Maximal extension of the Schwarzschild spacetime inspired by noncommutative geometry

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Abstract

We derive a transformation of the noncommutative geometry inspired Schwarzschild solution into new coordinates such that the apparent unphysical singularities of the metric are removed. Moreover, we give the maximal singularity-free atlas for the manifold with the metric under consideration. This atlas reveals many new features e.g. it turns out to describe an infinite lattice of asymptotically flat universes connected by black hole tunnels.

I. INTRODUCTION

The area of quantum gravity has not yet converged into a single theory and at present several rival theories co-exist. Nevertheless, certain common or global features like noncommutativity at lengths less than $10^{-16}cm$ [1, 2], a new uncertainty principle including gravity effects [3], the avoidance of physical singularities [5] (e.g. replaced in the noncommutative geometry by a deSitter core), black hole remnants [4] etc. are expected. The noncommutative aspect of spacetime has been recently applied to the final state of a black hole [1, 6–10]. The starting point of these new developments is the commutation relation $[x^\mu, x^\nu] = \theta^{\mu\nu}$. Based on such a commutation relation one can show that one of the effect of non commutativity is to replace point-like objects by de-localized matter sources which turn to be of Gaussian form. Following [10, 12, 13] we can take, instead of the point mass M, described by a δ-function distribution, a static, spherically symmetric, Gaussian-smeared matter source

$$\rho = \frac{M}{(4\pi\theta)^{D/2}} e^{-\frac{r^2}{4\theta}},$$

where $D > 0$ is the dimension of the underlying manifold. This observation gave rise to new models of mini black holes [1, 6–11] where the singularity at the origin is replaced by a self-gravitating droplet. Although the issue of smearing point-like structures might not be the only fingerprint of the noncommutative geometry, these models explicitly reveal its importance. For instance, the central singularity is replaced by a deSitter core (droplet) and the metric can have two horizons depending whether the black hole mass exceeds a certain critical mass [7]. In Schwarzschild coordinates, these horizons leave unphysical singularities in the metric components. We will present a coordinates extension in the following sections of the paper. It is therefore of some importance to find a maximal atlas for this metric and its interpretation. For any metric in General Relativity with apparent unphysical singularities there is a continued interest in finding maximal singularity-free extensions [14–16]. Such maximal atlases often shed new light on the manifold under consideration. In particular, this is true for metrics which are partly motivated by a quantum mechanical property such as the noncommutativity of the coordinate operators. New phenomena closely related to this quantum nature of spacetime can emerge. Indeed, for the metric inspired by noncommutative geometry which we study in the present paper, the maximal atlas reveals the existence of black hole tunnels connecting parallel universes.

The paper is organized as follows. In section II we will discuss the singularities of the metric which emerge in the framework of noncommutative geometry when a point-like structure is smeared by the so-called Gaussian prescription. Section III treats the new transformation of the Schwarzschild coordinates which leads to the maximal atlas. Section IV continues these considerations and discusses the determination of the constants in the transformation. Section V interprets the results in terms of a Penrose diagram. Section VI is devoted to the extreme case where the mass is equal to a critical mass. In section VII we draw our conclusions.

II. SINGULARITIES OF THE METRIC

The replacement of the sharp point-like structure discussed above suggests that the metric can be based on the Gaussian mass distribution,

$$\rho_\theta(r) = \frac{M}{(4\pi\theta)^{3/2}}\, e^{-r^2/(4\theta)}. \tag{1}$$

The ansatz of an anisotropic perfect fluid energy-momentum tensor taken together with two equations of state in which the pressure is determined by the Tolman-Oppenheimer- Volkov equation leads to the noncommutative geometry inspired Schwarzschild solution [7] given by

$$ds^2 = \left(1 - \frac{4M}{\sqrt{\pi}r}\gamma\left(\frac{3}{2},\frac{r^2}{4\theta}\right)\right)dt^2 - \left(1 - \frac{4M}{\sqrt{\pi}r}\gamma\left(\frac{3}{2},\frac{r^2}{4\theta}\right)\right)^{-1} dr^2 - r^2 d\vartheta^2 - r^2\sin^2\vartheta\, d\varphi^2, \tag{2}$$

where M is the mass of the black hole, $\theta > 0$ is a parameter encoding noncommutativity and γ is the incomplete lower gamma function. The singularities of the metric are determined by the equation

$$g_{00}(r) := 1 - \frac{4M}{\sqrt{\pi}r}\gamma\left(\frac{3}{2},\frac{r^2}{4\theta}\right) = 0. \tag{3}$$

