Thin-shell wormholes from charged black holes in generalized dilaton-axion gravity

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This paper discusses a new type of thin-shell wormhole constructed by applying the cut-and-paste technique to two copies of a charged black hole in generalized dilaton-axion gravity, which was inspired by low-energy string theory. After analyzing various aspects of this thin-shell wormhole, we discuss its stability to linearized spherically symmetric perturbations.

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I. INTRODUCTION

The study of traversable wormholes has received considerable attention from researchers for the past two decades. Although lacking observational evidence, wormholes are just as good a prediction of the general theory of relativity as black holes. In particular, we refer to the pioneering work of Visser [1], who proposed a theoretical method for constructing a new class of traversable Lorentzian wormholes from blackhole spacetimes. This construction proceeds by surgically grafting two Schwarzschild spacetimes together in such a way that no event horizon is permitted to form. The resulting structure is a wormhole spacetime in which the throat is a three-dimensional thin shell. In recent years, Visser's approach was adopted by various authors for constructing thin-shell wormholes by similar methods, generally requiring spherical symmetry [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20]. The approach is of special interest because it minimizes the amount of exotic matter required. All the exotic matter is confined to the shell.

More recently, Sur, Das and SenGupta [21] discovered a new black hole solution for Einstein-Maxwell scalar field systems inspired by low-energy string theory. They considered a generalized action in which two scalar fields are minimally coupled to an Einstein-Hilbert-Maxwell field in four dimensions,

$$I = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\left[R - \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - W\right], \qquad (1)$$

where

$$W = \frac{1}{2}\omega(\varphi)\partial_\mu\zeta\partial^\mu\zeta - \alpha(\varphi,\zeta)F_{\mu\nu}F^{\mu\nu} - \beta(\varphi,\zeta)F_{\mu\nu}F^{\mu\nu *},$$

$\kappa = 8\pi G$, R is the curvature scalar, $F_{\mu\nu}$ is the Maxwell field tensor, while φ and ζ are two massless scalar or pseudo scalar fields, which are coupled to the Maxwell field. This coupling is described by the functions α and β. Here ζ acquires a non-minimal kinetic term ω. In the context of low-energy string theory, fields ϕ and ξ can be identified as massless scalar dilaton and pseudo scalar axion fields, respectively.

With the above action, Eq. (1), Sur *et al.* [21] found the most general class of black hole solutions and obtained two types of metrics, classified as asymptotically flat and asymptotically non-flat. Since we are interested in obtaining a thin-shell wormhole from this new black hole, we adopt the asymptotically flat metric given by

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + h(r)(d\theta^2 + \sin^2\theta d\phi^2), \qquad (2)$$

where

$$f(r) = \frac{(r-r_-)(r-r_+)}{(r-r_0)^{2-2n}(r+r_0)^{2n}}, \qquad (3)$$

and

$$h(r) = \frac{(r+r_0)^{2n}}{(r-r_0)^{2n-2}}. \qquad (4)$$

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In addition, various other parameters are given by

$$r_{\pm} = m_0 \pm \sqrt{m_0^2 + r_0^2 - \frac{1}{8}\left(\frac{K_1}{n} + \frac{K_2}{1-n}\right)}, \quad (5)$$

$$r_0 = \frac{1}{16 m_0}\left(\frac{K_1}{n} - \frac{K_2}{1-n}\right), \quad (6)$$

$$m_0 = m - (2n-1) r_0, \quad (7)$$

$$K_1 = 4n[4r_0^2 + 2r_0(r_+ + r_-) + r_+ r_-], \quad (8)$$

$$K_2 = 4(1-n) r_+ r_-, \quad 0 < n < 1, \quad (9)$$

$$m = \frac{1}{16 r_0}\left(\frac{K_1}{n} - \frac{K_2}{1-n}\right) + (2n-1) r_0, \quad (10)$$

where m is the mass of the black hole. The parameters r_+ and r_- are the inner and outer event horizons, respectively. Also, $r = r_0$ is a curvature singularity; the parameters obey the condition $r_0 < r_- < r_+$.

