

# Expanding universes in the conformal frame of $f(R)$ gravity\*

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## Abstract

The late time evolution of Friedmann-Robertson-Walker (FRW) models with a perfect fluid matter source is studied in the conformal frame of  $f(R)$  gravity. We assume that the corresponding scalar field, nonminimally coupled to matter, has an arbitrary non-negative potential function  $V(\phi)$ . We prove that equilibria corresponding to non-negative local minima for  $V$  are asymptotically stable. We investigate all cases where one of the matter components eventually dominates. The results are valid for a large class of non-negative potentials without any particular assumptions about the behavior of the potential at infinity. In particular for a nondegenerate minimum of the potential with zero critical value we show that if  $\gamma$ , the parameter of the equation of state is larger than one, then there is a transfer of energy from the fluid to the scalar field and the later eventually dominates.

## 1 Introduction

The standard inflationary idea requires that there be a period of slow-roll evolution of a scalar field (the inflaton) during which its potential energy drives the universe

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in a quasi-exponential expansion. Besides a cosmological constant, a massive scalar field (quintessence) provides the simplest mechanism to obtain accelerated expansion of the universe within General Relativity. Therefore, scalar fields play a prominent role in the construction of cosmological scenarios aiming to describe the evolution of the early and the present universe. Since the nature of the scalar field supposed to cause accelerated expansion is unknown, it is important to investigate the general properties shared by all FRW models with a scalar field irrespectively of the particular choice of the potential.

In this paper we study the late time evolution of initially expanding flat and negatively curved FRW models with a scalar field having an arbitrary bounded from below potential function  $V(\phi)$ . The scalar field is nonminimally coupled to ordinary matter described by a barotropic fluid with equation of state

$$p = (\gamma - 1)\rho, \quad 0 < \gamma \leq 2.$$

The corresponding case of initially contracting models was studied in [1]. Nonminimally coupling occurs for example, in string theory [2], in scalar-tensor theories of gravity [3], in higher order gravity (HOG) theories [4] and in models of chameleon gravity [5]. In particular, for HOG theories derived from Lagrangians of the form

$$L = f(R) \sqrt{-g} + 2L_m(\Psi), \quad (1)$$

it is well known that under a suitable conformal transformation the field equations reduce to the Einstein field equations with a scalar field  $\phi$  as an additional matter source.

The conformal equivalence can be formally obtained by conformally transforming the Lagrangian (1) and the resulting action becomes [6],

$$\tilde{S} = \int d^4x \sqrt{-\tilde{g}} \left\{ \tilde{R} - [(\partial\phi)^2 + 2V(\phi)] + 2e^{-2\sqrt{2/3}\phi} \mathcal{L}_m \left( e^{-\sqrt{2/3}\phi} \tilde{g}, \Psi \right) \right\}.$$

Variation of  $\tilde{S}$  with respect to  $\tilde{g}$  yields the field equations,

$$\tilde{G}_{\mu\nu} = T_{\mu\nu}(\tilde{g}, \phi) + \tilde{T}_{\mu\nu}(\tilde{g}, \Psi), \quad (2)$$

and variation of  $\tilde{S}$  with respect to  $\phi$  yields the equation of motion of the scalar field,

$$\tilde{\square}\phi - \frac{dV}{d\phi} = \frac{1}{\sqrt{6}} e^{-2\sqrt{2/3}\phi} T_{\mu}^{\mu}(\tilde{g}, \Psi). \quad (3)$$

Note that the Bianchi identities imply that

$$\tilde{\nabla}^{\mu} \tilde{T}_{\mu\nu}(\tilde{g}, \Psi) \neq 0, \quad \tilde{\nabla}^{\mu} T_{\mu\nu}(\tilde{g}, \phi) \neq 0,$$

and therefore there is an energy exchange between the scalar field and ordinary matter.

## 2 Flat and negatively curved FRW with an arbitrary non-negative potential

For homogeneous and isotropic spacetimes the field equations (2) reduce to the Friedmann equation,

$$H^2 + \frac{k}{a^2} = \frac{1}{3} \left( \rho + \frac{1}{2} \dot{\phi}^2 + V(\phi) \right), \quad (4)$$

and the Raychaudhuri equation,

$$\dot{H} = -\frac{1}{2} \dot{\phi}^2 - \frac{\gamma}{2} \rho + \frac{k}{a^2}, \quad (5)$$

while the equation of motion of the scalar field (3), becomes

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = \frac{4 - 3\gamma}{\sqrt{6}} \rho. \quad (6)$$

The Bianchi identities yield the conservation equation,

$$\dot{\rho} + 3\gamma\rho H = -\frac{4 - 3\gamma}{\sqrt{6}} \rho \dot{\phi}, \quad (7)$$

(see for example [6, 7]). For simplicity we drop the tilde from all quantities. We adopt the metric and curvature conventions of [8].  $a(t)$  is the scale factor, an overdot denotes differentiation with respect to time  $t$ ,  $H = \dot{a}/a$  and units have been chosen so that  $c = 1 = 8\pi G$ . Here  $V(\phi)$  is the potential energy of the scalar field and  $V' = dV/d\phi$ .

