Propagation of Gravitational Waves in Generalized TeVeS

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Efforts are underway to improve the design and sensitivity of gravitational waves detectors, with the hope that the next generation of these detectors will observe a gravitational wave signal. Such a signal will not only provide information on dynamics in the strong gravity regime that characterizes potential sources of gravitational waves, but will also serve as a decisive test for alternative theories of gravitation that are consistent with all other current experimental observations. We study the linearized theory of the tensor-vector-scalar theory of gravity (TeVeS) with generalized vector action, an alternative theory of gravitation designed to explain the apparent deficit of visible matter in galaxies and clusters of galaxies without postulating yet undetected dark matter. We find the polarization states and propagation speeds for gravitational waves in vacuum, and show that in addition to the usual transverse-traceless propagation modes, there are two more transverse modes and two trace modes. Additionally, the propagation speeds are different from c.

I. INTRODUCTION

Gravitational wave (GW) science is raising in popularity, with several GW detectors in operation in the United States (LIGO), Europe (VIRGO and GEO) and Japan (TAMA), and efforts under way to improve their design and increase their sensitivity [1]. LISA, the laser interferometer space antenna, is expected to fly in the next decade, with LISA pathfinder marking the way [2]. The detection of gravitational waves will convey new information on dynamics of systems in the strong-field gravity limit, as well as in regions of the universe which are opaque to electromagnetic radiation. Gravitational waves will also serve as a test for alternative theories of gravity. Despite the outstanding success of General Relativity (GR) in the solar system, it fails to explain dynamics on galaxy and galaxy cluster scales without postulating large amounts of yet-undetected Dark Matter (DM). An alternative theory of gravitation that does not require DM is TeVeS, the tensor-vector-scalar theory of gravity, which was designed as a relativistic implementation of Milgrom's MOdified Newtonian Dynamics paradigm (MOND). MOND explains the asymptotical flatness of galaxy rotation curves without postulating the existence of yet undetected dark matter, as well as the sharpness of the Tully-Fisher relation which correlates luminosity of a disk galaxy with its asymptotic rotational velocity in a natural context. Milgrom proposed that Newtonian gravity progressively fails as accelerations drop below a characteristic scale $\mathfrak{a}_0 \simeq 10^{-10}\mathrm{m/s}^2$ which is typical of galaxy outskirts, and assumes that for accelerations of order $\mathfrak{a}_0$ or well below it, the Newtonian relation $\mathbf{a} = -\nabla\Phi_N$ is replaced by

$$\tilde{\mu}\left(|\mathbf{a}|/\mathfrak{a}_0\right)\mathbf{a} = -\nabla\Phi_N, \tag{1}$$

where the function $\tilde{\mu}(x)$ smoothly interpolates between $\tilde{\mu}(x) = x$ at $x \ll 1$ and the Newtonian expectation $\tilde{\mu}(x) = 1$ at $x \gg 1$. This relation with a suitable standard choice of $\tilde{\mu}(x)$ in the intermediate range has proved successful not only in justifying the flatness of galaxy rotation curves in regions where acceleration scales are much below $\mathfrak{a}_0$, but also in explaining detailed shapes of rotation curves in the inner parts in terms of the directly seen mass, and in giving a precise account of the observed Tully-Fisher law , $L \propto V_c^4$, predicting the relation $L = (Ga_0\Upsilon)^{-1}V_c^4$. This sharp relation, while obtained naturally in the framework of MOND, requires quite a fine tuning of dark halo parameters to be explained by the dark matter paradigm [3].

However, MOND alone is only a phenomenological prescription that does not fulfill the usual conservation laws, nor does it make clear if the departure from Newtonian physics is in the gravity or in the inertia side of the equation $\mathbf{F} = m\mathbf{a}$. Moreover, it is non relativistic, and as such it does not teach us how to handle gravitational lensing or cosmology in the weak acceleration regimes. To address these issues, Bekenstein designed the Tensor-Vector-Scalar theory of gravity (TeVeS) [4], a covariant field theory of gravity which has MOND as its low velocity, weak acceleration limit, while its nonrelativistic strong acceleration limit is Newtonian and its relativistic limit is GR. TeVeS sports two metrics, the "physical" metric on which all matter fields propagate, and the Einstein metric which interacts with the additional fields in the theory: a timelike dynamical vector field, A^α, and a scalar field, ϕ. The theory also involves

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a free function $\mathcal{F}$, a length scale ℓ, and two positive dimensionless constants k and K. The scalar field in TeVeS provides the additional gravitational potential for matter, whereas the vector field provides the desired light bending properties, in a fashion similar to the constant unit vector in Sanders' stratified theory [5]. The TeVeS equations are derived from an action principle, thus ensuring conservation of energy and momentum.

