\frac{\rho_0}{c_s} \right)^{-2/(d-1)} \left\{ -\frac{1}{c_s^2} V_0^a V_0^b + h^{ab} \right\}, \tag{89}$$

which can be inverted to yield

$$\mathcal{G}_{ab} = \left(\frac{\rho_0}{c_s} \right)^{2/(d-1)} \left[\begin{array}{c|c} -(c_s^2 - v^2) & -v^j \\ \hline -v^i & \delta^{ij} \end{array} \right]. \tag{90}$$

To see how and under what situations the general relativistic acoustic metric reduces to this non-relativistic acoustic metric first consider the conformal factor: In the non-relativistic limit $p_0 \ll \varrho_0$ and $\varrho_0 \approx \bar{m}\; n_0$, where $\bar{m}$ is the average mass of the particles making up to fluid (which by the barotropic assumption is a time-independent and position-independent constant). So in the non-relativistic limit we recover the standard result for the conformal factor [3, 7, 8, 9]

$$\frac{n_0^2\; c_s^{-1}}{\varrho_0 + p_0} \to \frac{n_0}{\bar{m} c_s} = \frac{1}{\bar{m}^2} \frac{\rho_0}{c_s} \propto \frac{\rho_0}{c_s}. \tag{91}$$

To now probe the tensor structure of the non-relativistic limit, let us recall that:

- c is the speed of light.
- c_s is the speed of sound.
- v is the three-velocity of the fluid.

Take conventions so that the physical spacetime metric and four-velocity are both dimensionless. In particular, the coordinates are chosen to be $x^a = (c\,t;\; x^i)$. Now adopting Gaussian normal coordinates at the point of interest

$$g_{ab} = \eta_{ab} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{92}$$

For the four-velocity we in general have

$$V_0^a = \gamma\left(1; v^i/c\right); \qquad [V_0]_a = \gamma\left(-1; v^i/c\right). \tag{93}$$

Remember that $\mathrm{d}x^0 = c\;\mathrm{d}t$, and note that the γ factor is defined using c the physical speed of light.

Now let us take the nonrelativistic limit. Ignoring the conformal factor, which comes along for the ride, we have

$$\mathcal{G}_{ab} \propto \left[g_{ab} + \left(1 - \frac{c_s^2}{c^2}\right) [V_0]_a [V_0]_b \right]. \tag{94}$$

- Then for the time-time component of the acoustic metric

$$\mathcal{G}_{00} \propto \left[-1 + \left(1 - \frac{c_s^2}{c^2}\right) \gamma^2 \right]. \tag{95}$$

This implies

$$\mathcal{G}_{00} \propto \left[-\frac{c_s^2 - v^2}{c^2} + \frac{\mathcal{O}(v^4, c_s^2 v^2)}{c^4} \right]. \tag{96}$$

So as long as *both* the speed of sound and the 3-velocity of the fluid are small compared to the speed of light we are justified in approximating

$$\mathcal{G}_{00} \propto -\frac{c_s^2 - v^2}{c^2} + \dots. \tag{97}$$

- In contrast for the time-space components of the acoustic metric

$$\mathcal{G}_{0i} \propto \left[\left(1 - \frac{c_s^2}{c^2}\right) \gamma^2 (-1) \left(\frac{+v^i}{c}\right) \right]. \tag{98}$$

This implies

$$\mathcal{G}_{0i} \propto -\left(1 - \frac{\mathcal{O}(c_s^2, v^2)}{c^2}\right) \frac{v^i}{c}. \tag{99}$$

So as long as *both* the speed of sound and the 3-velocity of the fluid are small compared to the speed of light we are justified in approximating

$$\mathcal{G}_{0i} \propto -\frac{v^i}{c} + \dots. \tag{100}$$

- Finally for the space-space components we note

$$\mathcal{G}_{ij} \propto \left[\delta_{ij} + \left(1 - \frac{c_s^2}{c^2}\right) \gamma^2 \frac{v^i \, v^j}{c^2} \right]. \tag{101}$$

This implies

$$\mathcal{G}_{ij} \propto \left[\delta_{ij} + \mathcal{O}(v^2/c^2) \right]. \tag{102}$$

So as long as the 3-velocity of the fluid is small compared to the speed of light we are justified in approximating

$$\mathcal{G}_{ij} \propto \delta_{ij} + \dots. \tag{103}$$

Collecting these results we see that in the nonrelativistic limit

$$\mathcal{G}_{ab} \propto \left[\begin{array}{c|c} -(c_s^2 - v^2)/c^2 & -v^i/c \\ \hline -v^j/c & \delta_{ij} \end{array} \right] + \dots \tag{104}$$

But now we realise that in the present context c is just some convenient fixed conversion constant from $\mathrm{d}x^0 = c\,\mathrm{d}t$ to $\mathrm{d}t$, so if we work in terms of the coordinates $(t;\ x^i)$, which are perhaps more natural in the non-relativistic limit, then

$$\mathcal{G}_{ab} \propto \left[\begin{array}{c|c} -(c_s^2 - v^2) & -v^i \\ \hline -v^j & \delta_{ij} \end{array}\right] + \dots \tag{105}$$

as required. Note that to do all this it is essential that *both* v and c_s are small compared to c, though there is no constraint on the relative sizes of v and c_s.