In the following we consider initially expanding solutions of (5)-(7), i.e.  $H(0) > 0$ . For flat,  $k = 0$ , models the state vector of the system (5)-(7) is  $(\phi, \dot{\phi}, \rho, H)$ , i.e. we have a four-dimensional dynamical system subject to the constraint (4). Defining  $y := \dot{\phi}$  and setting  $(4 - 3\gamma)/\sqrt{6} =: \alpha$ , we write the autonomous system as

$$\begin{aligned} \dot{\phi} &= y, \\ \dot{y} &= -3Hy - V'(\phi) + \alpha\rho, \\ \dot{\rho} &= -3\gamma\rho H - \alpha\rho y, \\ \dot{H} &= -\frac{1}{2}y^2 - \frac{\gamma}{2}\rho, \end{aligned} \quad (8)$$

subject to the constraint

$$3H^2 = \rho + \frac{1}{2}y^2 + V(\phi). \quad (9)$$

**Remark 1** *The function  $W$  defined by*

$$W(\phi, y, \rho, H) = H^2 - \frac{1}{3} \left( \frac{1}{2} y^2 + V(\phi) + \rho \right),$$

*satisfies*

$$\dot{W} = -2HW,$$

*and therefore, the hypersurface  $\{W = 0\}$ , representing flat,  $k = 0$ , cosmologies is invariant under the flow of (8). By standard arguments in ordinary differential equations theory it follows that also  $\text{sgn}(W)$  is invariant under the flow of (8). We deduce that solutions with  $W$  positive, null, or negative, represent scalar field cosmologies with  $k = -1, 0, 1$  respectively. Similar arguments applied to the third of (8) show that if  $\rho > 0$  at some initial time  $t_0$ , then  $\rho(t) > 0$  throughout the solution. Denoting by*

$$\epsilon = \frac{1}{2} y^2 + V(\phi),$$

*the energy density of the scalar field, we see that*

$$\dot{\epsilon} + \dot{\rho} = -3H(y^2 + \gamma\rho),$$

*which implies that, for expanding models the total energy  $\epsilon + \rho$  of the system decreases.*

The equilibria of (8) are given by  $(\phi = \phi_*, y = 0, \rho = 0, H = \pm \sqrt{V(\phi_*)/3})$  where  $V'(\phi_*) = 0$ . Regarding their stability for expanding cosmologies, we note the following facts. Critical points of  $V$  with negative critical value are not equilibria and they rather allow for recollapse of the model. Moreover, nondegenerate maximum points (with non negative critical value) for  $V$  are unstable, as can be easily seen by linearizing system (8) at the corresponding equilibria and verifying the existence of at least one eigenvalue with positive real part. It is interesting to study what happens near local minima of the potential with non negative critical value, and a stability result can be given when  $k = 0, -1$ .

**Proposition 2** *Let  $\phi_*$  a strict local minimum for  $V(\phi)$ , possibly degenerate, with nonnegative critical value. Then,  $\mathbf{p}_* = (\phi_*, y_* = 0, \rho_* = 0, H_* = \sqrt{\frac{V(\phi_*)}{3}})$  is an asymptotically stable equilibrium point for expanding cosmologies in the open spatial topologies  $k = 0$  and  $k = -1$ .*

**Sketch of the proof.** The proof consists in constructing a compact set  $\Omega$  in  $\mathbb{R}^4$  and showing that it is positively invariant. Applying LaSalle's invariance theorem to the functions  $W$  and  $(\rho + \epsilon)$  in  $\Omega$ , it is shown that every trajectory in  $\Omega$  is such that  $HW \rightarrow 0$  and  $H(y^2 + \gamma\rho) \rightarrow 0$  as  $t \rightarrow +\infty$ , which means  $y \rightarrow 0$ ,  $\rho \rightarrow 0$ , and

$H^2 - \frac{1}{3}V(\phi) \rightarrow 0$ . Since  $H$  is monotone and admits a limit,  $V(\phi)$  also admits a limit,  $V(\phi_*)$ , thus the solution approaches the equilibrium point  $\mathbf{p}_*$ . Details in [9]. ■

Similar results were proved in [10] for separately conserved scalar field and perfect fluid.