In this paper we present a new kind of thin-shell wormhole by surgically grafting two charged black holes in generalized dilaton-axion gravity. The exotic matter required for its physical existence may possibly be collected from scalar fields that built the black holes. Various aspects of this thin-shell wormhole are analyzed, particularly the equation of state relating pressure and density. Also discussed is the attractive or repulsive nature of the wormhole, as well as the energy conditions on the shell. Our final topic is a stability analysis to determine the conditions under which the wormhole is stable to linearized radial perturbations.

II. THIN-SHELL WORMHOLE CONSTRUCTION

The mathematical construction of our thin-shell wormhole begins by taking two copies of the black hole and removing from each the four-dimensional region

$$\Omega^{\pm} = \{r \leq a \mid a > r_+\}.$$

We now identify (in the sense of topology) the timelike hypersurfaces

$$\partial\Omega^{\pm} = \{r = a \mid a > r_+\},$$

denoted by Σ. The resulting manifold is geodesically complete and consists of two asymptotically flat regions connected by a throat. The induced metric on Σ is given by

$$ds^2 = -d\tau^2 + a(\tau)^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (11)$$

where τ is the proper time on the junction surface. Using the Lanczos equations [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20], one can obtain the surface stress energy tensor $S^i{}_j = \mathrm{diag}(-\sigma, p_\theta, p_\phi)$, where σ is the surface energy density and p_θ and p_ϕ are the surface pressures. The Lanczos equations now yield [7]

$$\sigma = -\frac{1}{4\pi}\frac{h'(a)}{h(a)}\sqrt{f(a) + \dot{a}^2} \quad (12)$$

and

$$p_\theta = p_\phi = p = \frac{1}{8\pi}\frac{h'(a)}{h(a)}\sqrt{f(a) + \dot{a}^2} + \frac{1}{8\pi}\frac{2\ddot{a} + f'(a)}{\sqrt{f(a) + \dot{a}^2}}. \quad (13)$$

To understand the dynamics of the wormhole, we assume the radius of the throat to be a function of proper time, or $a = a(\tau)$. Also, overdot and prime denote, respectively, the derivatives with respect to τ and a. For a static configuration of radius a, we obtain the respective values of the surface energy density and the surface pressures. For a static configuration of radius a, we obtain (assuming $\dot{a} = 0$ and $\ddot{a} = 0$) from Eqs. (12) and (13),

$$\sigma = -\frac{4[a + (1-2n) r_0]}{D}\frac{(a - r_-)(a - r_+)}{(a - r_0)(a + r_0)} \quad (14)$$

and

$$p_\theta = p_\phi = p = \frac{2a - r_- - r_+}{D} \quad (15)$$

where

$$D = 8\pi (a - r_0)^{1-n}(a + r_0)^n \sqrt{(a - r_-)(a - r_+)}. \quad (16)$$

Observe that the energy density σ is negative. The pressure p may be positive, however. This would depend on the position of the throat and hence on the physical parameters r_0, r_-, and r_+ defining the wormhole. Similarly, $p + \sigma$, $\sigma + 2p$, and $\sigma + 3p$, obtained by using the above equations, may also be positive under certain conditions, in which case the strong energy condition is satisfied.

Keeping in mind the condition $r_+ > r_- > r_0$ for different radii defining the wormhole, we plot p versus a in Fig. 1. We choose typical wormholes whose radii (r_0, r_-, and r_+) fall within the range 2 to 12 kms. Also taken into account is the sensitivity of the plots with respect to n, as described in the caption of the figure. We also plot σ versus a in Fig. 2.

III. THE GRAVITATIONAL FIELD

We now turn our attention to the attractive or repulsive nature of our wormhole. To perform the analysis, we calculate the observer's four-acceleration $a^\mu = u^\mu{}_{;\nu} u^\nu$, where $u^\nu = dx^\nu/d\tau = (1/\sqrt{f(r)}, 0, 0, 0)$. In view of the line element, Eq. (2), the only non-zero component is given by

$$a^r = \Gamma^r_{tt}\left(\frac{dt}{d\tau}\right)^2 = \frac{1}{2}\frac{Ar^2 - Br + C}{(r - r_0)^{3-2n}(r + r_0)^{2n+1}} \quad (17)$$