According to [7] there are three possible scenarios, namely

1. if $M < M_0 \approx 1.9\sqrt{\theta}$, the function g_{00} never vanishes,

2. if $M = M_0$, g_{00} vanishes just at one value $r_0 \approx 3.0\sqrt{\theta}$ (extremal black hole),

3. if $M > M_0$, there exist $r_\pm$ with $0 < r_- < r_+$ such that $g_{00}(r\pm) = 0$

A striking feature of the metric (2) is the absence of a true singularity at $r = 0$. Moreover, the above classification is of numerical nature since the equation $g_{00}(r) = 0$ cannot be solved in closed form. We prove below that in the case that the black hole mass exceeds the critical mass M_0 the roots of the equation $g_{00}(r) = 0$ are simple. This result will play an important role in the following considerations. First we need to clarify the nature of the critical mass M_0, which turns out to be the minimal mass to have horizons. Therefore, if we employ the horizon equation to define a function $M = M(r_H)$, where r_H is a solution of the equation $g_{00}(r) = 0$, we obtain

$$M(r_H) \equiv \frac{\sqrt{\pi}}{4} \frac{r_H}{\gamma\left(3/2; r_h^2/4\theta\right)}. \tag{4}$$

Thus, having considered the derivative $dM(r_H)/dr_H$, we can look for r_0 such that the latter vanishes, i.e. $dM/dr_H|_{r_0} = 0$. We do this in order to define $M_0 \equiv M(r_0)$. One can easily show that the above derivative vanishes if and only if

$$\gamma\left(3/2; r_H^2/4\theta\right) = r_H \frac{d\gamma}{dr_H}. \tag{5}$$

From the properties of the gamma function one can easily derive the following result

$$\gamma\left(3/2; r_H^2/4\theta\right) = \frac{1}{4\theta^{3/2}} \int_0^{r_H} dt\, t^2 e^{-t^2/4\theta} \tag{6}$$

and as a consequence

$$\gamma\left(3/2; r_H^2/4\theta\right) = \frac{1}{4\theta^{3/2}}\, r_H\, r_m^2 e^{-r_m^2/4\theta} \tag{7}$$

where $r_m \in [0, r_H]$. Eq. (5) can now be written as

$$\frac{1}{4\theta^{3/2}}\, r_H\, r_m^2 e^{-r_m^2/4\theta} = \frac{1}{4\theta^{3/2}}\, r_H^3 e^{-r_H^2/4\theta}, \tag{8}$$

which admits a unique solution if and only if $r_H = r_m = r_0$. We can conclude that there exists a unique horizon radius r_0, corresponding to the critical mass M_0. With this in mind we can establish the following lemma

Lemma 1 *Let $M > M_0$ and $0 < r_- < r_+$ such that $g_{00}(r_\pm) = 0$. Then, r_- and r_+ are simple zeroes of (3).*

Proof. We give the proof for a generic solution r_H of the horizon equation $g_{00}(r_H) = 0$. Therefore r_H corresponds either to r_+ or r_-. Notice that if the limit

$$\lim_{r \to r_H} \frac{g_{00}(r)}{r - r_H}$$

is finite, then r_H is a simple root. Since g_{00} is differentiable on the interval $[0,\infty)$ we can expand it in a Taylor series and we obtain

$$g_{00}(r) = (r - r_H)\left[g'_{00}(r_H) + \mathcal{O}(r - r_H)\right].$$

Hence,

$$\lim_{r\to r_H} \frac{g_{00}(r)}{r - r_H} = g'_{00}(r_H)$$

and in order to show that r_H is a simple zero we need to prove that $g'_{00}(r_H) \neq 0$. Taking into account the fact that

$$g'_{00}(r_H) = \frac{4M}{\sqrt{\pi} r_H}\left[\frac{1}{r_H}\gamma\left(\frac{3}{2}, \frac{r_H^2}{4\theta}\right) - \gamma'\left(\frac{3}{2}, \frac{r_H^2}{4\theta}\right)\right]$$

where a prime denotes differentiation with respect to the horizon radius and comparing the above equation with the Eq. (5), we can see that $g'_{00}(r_H)$ vanishes if and only if $r_H = r_0$. This implies that $M = M_0$ which is at variance with the initial assumption. As a result we can conclude that $g'_{00}(r_H) \neq 0$ for $M > M_0$ and $r_H \neq r_0$.