One aspect of gravitational waves in TeVeS has been investigated by Sotani, who calculated oscillation spectra of neutron stars in the theory generated by perturbations of the fluid [6], as well as by perturbations of the tensor and scalar fields [7], and compared them to the oscillation frequencies predicted in GR. Recently, Sagi [8] calculated the PPN parameters for TeVeS and showed that there is a link between the cosmological value of the scalar field, ϕ_c, and the coupling constant of the vector field, K, which prevents the scalar field from evolving with cosmological time as predicted by its equation of motion. This adds to existing evidence of dynamical problems in TeVeS with Maxwell-like vector action, that has been provided by Seifert [9] and by Contaldi et al. [10]. Consequently, in this work we investigate gravitational waves in vacuum for the generalized version of TeVeS introduced by Skordis [11], and determine the complete set of mode speeds and polarizations for generic values of the free parameters in the theory. This version of TeVeS has a vector action of the most general form which is quadratic in derivatives of the vector fields, and scalar and metric actions as in the original formulation of TeVeS, thus preserving the correct MOND and Newtonian limits. The vector field has four coupling constants associated with it, instead of just one in the original version of TeVeS. The results presented in this paper will be useful in determining the predictions of TeVeS with regard to gravitational wave emission from astrophysical sources.

II. GENERALIZED TEVES EQUATIONS

The metric, matter and scalar actions are given by [4]

$$S_g = \frac{1}{16\pi G}\int g^{\alpha\beta}R_{\alpha\beta}\sqrt{-g}\,d^4x, \tag{2}$$

$$S_m = \int \mathcal{L}\left(\tilde{g}_{\mu\nu}, f^\alpha, f^\alpha_{;\mu}, \cdots\right)\sqrt{-\tilde{g}}\,d^4x, \tag{3}$$

$$S_s = -\frac{1}{2k^2\ell^2 G}\int \mathcal{F}\left(k\ell^2\mathbf{h}^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta}\right)\sqrt{-g}\,d^4x, \tag{4}$$

Above $\mathbf{h}^{\alpha\beta} \equiv g^{\alpha\beta} - A^\alpha A^\beta$ with $A^\alpha \equiv g^{\alpha\beta}A_\beta$. In the scalar's action k is a dimensionless positive parameter while ℓ is a constant with the dimensions of length, and $\mathcal{F}$ a dimensionless free function. G is the gravitational coupling constant, and is not equal to the measured Newton's constant, which will be denoted G_N. The metric which couples to the gravitational fields is $g_{\mu\nu}$, dubbed here the Einstein metric. The matter fields, which include all standard model particles, couple to the physical metric, which is composed of the three gravitational fields

$$\tilde{g}_{\alpha\beta} = e^{-2\phi}g_{\alpha\beta} - 2A_\alpha A_\beta\sinh(2\phi). \tag{5}$$

The vector action is taken to be of the most general form quadratic in derivatives of the vector fields, as follows:

$$S_v = -\frac{1}{16\pi G}\int\sqrt{-g}d^4x\left(\frac{K}{2}F_{\alpha\beta}F^{\alpha\beta} + \frac{K_+}{2}S_{\alpha\beta}S^{\alpha\beta} + K_2\left(\nabla A\right)^2 + K_4\dot{A}_\alpha\dot{A}^\alpha - \lambda\left(A^\alpha A_\alpha + 1\right)\right) \tag{6}$$

where $F_{\alpha\beta} = A_{\alpha;\beta} - A_{\beta;\alpha}$, $S_{\alpha\beta} = A_{\alpha;\beta} + A_{\beta;\alpha}$ and $\dot{A}^\alpha = A^\beta A^\alpha_{;\beta}$. The K_i are dimensionless coupling constants. K is the coupling constant of the original version of TeVeS.

Variation of the action with respect to $g^{\alpha\beta}$ yields the TeVeS Einstein equations for $g_{\alpha\beta}$

$$G_{\alpha\beta} = 8\pi G\left(\tilde{T}_{\alpha\beta} + \left(1 - e^{-4\phi}\right)A^\mu\tilde{T}_{\mu(\alpha}A_{\beta)} + \tau_{\alpha\beta}\right) + \theta_{\alpha\beta}, \tag{7}$$

where $v_{(\alpha}A_{\beta)} \equiv v_\alpha A_\beta + A_\alpha v_\beta$, etc. The sources here are the usual matter energy-momentum tensor $\tilde{T}_{\alpha\beta}$ (related to the variational derivative of S_m with respect to $\tilde{g}^{\alpha\beta}$), as well as the energy-momentum tensors for the scalar and vector fields,

$$\tau_{\alpha\beta} \equiv \frac{\mu(y)}{kG}\left(\phi_{,\,\alpha}\phi_{,\,\beta} - A^{\mu}\phi_{,\mu}A_{(\alpha}\phi_{,\,\beta)}\right) - \frac{\mathcal{F}(y)g_{\alpha\beta}}{2k^2\ell^2 G} \tag{8}$$