6 Discussion

Under what conditions are the fully general relativistic derivation of this article necessary? (The non-relativistic analysis of [3, 7, 8, 9] is after all the basis of the bulk of the work in "analogue spacetimes", and is perfectly adequate for many purposes.) The current analysis will be needed in four separate situations:

- When working in an arbitrary curved relativistic background; (for example in the problems considered by Moncrief [1], and Bilic [2]).
- Whenever the fluid is flowing at relativistic speeds; (for example in the problems considered by Moncrief [1], and Bilic [2]).
- Less obviously, whenever the speed of sound is relativistic, even if background flows are non-relativistic; (for example a near-equilibrum photon gas where $c_s^2 = \frac{1}{3}c^2$ but flow velocities are all small $v \ll c$).
- Even less obviously, when the internal degrees of freedom of the fluid are relativistic, even if the overall fluid flow and speed of sound are non-relativistic. (That is, in situations where it is necessary to distinguish the energy density ϱ from the mass density ρ; this typically happens in situations where the fluid is strongly self coupled — for example in neutron star cores [13] or in relativistic BECs [4].)

In developing the current derivation, we have tried hard to be clear, explicit, and *minimal* — we have introduced only the absolute minimum of formalism that is requires to do the job, and have eschewed unnecessary side issues. We hope that the formalism will be useful to practitioners in the field of "analogue spacetimes", particularly with regard to ongoing and future developments [4, 5]. In particular, even in the non-relativistic case it is already known that adding vorticity greatly complicates the situation [14], and a deeper general relativistic analysis of this situation would be interesting. Looking further to the future, the “fluid-gravity correspondence” hints at even deeper connections between curved spacetimes and fluid dynamics [15, 16, 17].

Acknowledgements

This research was supported by the Marsden Fund administered by the Royal Society of New Zealand. The authors would like to thank Silke Weinfurtner, Bill Unruh, Stefano Liberati, Serena Fagnocchi, and Stefano Finazzi for their comments and suggestions.

Appendix: Specific enthalpy and thermodynamic considerations

We now consider some thermodynamics which for pedagogical purposes we have delayed as much as possible. Suppose we take the specific enthalpy as primary

$$w = \frac{\varrho + p}{n}, \tag{106}$$

and use the fact that we have already deduced

$$n = n_{(p=0)} \; \exp\left[\int_{\varrho_{(p=0)}}^{\varrho} \frac{\mathrm{d}\varrho}{\varrho + p(\varrho)} \right], \tag{107}$$

to then write

$$w = \exp[\log(\varrho+p)] n^{-1} = \exp\left\{ \int \frac{\mathrm{d}[\varrho+p]}{\varrho+p} \right\} \; \exp\left[- \int_{\varrho_{(p=0)}}^{\varrho} \frac{\mathrm{d}\varrho}{\varrho+p(\varrho)} \right]. \tag{108}$$

This implies

$$w = \frac{\varrho_{(p=0)}}{n_{(p=0)}} \exp\left[\int_0^p \frac{\mathrm{d}p}{\varrho+p} \right] = w_{(p=0)} \; \exp\left[\int_0^p \frac{\mathrm{d}p}{\varrho+p} \right]. \tag{109}$$

This so far is a purely (barotropic) thermodynamic result, independent of any irrotational condition.

A secondary result, now specifically tied to the Bernoulli equation (and hence to irrotational flow), is that

$$w = w_{(p=0)} \, ||\nabla\Theta||. \tag{110}$$

Note that the way we have presented the derivation we have been able to delay and avoid the need for thermodynamic arguments as far as possible. (In contrast, this equation is the starting point adopted by Bilic in his derivation of relativistic acoustic geometry [2], what for us is a peripheral result has in that analysis moved to centre stage — and gives we feel far too central a role to thermodynamic issues.)