III. A NEW TRANSFORMATION

We show that the singularities of (2) can be removed by a suitable coordinate transformation as in the case of the Reissner-Nordström solution. In order to do that we shall follow [15]. Like in the Kruskal approach [14] we introduce coordinates $u(t,r)$ and $v(t,r)$ such that the original metric goes over to

$$ds^2 = f^2(u,v)(dv^2 - du^2) - r^2(u,v)(d\vartheta^2 + \sin^2\vartheta\ d\varphi^2) \tag{9}$$

with the requirement that $f^2 \neq 0$. This will happen if u and v satisfy the non homogeneous system of first order nonlinear partial differential equations

$$f^2(u,v)\left[(\partial_t v)^2 - (\partial_t u)^2\right] = g_{00}(r), \tag{10}$$

$$f^2(u,v)\left[(\partial_r v)^2 - (\partial_r u)^2\right] = -g_{00}^{-1}(r), \tag{11}$$

$$\partial_r u\ \partial_t u - \partial_r v\ \partial_t v = 0. \tag{12}$$

The next step is to find a suitable transformation of the variable r such that the above system becomes a homogeneous system of PDEs. If we multiply (11) by g_{00}^2 and we introduce a new spatial variable $r_* = r_*(r)$ defined through

$$\frac{dr_*}{dr} = \frac{1}{g_{00}(r)} \tag{13}$$

then we have

$$(\partial_t v)^2 - (\partial_t u)^2 = F(r_*), \tag{14}$$

$$(\partial_{r_*} v)^2 - (\partial_{r_*} u)^2 = -F(r_*), \tag{15}$$

$$\partial_t v \; \partial_{r_*} v - \partial_t u \; \partial_{r_*} u = 0 \tag{16}$$

with $u = u(t, r_*)$, $v = v(t, r_*)$ and $F(r_*) := g_{00}(r)/f^2$ where r is a function of r^*. We want to show that u and v satisfy a wave equation. If we consider the combinations (14)+(15)±2(16) we arrive at the following equations

$$(\partial_t v + \partial_{r_*} v)^2 = (\partial_t u + \partial_{r_*} u)^2, \tag{17}$$

$$(\partial_t v - \partial_{r_*} v)^2 = (\partial_t u - \partial_{r_*} u)^2. \tag{18}$$

While taking the square roots of the above equations only those choices of the sign are allowed for which the determinant of the Jacobian of the transformation $\widetilde{x} := (v, u, \vartheta, \varphi) \longrightarrow x = (t, r, \vartheta, \varphi)$ does not vanish identically. Hence, we require

$$\det(J) = \det \begin{pmatrix} \partial_t v & \partial_{r_*} v \\ \partial_t u & \partial_{r_*} u \end{pmatrix} = \partial_t v \; \partial_{r_*} u - \partial_{r_*} v \; \partial_t u \neq 0. \tag{19}$$

Clearly, there are four possible choices of the sign. If we consider for example the case

$$\partial_t u + \partial_{r_*} u = \partial_t v + \partial_{r_*} v, \quad \partial_t u - \partial_{r_*} u = \partial_t v - \partial_{r_*} v,$$

we would obtain $\partial_t u = \partial_t v, \; \partial_{r_*} u = \partial_{r_*} v$ which implies $\det(J) = 0$. The same happens if we consider

$$\partial_t u + \partial_{r_*} u = -\partial_t v - \partial_{r_*} v, \quad \partial_t u - \partial_{r_*} u = -\partial_t v + \partial_{r_*} v.$$

On the other hand the choice

$$\partial_t u + \partial_{r_*} u = \partial_t v + \partial_{r_*} v, \quad \partial_t u - \partial_{r_*} u = -\partial_t v + \partial_{r_*} v,$$

leads to the equations

$$\partial_t u = \partial_{r_*} v, \qquad \partial_{r_*} u = \partial_t v. \tag{20}$$