$$\begin{aligned}\theta_{\alpha\beta} \equiv\ & K\left(F_{\sigma\alpha}F^{\sigma}{}_{\beta} - \frac{1}{4}F^2 g_{\alpha\beta}\right) + K_+\left(S_{\alpha\sigma}S_{\beta}{}^{\sigma} - \frac{1}{4}S^2 g_{\alpha\beta} + \nabla_\sigma\left[A^\sigma S_{\alpha\beta} - S^{\sigma}{}_{(\alpha}A_{\beta)}\right]\right)\\ &+ K_2\left(g_{\alpha\beta}\nabla_\sigma\left(A^\sigma\nabla\cdot A\right) - A_{(\alpha}\nabla_{\beta)}\nabla\cdot A - \frac{g_{\alpha\beta}}{2}(\nabla\cdot A)^2\right)\\ &+ K_4\left(\dot{A}_\beta\dot{A}_\alpha + \dot{A}_\sigma A_{(\alpha}\nabla_{\beta)}A^\sigma - \nabla_\sigma\left[\dot{A}^\sigma A_\alpha A_\beta\right] - \frac{g_{\alpha\beta}}{2}\dot{A}_\sigma\dot{A}^\sigma\right) - \lambda A_\alpha A_\beta,\end{aligned} \tag{9}$$

where $v_{[\alpha}A_{\beta]} \equiv v_\alpha A_\beta - A_\alpha v_\beta$, etc., and

$$\mu(y) \equiv \mathcal{F}'(y); \qquad y \equiv k\ell^2 \mathbf{h}^{\gamma\delta}\phi_{,\,\gamma}\phi_{,\,\delta}. \tag{10}$$

Each choice of the function $\mathcal{F}(y)$ defines a separate TeVeS theory. Its derivative $\mu(y)$ functions somewhat like the $\tilde{\mu}$ function in MOND. For $y > 0$, $\mu(y) \simeq 1$ corresponds to the high acceleration, i.e., Newtonian, limit, while the limit $0 < \mu(y) \ll 1$ corresponds to the deep MOND regime. In the MOND regime, $\mu(y) \sim \sqrt{y/D}$, with D a dimensionless constant. We shall only consider functions such that $\mathcal{F} > 0$ and $\mu > 0$ for either positive or negative arguments.

The equations of motion for the vector and scalar fields are obtained by varying the action with respect to ϕ and A_α, respectively. We have

$$\left[\mu(y)\mathbf{h}^{\alpha\beta}\phi_{,\,\alpha}\right]_{;\,\beta} = kG\left[g^{\alpha\beta} + \left(1 + e^{-4\phi}\right)A^\alpha A^\beta\right]\tilde{T}_{\alpha\beta}\,, \tag{11}$$

for the scalar and

$$\begin{aligned}&K\nabla_\alpha F^{\alpha\beta} + K_+\nabla_\alpha S^{\alpha\beta} + K_2\nabla^\beta\left(\nabla\cdot A\right) - K_4\dot{A}^\sigma\nabla^\beta A_\sigma + K_4\nabla_\sigma\left(\dot{A}^\beta A^\sigma\right) + \lambda A^\beta + \frac{8\pi}{k}\mu A^\alpha\phi_{,\alpha}g^{\beta\gamma}\phi_{,\gamma}\\ &= 8\pi G\left(1 - e^{-4\phi}\right)g^{\beta\alpha}\tilde{T}_{\alpha\gamma}A^\gamma\end{aligned} \tag{12}$$

for the vector. Additionally, there is the normalization condition on the vector field

$$A^\alpha A_\alpha = g_{\alpha\beta}\,A^\alpha A^\beta = -1. \tag{13}$$

The λ in Eq. (12), the lagrange multiplier charged with the enforcement of the normalization condition, can be calculated from the vector equation.

III. METRIC, VECTOR AND SCALAR PERTURBATIONS ON A CURVED BACKGROUND IN TEVES

We start by considering metric, vector and scalar perturbations on a curved background. One can derive results on the structure of the solutions of the TeVeS equations, following the accounting system described in [12], Ch. 2. In order to discern the perturbation from the background we assume that in some coordinate system we can write the metric, vector and scalar as

$$g_{\alpha\beta} = \bar{g}_{\alpha\beta} + h_{\alpha\beta} \tag{14}$$
$$A^\alpha = \bar{A}^\alpha + u^\alpha \tag{15}$$
$$\phi = \phi_B + \delta\phi, \tag{16}$$

where $\bar{g}_{\alpha\beta}$, $\bar{A}^\alpha$ and ϕ_B have a typical scale of variation L_B, on top of which small amplitude perturbations are superimposed, characterized by a scale ℓ_g satisfying $\ell_g \ll L_B$ (alternatively, the distinction can be made in frequency space, with the background characterized by a frequency much lower that the perturbation). Additionally, we assume that the background metric, vector and scalar are $O(1)$, whereas the perturbations are of order $\epsilon \ll 1$. Writing the vacuum TeVeS Einstein equations in the form