In a similar vein, the energy equation

$$\nabla_a(\varrho V^a) + p(\nabla_a V^a) = 0, \tag{111}$$

can be combined with the conservation equation

$$\nabla_a(n V^a) = 0, \tag{112}$$

as follows:

$$\nabla_a V^a = -\frac{1}{\varrho + p} V^a \nabla_a \varrho = -\frac{1}{\varrho + p} \frac{\mathrm{d}\varrho}{\mathrm{d}\tau}, \tag{113}$$

$$\nabla_a V^a = -\frac{1}{n} V^a \nabla_a n = -\frac{1}{n} \frac{\mathrm{d}n}{\mathrm{d}\tau}, \tag{114}$$

where $\mathrm{d}/\mathrm{d}\tau$ now refers to a material derivative along the flow. Eliminating the divergence we have

$$\frac{1}{\varrho + p} \frac{\mathrm{d}\varrho}{\mathrm{d}\tau} = \frac{1}{n} \frac{\mathrm{d}n}{\mathrm{d}\tau}. \tag{115}$$

More formally, by invoking the barotropic condition we see that for any "fluid element" (*i.e.*, tiny lump of fluid) we have

$$\frac{\mathrm{d}\varrho}{\varrho + p} = \frac{\mathrm{d}n}{n}, \tag{116}$$

which can be rearranged as

$$\mathrm{d}(\varrho/n) = -p \, \mathrm{d}(1/n). \tag{117}$$

In terms of $\mathcal{V}$, the "specific volume" of a fluid element, we have $\mathcal{V} \propto 1/n$ and so

$$\mathrm{d}(\varrho \, \mathcal{V}) = -p \, \mathrm{d}\mathcal{V}, \tag{118}$$

which connects back to basic thermodynamics and again clearly verifies that in a relativistic setting ϱ is the energy density, while n is the number density.

References

[1] Vincent Moncrief, "Stability of stationary, spherical accretion onto a Schwarzschild black hole", Astrophysical Journal **235** (1980) 1038–1046.

[2] Neven Bilic, "Relativistic acoustic geometry", Classical and Quantum Gravity **16** (1999) 3953–3964 [gr-qc/9908002].

[3] C. Barceló, S. Liberati and M. Visser, "Analogue gravity," Living Rev. Rel. **8** (2005) 12 [arXiv:gr-qc/0505065].

[4] Serena Fagnocchi, Stefano Finazzi, Stefano Liberati, Marton Kormos, Andrea Trombettoni, "Relativistic Bose-Einstein Condensates: a New System for Analogue Models of Gravity," arXiv:1001.1044 [gr-qc].

[5] Silke Weinfurtner, Bill Unruh, Jason Penner, work in progress.

[6] W. Gordon, "Zur lichtfortplanzung nach der relativitätstheorie", Ann. Phys., Leipzig, **72** (1923) 421–456.

[7] W. G. Unruh, "Experimental black hole evaporation," Phys. Rev. Lett. **46** (1981) 1351.

[8] M. Visser, "Acoustic propagation in fluids: An Unexpected example of Lorentzian geometry," arXiv:gr-qc/9311028.

[9] M. Visser, "Acoustic black holes: Horizons, ergospheres, and Hawking radiation," Class. Quant. Grav. **15**, 1767 (1998) [arXiv:gr-qc/9712010].

[10] S. W. Hawking and G. F. R. Ellis, "The large scale structure of space-time", (Cambridge University Press, England, 1973).

[11] C. Barceló, S. Liberati and M. Visser, "Analog gravity from field theory normal modes?," Class. Quant. Grav. **18** (2001) 3595 [arXiv:gr-qc/0104001].

[12] C. Barceló, S. Liberati and M. Visser, "Refringence, field theory, and normal modes," Class. Quant. Grav. **19** (2002) 2961 [arXiv:gr-qc/0111059].

[13] S. L. Shapiro and S. A. Teukolsky, "Black holes, white dwarfs, and neutron stars: The physics of compact objects," *New York, USA: Wiley (1983) 645 p*

[14] S. E. Perez Bergliaffa, K. Hibberd, M. Stone and M. Visser, "Wave Equation for Sound in Fluids with Vorticity," Physica D **191** (2004) 121–136. [arXiv:cond-mat/0106255].

[15] S. Bhattacharyya *et al.*, "Local Fluid Dynamical Entropy from Gravity," JHEP **0806** (2008) 055 [arXiv:0803.2526 [hep-th]].

[16] M. Rangamani, "Gravity & Hydrodynamics: Lectures on the fluid-gravity correspondence," Class. Quant. Grav. **26** (2009) 224003 [arXiv:0905.4352 [hep-th]].

[17] N. Ambrosetti, J. Charbonneau and S. Weinfurtner, "The fluid/gravity correspondence: Lectures notes from the 2008 Summer School on Particles, Fields, and Strings," arXiv:0810.2631 [gr-qc].