We require that

$$\det(J) = (\partial_{r_*} u)^2 - (\partial_t u)^2 \neq 0 \tag{21}$$

in that part of the manifold which is described by the coordinates $(v, u, \vartheta, \varphi)$. In the next section we will show that the above condition is indeed satisfied. Finally, it is not difficult to verify that the choice

$$\partial_t u + \partial_{r_*} u = -\partial_t v - \partial_{r_*} v, \quad \partial_t u - \partial_{r_*} u = \partial_t v - \partial_{r_*} v$$

is equivalent to the previous one in the sense that both give rise to the same wave equations. Thus, from (20) we can derive the following wave equations:

$$\partial_{tt} u - \partial_{r_* r_*} u = 0, \qquad \partial_{tt} v - \partial_{r_* r_*} v = 0,$$

with the solutions

$$u(t, r_*) = h(r_* + t) + g(r_* - t), \qquad v(t, r_*) = h(r_* + t) - g(r_* - t). \tag{22}$$

Substituting (22) into (14) or (15) gives

$$4 \frac{dh}{dy} \frac{dg}{dz} = F(r_*) \tag{23}$$

with $y := r_* + t$, $z := r_* - t$ whereas (16) gives a trivial identity. Note that (23) fixes the relative signs of the functions h and g since by definition of the function $F(r_*)$ we have $F(r_*) > 0$ for $r > r_+$ and $F(r_*) < 0$ for $r_- < r < r_+$. Moreover, if we substitute (22) into (21) the invertibility condition simplifies to the requirement

$$F(r_*) \neq 0. \tag{24}$$

Clearly, (24) is not satisfied for $r = r_\pm$. This means that on the spheres with radius $r_\pm$ the transformations from spherical coordinates to ones which we are constructing, are not invertible. However, this is not really a problem since our goal is to construct several charts patching different regions of the manifold and by construction we will see that the transfer functions between these charts are always invertible. Finally, if we compute $\partial_{r_*}(23)/(23)$ and $\partial_t(23)/(23)$ with the requirement that $r \neq r_\pm$ we end up with the following equations:

$$\left(\frac{d^2 h}{dy^2}\right) / \left(\frac{dh}{dy}\right) + \left(\frac{d^2 g}{dz^2}\right) / \left(\frac{dg}{dz}\right) = \frac{1}{F} \frac{dF}{dr_*}, \qquad \left(\frac{d^2 h}{dy^2}\right) / \left(\frac{dh}{dy}\right) - \left(\frac{d^2 g}{dz^2}\right) / \left(\frac{dg}{dz}\right) = 0.$$

Summing the above equations we obtain

$$2\frac{d}{dy}\left(\ln\frac{dh}{dy}\right)=\frac{d}{dr_*}\left(\ln F\right). \tag{25}$$

Hence, once we solve for h the unknown function g can be determined from

$$\frac{d}{dz}\left(\ln\frac{dg}{dz}\right)=\frac{d}{dy}\left(\ln\frac{dh}{dy}\right). \tag{26}$$

Since the variables y and r_* in (25) can be regarded as independent variables, we can set both sides in (25) equal to a separation constant 2γ. The factor 2 has been introduced in order to simplify the left hand side of (25). Hence, the solutions read

$$F(r_*)=c_1 e^{2\gamma r_*},\qquad h(y)=\frac{c_2}{\gamma}\,e^{\gamma y}+c_3,\quad g(z)=\frac{c_4}{\gamma}\,e^{\gamma z}+c_5,$$

with $\gamma\neq 0$. However, this condition on γ is always satisfied as we shall see in the next section. Guided by the principle that we wish to derive the most simple expressions for u and v we can choose without loss of generality, $c_3=c_5=0$. Taking into account the definition of the tortoise coordinate r_* we obtain,

$$v(t,r_*)=\frac{e^{\gamma r_*}}{\gamma}\left(c_2\,e^{\gamma t}-c_4\,e^{-\gamma t}\right),\quad u(t,r_*)=\frac{e^{\gamma r_*}}{\gamma}\left(c_2\,e^{\gamma t}+c_4\,e^{-\gamma t}\right).$$