$$R_{\alpha\beta} = \left(8\pi G\tau_{\mu\nu} + \theta_{\mu\nu}\right)\left(\delta^\mu_\alpha\delta^\nu_\beta - \frac{1}{2}g_{\alpha\beta}g^{\mu\nu}\right), \tag{17}$$

and expanding the Ricci tensor to $O(\epsilon^2)$, we can split the TeVeS equations into two parts: a low frequency, long wavelength part which describes how the background is affected by the perturbations, and a high frequency, short wavelength part which describes the propagation of the perturbations on the background, as follows

$$\bar{R}_{\alpha\beta} = -[R^{(2)}_{\alpha\beta}]^{Low} + \left((8\pi G\tau_{\mu\nu} + \theta_{\mu\nu}) \left(\delta^\mu_\alpha \delta^\nu_\beta - \frac{1}{2} g_{\alpha\beta} g^{\mu\nu}\right) \right)^{Low} \tag{18}$$

and

$$R^{(1)}_{\alpha\beta} = -[R^{(2)}_{\alpha\beta}]^{High} + \left((8\pi G\tau_{\mu\nu} + \theta_{\mu\nu}) \left(\delta^\mu_\alpha \delta^\nu_\beta - \frac{1}{2} g_{\alpha\beta} g^{\mu\nu}\right) \right)^{High} \tag{19}$$

with $\theta_{\mu\nu}$ given by Eq. (9) and $\tau_{\mu\nu}$ by Eq. (8). In the above, $\bar{R}_{\alpha\beta}$ is constructed from $\bar{g}_{\alpha\beta}$, and contains only low frequency modes. $R^{(1)}_{\alpha\beta}$ is linear in $h_{\alpha\beta}$, and is thus high frequency, whereas $R^{(2)}_{\alpha\beta}$, which is quadratic in $h_{\alpha\beta}$, can contain both low-frequency modes generated from terms with nearly equal but opposite high wave-vectors, as well as high-frequency modes. In the absence of matter sources, the equations of motion force the amplitude of the perturbations ϵ to be equal to ℓ_g/L_B (this is true only in the absence of sources; if sources are present then they determine the background curvature whereas $\epsilon \ll \ell_g/L_B$). This comes about since the Ricci tensor contains terms which are quadratic in derivatives of the metric, so that on the left hand side of Eq. (18) we have terms quadratic in derivatives of the background metric, which are proportional to $1/L_B^2$, and on the right hand side we have terms quadratic in derivatives of the perturbation, which are proportional to $(\epsilon/\ell_g)^2$. One can then set $L_B = 1$ and use a single expansion parameter $|h_{\alpha\beta}|, |u^\alpha|, |\delta\phi| \sim \epsilon \sim \ell_g \ll 1$. From Eq. (18) we see that terms quadratic in the perturbations induce changes in the background, whereas the propagation equations, Eq. (19), are first order in the perturbation. The leading term in the propagation equations, which contain second derivatives of the perturbations, is then $O(1/\epsilon)$, and first derivatives of the perturbations or of the background metric are $O(1)$.

Regardless of the form of $\mu(y)$, the scalar equation separates from the vector-tensor equations, since for all y $0 < \mu(y) \leq 1$ and $0 < \mathcal{F} \leq y$. We can thus evaluate the order of magnitude of $\tau_{\alpha\beta}$ by taking $\mu = 1$ and $\mathcal{F} = y$; this will give us a bound from above: $\tau_{\alpha\beta} \sim \phi_{,\alpha}\phi_{,\beta}$. Then $\phi_{,\alpha}\phi_{,\beta} \sim O((\epsilon/\ell_g)^2) = O(1)$. The contribution of the scalar field to the vector equation, $\frac{8\pi}{k}\mu A^\alpha \phi_{,\alpha} g^{\beta\gamma}\phi_{,\gamma}$, is of the same order of magnitude. Hence the scalar contributions to the equations for the vector and the metric are $O(1)$, and are an order ϵ smaller than the leading contributions, containing second derivatives of perturbations. Consequently, the vector-metric system separates from the scalar equation on a curved background, and one can treat the two separately.

A. Vector-metric perturbations

We are interested in the lowest order terms in the metric and vector equations, which are $O(1/\epsilon)$. To this order, we can approximate $\bar{g}_{\alpha\beta}$ by the Minkowski metric $\eta_{\alpha\beta}$, and the background scalar and vector fields by $\bar{A}^\alpha \approx (1, 0, 0, 0)$ and $\phi_B = \phi_c \approx const$. This approximation is valid only in the absence of matter, where the background curvature can be assumed to be almost absent. To avoid carrying factors of $e^{2\phi_c}$ throughout the calculation, we chose coordinates in which the background Einstein metric is Minkowski; at the end of the calculation we will switch to coordinates in which the background physical metric is Minkowski. To first order in the perturbations, the difference between the two coordinate systems is only one of scale. For simplicity, we elected to work in the reference frame in which the vector field is at rest; when including matter content, the velocity of the matter frame with respect to the vector frame, v, would have to be accounted for, and the vector field would acquire a temporal component of the order of v^2 and a spatial component of the order of v. We will work in units in which the speed of light is unity.