where,

$$A = r_- + r_+ + 4nr_0 - 2r_0,$$

$$B = 2r_0^2 + (r_- + r_+)(4nr_0 - 2r_0) + 2r_- r_+,$$

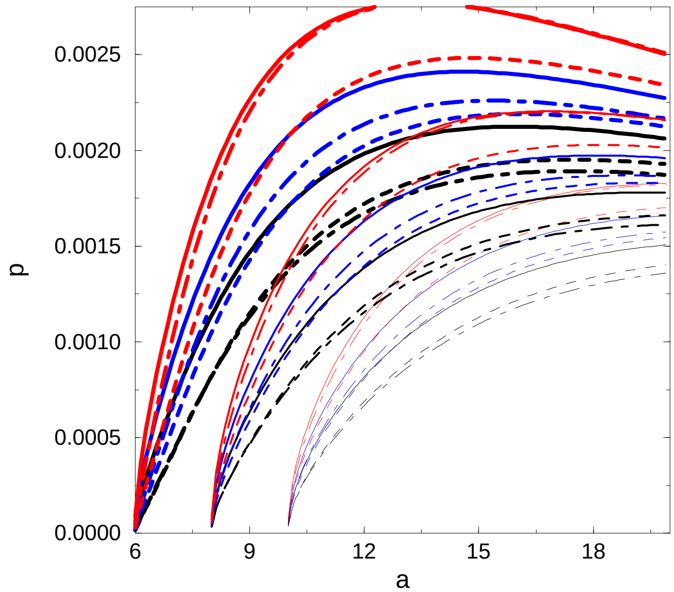

FIG. 1: Plot for p versus a. The black, blue, and red colors represent $n = 0.98$, 0.5 and 0.02, respectively. For every color, thin, thick, and thicker curves, respectively, represent $r_+ = 10$, 8, and 6. For every combination of r_+ and n, we plot three different sets, ($r_- = 5$, $r_0 = 2$), ($r_- = 5$, $r_0 = 3$), and ($r_- = 4$, $r_0 = 2$), which are represented by chain and solid curves, respectively.

and

$$C = r_0^2(r_- + r_+) + (4nr_0 - 2r_0)r_- r_+.$$

A radially moving test particle initially at rest obeys the equation of motion

$$\frac{d^2r}{d\tau^2} = -\Gamma^r_{tt}\left(\frac{dt}{d\tau}\right)^2 = -a^r. \tag{18}$$

If $a^r = 0$, we obtain the geodesic equation. Moreover, a wormhole is attractive if $a^r > 0$ and repulsive if $a^r < 0$. These characteristics depend on the parameters r_0, r_-, r_+, and n, the conditions on which can be conveniently expressed in terms of the coefficients A, B, and C. To avoid negative values for r, let us consider only the root $r = (B + \sqrt{B^2 - 4AC})/(2A)$ of the quadratic equation $Ar^2 - Br + C = 0$. It now follows from Eq. (17) that $a^r = 0$ whenever

$$\left(r - \frac{B}{2A}\right)^2 = \frac{B^2 - 4AC}{4A^2}.$$

For the attractive case, $a^r > 0$, the condition becomes

$$\left(r - \frac{B}{2A}\right)^2 > \frac{B^2 - 4AC}{4A^2}.$$

For the repulsive case, $a^r < 0$, the sense of the inequality is reversed.

IV. THE TOTAL AMOUNT OF EXOTIC MATTER

In this section we determine the total amount of exotic matter for the thin-shell wormhole. This total can be quantified by the integral [7, 8, 9, 10, 11, 12, 13]

$$\Omega_\sigma = \int [\rho + p]\sqrt{-g}d^3x. \tag{19}$$

By introducing the radial coordinate $R = r - a$, w get

$$\Omega_\sigma = \int_0^{2\pi}\int_0^{\pi}\int_{-\infty}^{\infty}[\rho + p]\sqrt{-g}\,dR\,d\theta\,d\phi.$$

Since the shell is infinitely thin, it does not exert any radial pressure. Moreover, $\rho = \delta(R)\sigma(a)$. So

$$\Omega_\sigma = \int_0^{2\pi}\int_0^{\pi}\left[\rho\sqrt{-g}\right]\Big|_{r=a} d\theta\,d\phi = 4\pi h(a)\sigma(a)$$
$$= -\frac{16\pi[a + (1-2n)r_0]}{D}\left[\frac{(a - r_-)(a - r_+)}{(a - r_0)^{2n-1}(a + r_0)^{1-2n}}\right]. \tag{20}$$

Here D is given in Eq. (16).