1. *Regions I and III ($r>r_+$)*

Let (v_I,u_I) denote the specific coordinates (v,u) specialized for the region I characterized by $r>r_+$. In this region $F(r_*)>0$ and equation (23) requires that we choose h and g with a positive relative sign. This implies that the constants c_2 and c_4 have to be chosen with the same sign. The simplest choice is $c_2=\gamma/2=c_4$. Thus, we have

$$v_I(t,r_*)=e^{\gamma r_*}\sinh(\gamma t),\quad u_I(t,r_*)=e^{\gamma r_*}\cosh(\gamma t)$$

and

$$f^2=\frac{g_{00}(r)}{c_1}\,e^{-2\gamma r_*}.$$

In the next section we shall see that the requirement $f^2\neq 0$ fixes the value of the separation constant γ. The constant c_1 can be fixed by requiring that the inverse transformation from the coordinates $(v_I,u_I,\vartheta,\varphi)$ to the coordinates (t,r,ϑ,φ) gives (2) again and we find that $c_1=\gamma^2$. Thus, we conclude that

$$f^2=\frac{g_{00}(r)}{\gamma^2}\,e^{-2\gamma r_*}.$$

Clearly, we still have the possibility to choose the functions h and g both with negative signs so that their relative sign is again positive. In this case we get a new set of coordinates representing a portion of the manifold (we call it region III) which is isometric to region I. In particular, the coordinates which map such a region are

$$v_{III}(t,r_*) = -e^{\gamma r_*}\ \sinh(\gamma t),\quad u_{III}(t,r_*) = -e^{\gamma r_*}\ \cosh(\gamma t).$$

Note that u_{III} is always negative as it should be according to the present choice of the relative sign of the functions h and g whereas v_{III} can assume both negative and positive values.

2. *Regions II and IV ($r_- < r < r_+$)*

Region II is obtained by deriving the corresponding transformation when the radial coordinate varies in the interval (r_-, r_+). In this case $F(r^*) < 0$ and the relative sign of the functions h and g must be negative. If we choose $c_4 = -\gamma/2 = -c_2$ we get

$$v_{II}(t,r_*) = e^{\gamma r_*}\ \cosh(\gamma\ t),\quad u_{II}(t,r_*) = e^{\gamma r_*}\ \sinh(\gamma\ t).$$

Clearly, we are also free to make the opposite choice. This is equivalent to the transformation $(v,u) \longrightarrow (-v,-u)$. Thus, region IV is isometric to region II and it is described by the coordinates

$$v_{IV}(t,r_*) = -e^{\gamma r_*}\ \cosh(\gamma\ t),\quad u_{IV}(t,r_*) = -e^{\gamma r_*}\ \sinh(\gamma\ t).$$

In the next section we shall discuss that the overlapping conditions at $r = r_\pm$ are satisfied. It is evident that the inverse transformations can only be given implicitly since from (13) we see that $r_*(r)$ cannot be inverted in terms of elementary functions. However, from the following relations

$$u^2 - v^2 = e^{2\gamma r_*},\quad \frac{1}{\gamma}\tanh^{-1}\left(\frac{v}{u}\right) = t, \tag{27}$$

we see that in the (v,u)-plane,the lines t =const are straight lines v/u =const whereas lines r =const are represented by the hyperbolae $u^2 - v^2$ =const.

IV. DETERMINATION OF THE SEPARATION CONSTANT γ

Let us first consider the singularity at r_+. Lemma 1 ensures that r_+ is a simple zero of the metric coefficient g_{00}. Thus, the following Taylor expansion holds in a neighborhood of

r_+, namely

$$g_{00}(r) = \alpha(r - r_+) + \beta(r - r_+)^2 + \mathcal{O}(r - r_+)^3$$

with

$$\alpha := g_{00}^{'}(r_+), \quad \beta := \frac{1}{2} g_{00}^{''}(r_+).$$

On the other hand the expansions of the tortoise coordinate r_* and of the functions $e^{\pm 2\gamma r_*}$ are

$$\begin{aligned} r_*(r) &= \frac{1}{\alpha} \ln(r - r_+) - \frac{\beta}{\alpha^2}(r - r_+) + \mathcal{O}(r - r_+)^2, \\ e^{\pm 2\gamma r_*} &= (r - r_+)^{2\gamma/\alpha} \left[1 \mp \frac{2\beta\gamma}{\alpha^2}(r - r_+) + \mathcal{O}(r - r_+)^2 \right]. \end{aligned}$$