We substitute Eqs. (14,15,16) into the metric and vector TeVeS field equations in vacuum, Eq. (17) and

$$K\nabla_\alpha F^{\alpha\beta} + K_+ \nabla_\alpha S^{\alpha\beta} + K_2 \nabla^\beta (\nabla \cdot A) - K_4 \dot{A}^\sigma \nabla^\beta A_\sigma + K_4 \nabla_\sigma \left(\dot{A}^\beta A^\sigma\right) + \lambda A^\beta + \frac{8\pi}{k}\mu A^\alpha \phi_{,\alpha} g^{\beta\gamma}\phi_{,\gamma} = 0. \tag{20}$$

To order $O(1/\epsilon)$, indices are raised and lowered with $\eta_{\alpha\beta}$, so that for example $u^i = u_i$, etc.

The temporal component of the vector equation gives λ, which is first order in the perturbation, since the background lagrange multiplier is zero. The spatial components are

$$K_2 h_{jj,0i} + 2K_+ h_{ij,j0} + (K + K_+ - K_4)\left(h_{00,0i} - 2u^i_{,00}\right) + 2(K + K_+) u^i_{,jj} = 0 \tag{21}$$

The metric equations are:

$$00: h_{ij,ij} - h_{ii,jj} = 2(K + K_+ - K_4)\left(h_{0i,0i} - \frac{1}{2}h_{00,ii} + u^i_{,0i}\right) \tag{22}$$

$$0i: h_{0j,ji} + h_{ij,j0} - h_{0i,jj} - h_{jj,i0} = K_2 h_{jj,i0} + 2K_+(h_{ij,j0} + u^i_{,jj}) + 2(K_2 + K_+)u^j_{,ji} \tag{23}$$

$$ij: h_{k(i,j)k} - h_{0(j,i)0} - h_{,ij} - \Box h_{ij} - \delta_{ij}\left(h_{00,00} + h_{kl,kl} - \Box h\right) = K_+\left(h_{ij,00} + u^j_{,i0} + u^i_{,j0}\right) + K_2\delta_{ij}(h_{kk,00} + 2u^k_{,k0}) \tag{24}$$

Round brackets denote symmetrization without a factor 1/2. h is the trace of the metric perturbation, and $\Box$ is the flat space d'Alembertian. The temporal component of the vector field perturbation is determined from the normalization condition:

$$u^0 = \frac{1}{2}h_{00} \tag{25}$$

This system of equations is very similar to the system obtained in Æther linearized theory, therefore we will follow the analysis in [13].

As in GR, after choosing a frame within which the metric, vector and scalar fields have the form (14, 15, 16), we are left with a residual gauge symmetry. Under an infinitesimal transformation of coordinates

$$x^\mu \to x'^\mu = x^\mu + \xi^\mu(x), \tag{26}$$

the fields transform as

$$h'_{\mu\nu} = h_{\mu\nu} - \left(\partial_\mu\xi_\nu + \partial_\nu\xi_\mu\right), \tag{27}$$

$$u'^\mu = u^\mu + \partial_0\xi^\mu, \tag{28}$$

$$\delta\phi' = \delta\phi. \tag{29}$$

If $|\partial_\mu\xi_\nu| \sim |h_{\mu\nu}|$, then the condition $|h_{\mu\nu}| \ll 1$ is preserved. We thus have the freedom to perform a linearized gauge transformation on the fields. The Lorentz gauge usually chosen in GR is of little use to us, since the additional terms in the equations that stem from the vector field stress-energy tensor are not simplified in this gauge. Instead, we choose as in [13] to impose the four conditions $h_{0i} = 0$ and $u^i_{,i} = 0$. It is easy to show that they can be obtained from a gauge transformation of the form (26); starting from an arbitrary gauge that satisfies (14, 15, 16), one has to elect a vector ξ^μ that satisfies

$$h'_{0i} = h_{0i} - (\xi_{i,0} + \xi_{0,i}) = 0 \tag{30}$$

and

$$u'_i = u_i + \xi_{i,0} = 0 \tag{31}$$

Adding the vector equation and the spatial divergence of the metric equation, one gets a Poisson equation for ξ_0, and ξ_i can then be found by integrating the metric equation with respect to time.