### 3 Energy exchange

In the following we restrict ourselves to the flat,  $k = 0$ , case and study the energy transfer from the perfect fluid to the scalar field. We are interested to study the late time behavior near the equilibrium point  $(\phi = \phi_*, y = 0, \rho = 0, H = \sqrt{\frac{V(\phi_*)}{3}})$ , which, by the previous Proposition is asymptotically stable. We suppose that the initial data in the basin of attraction of this equilibrium are such that the fluid is the dominant matter component, i.e.

$$\rho_0 > \epsilon_0,$$

and we investigate whether there is a time  $t_1$  such that

$$\epsilon(t) > \rho(t), \quad \forall t > t_1. \quad (10)$$

This question is relevant to the coincidence problem, that is, why dark energy and matter appear to have roughly the same energy density today (see [11] and references therein).

If  $V(\phi_*) > 0$ , the transition (10) does always happen. In fact, it easily follows that if the critical value of the potential is strictly positive, it behaves as an effective cosmological constant and the energy of the scalar field tends to this value whereas the energy of the fluid tends to zero. We conclude that the relevant case is when  $V(\phi_*) = 0$  and hereafter we will focus on the case when  $\phi_*$  is a nondegenerate minimum. Without loss of generality we suppose that  $\phi_* = 0$  and therefore, the general form of the potential studied can be written in a neighborhood of  $\phi = 0$  as

$$V(\phi) = \frac{1}{2}\omega^2\phi^2 + \mathcal{O}(\phi^3), \quad \omega > 0. \quad (11)$$

Integrating (7), we get

$$\rho(t) = ce^{-\alpha\phi(t)}a(t)^{-3\gamma}. \quad (12)$$

From the Proposition in the previous Section we know *a-priori* that  $(\phi, \dot{\phi}, \rho, H) \rightarrow (0, 0, 0, 0)$  as  $t \rightarrow \infty$ , hence we can write  $\rho(t) \simeq ca(t)^{-3\gamma}$  as  $t \rightarrow \infty$ .

We now turn our attention to the equation of motion of the scalar field (6) with the potential (11), namely

$$\ddot{\phi} + 3H\dot{\phi} + \omega^2\phi + \mathcal{O}(\phi^2) = \alpha\rho. \quad (13)$$

This equation can be solved by the Kryloff-Bogoliuboff (KB) approximation [12]. We present an outline of the method for the convenience of readers with no previous knowledge of the KB approximation. Consider the differential equation

$$\ddot{\phi} + \eta f(\phi, \dot{\phi}) + \omega^2 \phi = 0, \quad 0 < \eta \ll 1. \quad (14)$$

If  $\eta = 0$ , the solution can be written as

$$\phi(t) = \frac{1}{\omega} r \cos(-\omega t + \chi) \quad \text{and} \quad \dot{\phi}(t) = r \sin(-\omega t + \chi),$$

where  $r$  and  $\chi$  are arbitrary constants. We are looking for a solution of (14) which resembles to the form of the simple harmonic oscillator, that is,

$$\phi(t) = \frac{1}{\omega} r(t) \cos(-\omega t + \chi(t)) \quad \text{and} \quad \dot{\phi}(t) = r(t) \sin(-\omega t + \chi(t)). \quad (15)$$

Setting  $\theta(t) = -\omega t + \chi(t)$  and substituting (15) in (14) yields

$$\begin{aligned} \frac{dr}{dt} &= -\eta f\left(\frac{1}{\omega} r \cos \theta, r \sin \theta\right) \sin \theta, \\ \frac{d\chi}{dt} &= -\frac{\eta}{r} f\left(\frac{1}{\omega} r \cos \theta, r \sin \theta\right) \cos \theta. \end{aligned} \quad (16)$$

For  $\eta$  small,  $r(t)$  and  $\chi(t)$  are slowly varying functions of  $t$ . Consequently, in a time  $T = 2\pi/\omega$ ,  $r(t)$  and  $\chi(t)$  have not changed appreciably, while  $\theta(t) = -\omega t + \chi(t)$  will increase approximately by  $-2\pi$ . Therefore, we replace the rhs of (16) by their average values over a range of  $2\pi$  of  $\theta$ , i.e. the amplitude  $r(t)$  is *regarded as a constant* in taking the average (this is the essence of the KB approximation). This leads to the equations

$$\begin{aligned} \frac{dr}{dt} &= -\frac{\eta}{2\pi} \int_0^{2\pi} f\left(\frac{1}{\omega} r \cos \theta, r \sin \theta\right) \sin \theta d\theta, \\ \frac{d\chi}{dt} &= -\frac{\eta}{2\pi r} \int_0^{2\pi} f\left(\frac{1}{\omega} r \cos \theta, r \sin \theta\right) \cos \theta d\theta. \end{aligned}$$