This NEC violating matter can be reduced by taking the value of a closer to r_+, the location of the outer event horizon. The closer a is to r_+, however, the closer the wormhole is to a black hole: incoming microwave background radiation would get blueshifted to an extremely high temperature [22]. On the other hand, it follows from Eq. (20) that for $a \gg r_+$, Ω_σ will depend linearly on a:

$$\Omega_\sigma \approx -2a. \tag{21}$$

V. AN EQUATION OF STATE

Taking the form of the equation of state (EoS) to be $p = w\sigma$, we obtain from Eqs. (14) and (15),

$$\frac{p}{\sigma} = w = \frac{1}{4}\frac{(2a - r_- - r_+)(r_0^2 - a^2)}{(a - r_-)(a - r_+)[a + (1-2n)r_0]}. \tag{22}$$

Observe that whenever the location of the wormhole throat is very large, then $w \to -1$, i.e., the distribution of matter in the shell is of the dark-energy type. When $a \to \frac{1}{2}(r_- + r_+)$, then $p \to 0$, so there is a dust shell situated at $a = \frac{1}{2}(r_- + r_+)$. In Fig. 3, we plot equation of state papameter, ω, with respect to a, wherein we choose same condition, $r_+ > r_- > r_0$, for typical wormholes whose radii fall within the range of 2 and 12 kms.

Our spacetime metric implies that the surface mass of this thin shell is given by $M_{shell} = 4\pi h(a)\sigma$. (For a static solution, we have $\dot{a} = 0$ and $\ddot{a} = 0$.) Thus

$$M_{shell} = 2\frac{[a + (1-2n)r_0]}{(r_0^2 - a^2)}\times$$
$$(a - r_0)^{1-n}(a + r_0)^n\sqrt{(a - r_-)(a - r_+)}. \tag{23}$$

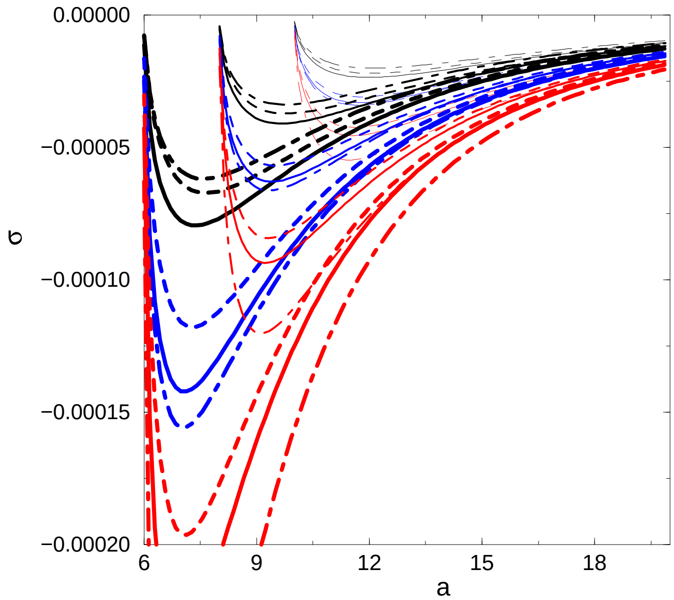

FIG. 2: Plot for σ versus a. The description of the curves is the same as in FIG. 1.

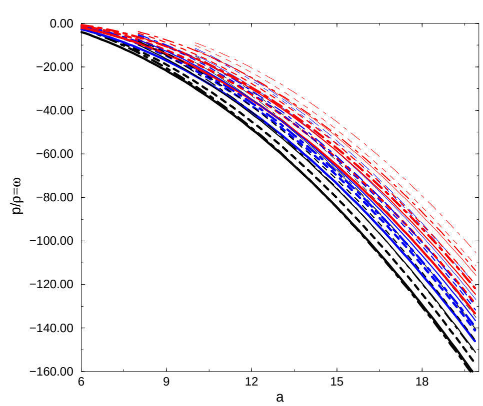

FIG. 3: Plot of the equation of state parameter $w = p/\sigma$. The description of the curves is the same as in FIG. 1.