In the gauge in which $h_{0i} = 0$ and $u^i_{,i} = 0$, the metric-vector system of equations takes the form:

$$h_{ij,ij} - h_{ii,jj} = -(K + K_+ - K_4)h_{00,ii} \tag{32}$$

$$h_{ij,j0} - h_{jj,i0} = K_2 h_{jj,i0} + 2K_+(h_{ij,j0} + u^i_{,jj}) \tag{33}$$

$$h_{k(i,j)k} - h_{,ij} - \Box h_{ij} - \delta_{ij}\left(h_{00,00} + h_{kl,kl} - \Box h\right) = K_+\left(h_{ij,00} + u^j_{,i0} + u^i_{,j0}\right) + K_2\delta_{ij}h_{kk,00} \tag{34}$$

$$K_2 h_{jj,0i} + 2K_+ h_{ij,j0} + (K + K_+ - K_4)\left(h_{00,0i} - 2u^i_{,00}\right) + 2(K + K_+)u^i_{,jj} = 0 \tag{35}$$

we now have thirteen equations and nine unknown functions (the divergence condition on the spatial part of the vector field leaves only two independent vector components), hence four of the equations above are redundant, and will serve as checks for our calculation. We will work with the six equations (34) and with the three vector equations (35), since they are the equations controlling the dynamics of the system, whereas the rest of the equations are constraint equations. To solve the system, we assume plane wave solutions for the perturbations, in coordinates such that the wave vector is $(k_0, 0, 0, k_3)$:

$$h_{\alpha\beta} = \epsilon_{\alpha\beta}e^{ik_\mu x^\mu} \tag{36}$$

$$u^a = \epsilon_a e^{ik_\mu x^\mu}. \tag{37}$$

The equations become:

$$
\begin{aligned}
u^i &: k_0 k_3 K_+ \epsilon_{i3} + k_0^2 (K + K_+ - K_4)\epsilon_i - k_3^2 (K + K_+)\epsilon_i = 0 && (38)\\
u^3 &: K_2 \epsilon_{ii} + (2K_+ + K_2)\epsilon_{33} + (K + K_+ - K_4)\epsilon_{00} = 0 && (39)\\
E_{11} &: k_3^2 \epsilon_{00} + ((1+K_2)k_0^2 - k_3^2)\epsilon_{22} + (2K_+ + K_2)k_0^2 \epsilon_{11} + (1+K_2)k_0^2 \epsilon_{33} = 0 && (40)\\
E_{22} &: k_3^2 \epsilon_{00} + ((1+K_2)k_0^2 - k_3^2)\epsilon_{11} + (2K_+ + K_2)k_0^2 \epsilon_{22} + (1+K_2)k_0^2 \epsilon_{33} = 0 && (41)\\
E_{33} &: (1+K_2)(\epsilon_{11} + \epsilon_{22}) + (2K_+ + K_2)\epsilon_{33} = 0 && (42)\\
E_{12} &: \left((2K_+ - 1)k_0^2 + k_3^2\right)\epsilon_{12} = 0 && (43)\\
E_{i3} &: (2K_+ - 1)k_0^2 \epsilon_{i3} - 2K_+ k_0 k_3 \epsilon_i = 0 && (44)
\end{aligned}
$$

in all the above $i = 1, 2$ and double indices imply summation.

The above equations are homogeneous in k_μ, and therefore there is no dispersion of the waves. If we define the wave speed to be $s = k_0/k_3$ (it will be a real wave speed only if $s^2 > 0$), from the requirement that the determinant of this homogeneous linear system of equations is zero we obtain three wave speeds:

$$
\begin{aligned}
s_1^2 &= \frac{1}{1-2K_+}\\
s_2^2 &= \frac{K + K_+ - 2KK_+}{(1-2K_+)(K+K_+-K_4)}\\
s_3^2 &= \frac{(K_2+2K_+)(2-(K+K_+)+K_4)}{(1-2K_+)(K+K_+-K_4)(2+3K_2+2K_+)} && (45)
\end{aligned}
$$

Here we see that for $K_+ = 1/2$ and for $K + K_+ - K_4 = 0$ the wave speeds are infinite. That $K + K_+ - K_4 = 0$ is disallowed is consistent with Skordis' result, that $K + K_+ - K_4$ is the coefficient of the time derivative term in the vector cosmological perturbation equation [11], meaning that when $K + K_+ - K_4 = 0$ there is no growing mode in the vector field that can assist structure formation. For values of the coupling constants K_i for which s^2 is positive and finite, the dispersion relation obtained is linear, and $|s|$ represents the propagation speed of gravitational disturbances. For these values, the theory has well defined propagating waves solutions. If s^2 is negative for a mode, then the frequency k_0 is imaginary, indicating the existence of exponentially growing or decaying solutions. In such a case the theory is unstable and hence presumably unphysical.