If we could apply the KB approximation to (13) with

$$\eta f(\phi, \dot{\phi}) = 3H\dot{\phi} + \mathcal{O}(\phi^2) - \alpha\rho, \quad (17)$$

then we should obtain for the amplitude of  $\phi$

$$\frac{dr}{dt} = -\frac{1}{2\pi} \int_0^{2\pi} (3H\omega r \cos \theta + \mathcal{O}(r^2 \cos^2 \theta) - \alpha c e^{-(\alpha/\omega)r \cos \theta} a^{-3\gamma}) \sin \theta d\theta. \quad (18)$$

The integral

$$\int_0^{2\pi} e^{-(\alpha/\omega)r \cos \theta} \sin \theta d\theta,$$

vanishes and since  $H$  is a slow varying function compared to the rapid oscillations of the scalar field, it can be considered as constant during one period,  $T = 2\pi/\omega$ . Therefore, (18) becomes

$$\frac{dr}{dt} = -\frac{3}{2}Hr + \mathcal{O}(r^3), \quad (19)$$

since the integration of the quadratic terms vanish. Ignoring third-order terms, eq. (19) can be integrated to give  $r(t) = r_0(t)a^{-3/2}$ , where  $r_0(t)$  is bounded and bounded away from zero uniformly as  $t \rightarrow +\infty$ .

Hence we find that for large  $t$  the amplitude of the scalar field varies as  $a^{-3/2}$ . Since the amplitude of  $\dot{\phi}$  has by (15) the same time dependence as the amplitude of  $\phi$  and in our approximation

$$\epsilon \cong \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\omega^2\phi^2,$$

it follows that

$$\epsilon \cong a^{-3}.$$

Comparing this result with the time dependence of the fluid density  $\rho \cong a^{-3\gamma}$ , we arrive at the following picture for the evolution of the universe.

*If  $\gamma < 1$  the energy density  $\rho$  eventually dominates over the energy density of the scalar field  $\epsilon$  and this universe follows the classical Friedmannian evolution. For  $\gamma > 1$ , since  $\rho$  decreases faster than the energy density  $\epsilon$  of the scalar field,  $\epsilon$  eventually dominates over  $\rho$ .*

A rigorous proof of these results is given in [9]. The main obstruction to apply the KB approximation is due to (17), indeed, the above argument applies if  $\rho/r$  goes to zero. Otherwise, the rhs in (16) is not infinitesimal and so  $\chi(t)$  could in principle be comparable to  $\omega t$ .

Our results could be interesting in investigations of cosmological scenarios in which the energy density of the scalar field mimics the background energy density. For viable dark energy models, it is necessary that the energy density of the scalar field remains insignificant during most of the history of the universe and emerges only at late times to account for the current acceleration of the universe.

## References

- [1] R. Giambò, *J. Math. Phys.* **50** 012501, 2009.
- [2] M. Gasperini, *Elements of String Cosmology*, Cambridge University Press, 2007.

- [3] Y. Fuji and K. Maeda, *The Scalar-Tensor Theory of Gravitation*, Cambridge University Press, 2003; V. Faraoni, *Cosmology in Scalar-Tensor Gravity*, Springer, 2004.
- [4] S. Capozziello and M. Francaviglia, Extended theories of gravity and their cosmological and astrophysical applications, *Gen. Rel. Grav.* **40** 357, 2008.
- [5] T. P. Waterhouse, *An Introduction to Chameleon Gravity*, arXiv:astro-ph/0611816, 2006.
- [6] R. Bean, B. D. L. Pogosian, A. Silvestri and M. Trodden, *Phys. Rev. D* **75** 064020, 2007.
- [7] G. Magnano and L. M. Sokolowski, *Phys. Rev. D* **50** 5039, 1994.
- [8] J. Wainwright and G. F. R. Ellis, *Dynamical Systems in Cosmology*, Cambridge University Press, 1997.
- [9] R. Giambò and J. Miritzis, arXiv:0908.3452, 2009.
- [10] J. Miritzis, *Class. Quantum Grav.* **20** 2981, 2003.
- [11] L. Amendola, M. Quartin, S. Tsujikawa and I. Waga, *Phys. Rev. D* **74** 023525, 2006.
- [12] N. Kryloff and N. Bogoliuboff, *Introduction to Nonlinear Mechanics*, Princeton University Press, 1943.