One can note that the mass of the black hole in Eq. (10) is increasing with r_0 and at the same time mass of the thin shell is decreasing with r_0. In other words, mass confined within the thin shell would be reduced by increasing of the black hole mass.

VI. STABILITY

Now we will focus our attention on the stability of the configuration under small perturbations around a static solution at a_0. The starting point is the definition of a potential, extended to our metric [Eq. (2)]. Rearranging Eq. (12), we obtain the thin shell's equation of motion

$$\dot{a}^2 + V(a) = 0. \tag{24}$$

Here the potential $V(a)$ is defined as

$$V(a) = f(a) - \left[\frac{4\pi h(a)\sigma(a)}{h'(a)}\right]^2. \tag{25}$$

Expanding $V(a)$ around a_0, we obtain

$$\begin{aligned} V(a) &= V(a_0) + V'(a_0)(a-a_0) + \frac{1}{2}V''(a_0)(a-a_0)^2 \\ &\quad + O\left[(a-a_0)^3\right], \end{aligned} \tag{26}$$

where the prime denotes the derivative with respect to a, assuming a static solution situated at a_0. Since we are linearizing around $a = a_0$, we must have $V(a_0) = 0$ and $V'(a_0) = 0$. The configuration will then be in stable equilibrium if $V''(a_0) > 0$.

To carry out this analysis, we start with the energy conservation equation. Using Eqs. (12) and (13), one can verify that

$$\frac{d}{d\tau}(\sigma\mathcal{A}) + p\frac{d\mathcal{A}}{d\tau} = \{[h'(a)]^2 - 2h(a)h''(a)\}\frac{\dot{a}\sqrt{f(a)+\dot{a}^2}}{2h(a)}, \tag{27}$$

where $\mathcal{A} = 4\pi h(a)$ by Eq. (2). The first term on the left side corresponds to a change in the throat's internal energy, while the second term corresponds to the work done by the throat's internal forces. According to Ref. [19], the right side represents a flux. It is shown next that Eq. (27) can be written

$$h(a)\sigma' + h'(a)(\sigma+p) + \{[h'(a)]^2 - 2h(a)h''(a)\}\frac{\sigma}{2h'(a)} = 0. \tag{28}$$

It is also shown in Ref. [19] that

$$\begin{aligned} V''(a) &= f''(a) + 16\pi^2 \times \\ &\left\{\left[\frac{h(a)}{h'(a)}\sigma'(a) + \left(1 - \frac{h(a)h''(a)}{[h'(a)]^2}\right)\sigma(a)\right][\sigma(a)+2p(a)] \right. \\ &\left. + \frac{h(a)}{h'(a)}\sigma(a)[\sigma'(a)+2p'(a)]\right\}. \end{aligned} \tag{29}$$

Next, we define a parameter β, which is interpreted as the subluminal sound speed, by the relation

$$\beta^2(\sigma) = \left.\frac{\partial p}{\partial\sigma}\right|_\sigma. \tag{30}$$

To do so, observe that

$$\begin{aligned} &\sigma'(a) + 2p'(a) \\ &\quad = \sigma'(a)[1 + 2p'(a)/\sigma'(a)] = \sigma'(a)(1+\beta^2). \end{aligned}$$

Using Eq. (28), we can now rewrite $V''(a)$ as follows:

$$V''(a) = f''(a) - 8\pi^2 \left\{ [\sigma(a) + 2p(a)]^2 \right.$$
$$\left. +2\sigma(a) \left[\left(\frac{3}{2} - \frac{h(a)h''(a)}{[h'(a)]^2} \right) \sigma(a) + p(a) \right] (1 + 2\beta^2) \right\}. \tag{31}$$

At the static solution $a = a_0$, the conditions $V(a_0) = 0$ and $V'(a_0) = 0$ are indeed met. Using $V''(a_0) > 0$, our final step is to solve for β^2. As long as

$$4\sigma \left[\left(\frac{3}{2} - \frac{hh''}{(h')^2} \right) \sigma + p \right] > 0, \tag{32}$$

we obtain at $a = a_0$

$$\beta^2 < -\frac{1}{2} + \frac{f''/8\pi^2 - (\sigma + 2p)^2}{4\sigma \left[\left(\frac{3}{2} - \frac{hh''}{(h')^2} \right) \sigma + p \right]}. \tag{33}$$