The fields excitations corresponding to the wave speeds (45) are:

- Two transverse-traceless modes corresponding to s_1: $\epsilon_{22} = -\epsilon_{11} \neq 0, \epsilon_{12} \neq 0$.
- Two transverse vector-tensor modes corresponding to s_2: $\epsilon_i = \frac{1}{2}\sqrt{\frac{(1-2K_+)(2KK_+-(K+K_+))}{K_+(K+K_+-K_4)}}\epsilon_{i3}$, with $i = 1, 2$.
- A trace mode involving the metric trace and the vector temporal component through the normalization condition, corresponding to s_3: $\epsilon_0 = \frac{1}{2}\epsilon_{00}$, $\epsilon_{11} = \epsilon_{22} = \frac{1}{2}(K + K_+ - K_4)\epsilon_{00}$, $\epsilon_{33} = \frac{(1+K_2)(K_4-(K+K_+))}{K_2+2K_+}\epsilon_{00}$.

The remaining metric equations agree with the above results. The modes are easily classified by their different propagation speeds, allowing us to naturally obtain the modes for the physical metric by simply substituting our result in the expression for the physical metric. To linear order, the physical metric is:

$$
\begin{aligned}
\tilde{g}_{00} &= e^{2\phi_c}(-1 + h_{00} - 2\delta\phi)\\
\tilde{g}_{0i} &= -2u^i \sinh(2\phi_c)\\
\tilde{g}_{ij} &= e^{-2\phi_c}(\delta_{ij}(1-2\delta\phi) + h_{ij}). && (46)
\end{aligned}
$$

Going to Minkowski coordinates through the following coordinate transformation

$$
x^{\bar{0}} = e^{\phi_c} x^0 \ , x^{\bar{j}} = e^{-\phi_c} x^j, \tag{47}
$$

we get

$$
\begin{aligned}
\tilde{g}_{00} &= -1 + h_{00} - 2\delta\phi\\
\tilde{g}_{0i} &= -2u^i \sinh(2\phi_c)\\
\tilde{g}_{ij} &= \delta_{ij}(1-2\delta\phi) + h_{ij}. && (48)
\end{aligned}
$$

We must remember to transform the wave speeds to Minkowski coordinates as well:

$$\begin{aligned} s_1^2 &= \frac{e^{-4\phi_c}}{1-2K_+} \\ s_2^2 &= \frac{e^{-4\phi_c}\left(K+K_+-2KK_+\right)}{(1-2K_+)(K+K_+-K_4)} \\ s_3^2 &= \frac{e^{-4\phi_c}(K_2+2K_+)(2-(K+K_+)+K_4)}{(1-2K_+)(K+K_+-K_4)(2+3K_2+2K_+)} \end{aligned} \tag{49}$$

We can then write the physical metric as $\tilde{\eta}_{\mu\nu}+\tilde{h}_{\mu\nu}$, with the physical perturbation tensor given by

$$\tilde{h}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \epsilon_{11} & \epsilon_{12} & 0 \\ 0 & \epsilon_{12} & -\epsilon_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cos\left[\omega(t-z/s_1)\right] + \begin{pmatrix} 0 & d\epsilon_{13} & d\epsilon_{23} & 0 \\ d\epsilon_{13} & 0 & 0 & \epsilon_{13} \\ d\epsilon_{23} & 0 & 0 & \epsilon_{23} \\ 0 & \epsilon_{13} & \epsilon_{23} & 0 \end{pmatrix} \cos\left[\omega(t-z/s_2)\right] \tag{50}$$

$$+ \begin{pmatrix} \epsilon_{00} & 0 & 0 & 0 \\ 0 & a\epsilon_{00} & 0 & 0 \\ 0 & 0 & a\epsilon_{00} & 0 \\ 0 & 0 & 0 & b\epsilon_{00} \end{pmatrix} \cos\left[\omega(t-z/s_3)\right] \tag{51}$$

with $d=-\sqrt{\frac{(1-2K_+)(2KK_+-(K+K_+))}{K_+(K+K_+-K_4)}}\sinh\left(2\phi_c\right)$, $a=\frac{1}{2}(K+K_+-K_4)$ and $b=\frac{(1+K_2)(K_4-(K+K_+))}{K_2+2K_+}$.

TeVeS exhibits the usual transverse-traceless propagation mode of GR, but at a speed different from c. Additionally, it has two more modes which excite both spatial and temporal directions, and one trace mode, which is not symmetric in all spatial directions. We will show in the next section that the scalar equation generates an additional trace mode.

Incidentally, in the original version of TeVeS, which had all K_i but K zero, $s_1=s_2=e^{-2\phi_c}$, showing no dependence on the coupling constant of the theory, whereas $s_3=0$, meaning that the theory has no propagating scalar mode. This is consistent with the results in [4].

B. Scalar perturbations

Since the scalar field perturbations appear in the trace of the physical metric, we expect the scalar field to give rise to another, second trace mode. However, there is a subtlety in the derivation of the lowest order contribution of the scalar equation. The scalar equation in vacuum is

$$\left[\mu(y)\mathbf{h}^{\alpha\beta}\phi_{,\,\alpha}\right]_{;\,\beta}=0\,,. \tag{52}$$

To linear order in the perturbation, $\mu(y)\approx\mu(y_B)+\mu'(y_B)(y-y_B)$. With $\phi_B=\phi_c\approx const.$, $y_B=0$ and since for small y, $\mu(y)\sim\sqrt{y}$, then $\mu'(y_B)$ diverges. Therefore, to extract information on the propagation of scalar waves we have to relax the assumption $\phi_B\approx const.$ and allow it to depend on the space coordinates. This case has already been analyzed in [4], using the WKB approximation, as is common practice for waves on a curved background; the scalar equation was shown to acquire the form

$$0=\left(\mathbf{h}^{\alpha\beta}+2\xi H^\alpha H^\beta\right)\delta\phi_{;\alpha\beta} \tag{53}$$

$$H^\alpha \equiv \frac{\mathbf{h}^{\alpha\beta}\phi_{B,\beta}}{\sqrt{\mathbf{h}^{\mu\nu}\phi_{B,\mu}\phi_{B,\nu}}} \tag{54}$$

$$\xi \equiv \frac{d\ln\mu(y)}{d\ln y} \tag{55}$$

here terms containing first derivatives of the scalar field perturbation were omitted, being $O(1)$. Note that although originally this derivation was made assuming only scalar perturbations, and ignoring vector and metric perturbations, it is valid in the presence of metric and vector perturbations, owing to the separation of the metric-vector system from the scalar equation. Contributions of metric or vector perturbations to the scalar equation are also higher order, since there is no way of forming second-derivative terms of the metric or the vector perturbations in the scalar equation.

The wave speed was shown to be $s_4 \leq e^{-2\phi_B}$ in the deep MOND regime, $s_4 = e^{-2\phi_B}/\sqrt{2}$ in the Newtonian regime, and $e^{-2\phi_B}/\sqrt{2} \leq s_4 \leq \sqrt{1+2\xi}e^{-2\phi_B}/\sqrt{2}$ in the intermediate regime. Going back to the physical metric in Minkowski coordinates, Eq. (48), we see that this generates an additional trace mode in the physical metric, at a speed different from that of the vector modes. In the notation of subsection (III A), it can be displayed as

$$\tilde{h}_{\mu\nu} = \begin{pmatrix} -2\delta\phi & 0 & 0 & 0 \\ 0 & -2\delta\phi & 0 & 0 \\ 0 & 0 & -2\delta\phi & 0 \\ 0 & 0 & 0 & -2\delta\phi \end{pmatrix} \cos\left[\omega(t - z/s_4)\right] \tag{56}$$

IV. CONCLUSIONS

We investigated the propagation of gravitational perturbations in vacuum for the tensor-vector-scalar theory of gravity. We found that in the linear approximation on a curved background, the scalar equation separates from the vector-metric system of equations. We solved the vector-metric system of equations to lowest order in the background curvature, and obtained propagating wave solutions, with a linear dispersion relation, and three distinct wave speeds depending on the coupling constants of the theory and on the background value of the scalar field. The corresponding physical metric perturbations can be classified into a pair of transverse-traceless modes, another pair of excitations of the temporal-spatial components of the physical metric, and an asymmetric trace mode. Perturbations of the scalar equation were ill defined to lowest order in the background curvature; relaxing the restriction on the background scalar field, we were able to deduce from the analysis in [4] that the scalar field gives rise to an additional trace mode, at a different propagation speed which depends on the background value of the scalar field and on the free function of the theory.

TeVeS thus predicts six different modes of propagation for the gravitational field, at four distinct speeds, all different from the speed of light. Additionally, all speeds depend on a factor $e^{-2\phi_B}$, which is expected to be close to unity when $\phi_B = \phi_c$, but might induce a significant lag with respect to the speed of light for large values of the background scalar field. Such a lag might pose a problem for TeVeS; since the scalar and vector fields are coupled to matter via the physical metric, one would expect ultra high energy cosmic rays, whose velocity is close to c, to emit Cherenkov-like radiation of scalar and vector particles, if they move at a velocity higher than those scalar and vector particles. Such an emission would cause the cosmic rays to lose energy, and how much energy is lost would depend on the scalar and vector particles emission rate, on the distance traveled by the cosmic rays from their sources, and on the strength of the matter-fields coupling. Such Cherenkov-like radiation might pose very stringent restrictions on TeVeS' parameters, as was the case for Einstein-Æther theory [14].

That problem could be avoided, at least for the physical perturbations originating in the vector-metric system, if the values of the coupling constants of the theory were such as to make the propagation speeds larger than c. The question of whether superluminal propagation in a theory with two metrics is allowed is still open [15–17]. If superluminal propagation could be allowed in TeVeS without disturbing causality, then one could also think of relinquishing the unconventional kinetic term in the action of the scalar field, which was introduced to prevent faster than light scalar waves.

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