If inequality (32) is reversed, we get at $a = a_0$

$$\beta^2 > -\frac{1}{2} + \frac{f''/8\pi^2 - (\sigma + 2p)^2}{4\sigma \left[\left(\frac{3}{2} - \frac{hh''}{(h')^2} \right) \sigma + p \right]}. \tag{34}$$

Fig. 4 shows typical regions of stability using somewhat arbitrary values of the various parameters: $r_0 = 1$, $r_- = 2$, $r_+ = 3$, and $n = 1/2$. The region is below the curve on the right and above the curve on the left. The sign change is determined by inequality (32). These results are similar to those in Refs. [2] and [19] in the sense that the regions do not correspond to any value in the interval $0 < \beta^2 \leq 1$. Since β is ordinarily interpreted as the speed of sound, it is highly desirable to obtain a region for which $\beta^2 < 1$. This is indeed possible for our wormhole: if we choose $r = r_-$ close to $r = r_+$, then we typically get a region of stability for $\beta^2 < 1$. For example, in Fig. 5, $r_0 = 1$, $r_- = 2$, $r_+ = 2.05$, and $n = 0.8$. The closer r_- is to r_+, the more the region of stability extends below $\beta^2 = 1$.

VII. CONCLUSION

A new black-hole solution by Sur *et al.* [21], for Einstein-Maxwell scalar fields was inspired by low-energy string theory. This paper discusses a new thin-shell wormhole constructed by applying the cut-and-paste technique to two copies of such black holes. We analyzed various aspects of this wormhole, such as the amount of exotic matter required, the attractive or repulsive nature of the wormhole, and a possible equation of state for the thin shell. The stability analysis concentrated on the parameter β, normally interpreted as the speed of sound. It was found that whenever the two event horizons are close together, a stability region exists for some values of β^2 less than unity, unlike the cases discussed in Refs. [2] and [19].

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[1] M. Visser, Nucl. Phys. B **328**, 203 (1989).
[2] E. Poisson and M. Visser, Phys. Rev. D **52**, 7318 (1995).
[3] F. S. N. Lobo and P. Crawford, Class. Quant. Grav. **21**, 391 (2004).
[4] F. S. N. Lobo, Class. Quant. Grav. **21**, 4811 (2004).
[5] E. F. Eiroa and G. Romero, Gen. Rel. Grav. **36**, 651 (2004).
[6] E. F. Eiroa and C. Simeone, Phys. Rev. D **70**, 044008 (2004).
[7] E. F. Eiroa and C. Simeone, Phys. Rev. D **71**, 127501 (2005).
[8] M. Thibeault, C. Simeone and E. F. Eiroa, Gen. Rel. Grav. **38**, 1593 (2006).
[9] F. S. N. Lobo, Phys. Rev. D **71**, 124022 (2005).
[10] F. Rahaman et al., Gen. Rel. Grav. **38**, 1687 (2006).
[11] E. Eiroa and C. Simeone, Phys. Rev. D **76**, 024021 (2007).
[12] F. Rahaman et al., Int. J. Mod. Phys. D **16**, 1669 (2007).
[13] F. Rahaman et al., Gen. Rel. Grav. **39**, 945 (2007).
[14] F. Rahaman et al., Chin. J. Phys. **45**, 518 (2007) e-Print: arXiv:0705.0740[gr-qc].
[15] J. P. S. Lemos and F. S. N. Lobo, Phys. Rev. D **78**, 044030 (2008).
[16] M. G. Richarte and C. Simeone, Phys. Rev. D **76**, 087502 (2007).
[17] F. Rahaman et al., Acta Phys. Polon. B **40**, 1575 (2009) arXiv:0804.3852[gr-qc].
[18] F. Rahaman et al., Mod. Phys. Lett. A **24**, 53 (2009) arXiv:0806.1391[gr-qc]; F. Rahaman et al., arXiv:0909.1071[gr-qc]
[19] E. F. Eiroa, Phys. Rev. D **78**, 024018 (2008).
[20] E. F. Eiroa, M. G. Richarte, and C. Simeone, Phys. Lett. A **373** 1 (2008).
[21] S. Sur, S. Das and S. SenGupta, JHEP **0510**, 064 (2005).
[22] T.A. Roman, Phys. Rev. D **53**, 5496 (1993).