Hence, we have

$$\frac{g_{00}(r)}{e^{2\gamma r_*}} = (r - r_+)^{(\alpha - 2\gamma)/\alpha} \left[\alpha + \left(\beta + \frac{2\beta\gamma}{\alpha} \right) (r - r_+)^2 + \mathcal{O}(r - r_+)^2 \right].$$

From Lemma 1 it follows that α can never vanish and we can choose γ so that the singularity at r_+ is cancelled. This happens if γ is chosen to be

$$\gamma_+ = \frac{\alpha}{2} = \frac{1}{2} g_{00}^{'}(r_+).$$

Clearly, it is not possible to cancel both singularities at once. In order to remove the singularity at r_- we can proceed as we did above and we find that

$$\gamma_- = \frac{1}{2} g_{00}^{'}(r_-).$$

If we choose $\gamma = \gamma_+$ in the (v, u) coordinates we can proceed from an arbitrarily large r towards smaller r across the set $r \in (r_-, r_+$. If we want to continue further across $r = r_-$ we have to go back to the coordinates (v, u) and choose $\gamma = \gamma_-$. In this way we can continue across $r = r_-$ and reach $r = 0$ since it is not a singularity for the metric (2) and even continue through this timelike surface.

V. RADIAL OBSERVERS

We now analyze the motion of a radial observer inside the noncommutative geometry inspired Schwarzschild black hole. In doing so we will not address issues of stability which we postpone to a future investigation. When the observer enters the event horizon at r_+

but has still not crossed the second horizon at r_-, the radial coordinate r becomes timelike, implying that the motion proceeds with decreasing r. Once the observer has crossed the Cauchy horizon at r_-, the coordinate r becomes spacelike again. This means that we have two possible kinds of motion given by increasing and decreasing values of r. At this point the observer can take two decisions: either to cross the timelike surface $r = 0$ in order to approach an asymptotically flat region or to reverse his/her course as is the case in the Reissner-Nordström metric [18]. By taking the latter decision, the observer will cross another copy of the surface $r = r_-$. Having entered the new region $r_- < r < r_+$ the radial coordinate becomes again timelike and the observer is forced to cross a new copy of the event horizon r_+. In this way the observer will emerge out of a white hole in an asymptotically flat universe. However, the journey of the observer might not finish here since he still has the possibility to enter the noncommutative inspired Schwarzschild black hole living in this new universe.

The result is the possibility to make a trip through an infinite number of universes connected by black hole tunnels. In order to depict the dynamics described above we shall construct a Penrose diagram (see Fig. 1) for the spacetime structure of the maximal extended Schwarzschild solution inspired by noncommutative geometry. First of all, we observe that the radial null geodesics in the metric (9) are $du/dv = \pm 1$. For this reason it is convenient to switch to null coordinates $p = u + v$, $q = u - v$ with $v = e^{\gamma_+ r_*} \sinh(\gamma_+ t)$ and $u = e^{\gamma_+ r_*} \cosh(\gamma_+ t)$ in order to apply the so-called Penrose transformation $P = \tanh p$ and $Q = \tanh q$. It is clear from (27) that the equation of the apparent singularity at r_+ reads $u^2 - v^2 = 0$. Thus, in the (Q, P)-plane the equation $u = v$ becomes $Q = 0$ whereas $u = -v$ reads $P = 0$. Moreover, the null infinities $p = \pm\infty$ and $q = \pm\infty$ are mapped into $P = \pm 1$ and $Q = \pm 1$. In this way we mapped the (v, u)-plane in the square $[-1, 1] \times [-1, 1]$ where as usual we can introduce new coordinates $U = (P + Q)/2$ and $V = (P - Q)/2$. We recall that the coordinate singularity at r_+ has equation $U = \pm V$. The null infinities are now straight line segments with equations $U + V = P = \pm 1$ and $U - V = Q = \pm 1$, while the lines r =const in the subspace $\{\vartheta = \text{const},\ \varphi = \text{const}\}$ which are represented by the hyperbolae $u^2 - v^2$ =const are still hyperbolae in the (U, V)-plane with equation [19]

$$\alpha^2 \left[\left(U + \frac{1}{\alpha} \right)^2 - V^2 \right] = 1, \quad \alpha := \frac{1 - C}{1 + C}, \quad C := e^{2pq} = \text{const}.$$

Concerning the regions $r < r_+$ we just repeat the above procedure with $v = e^{\gamma_- r_*} \sinh(\gamma_- t)$

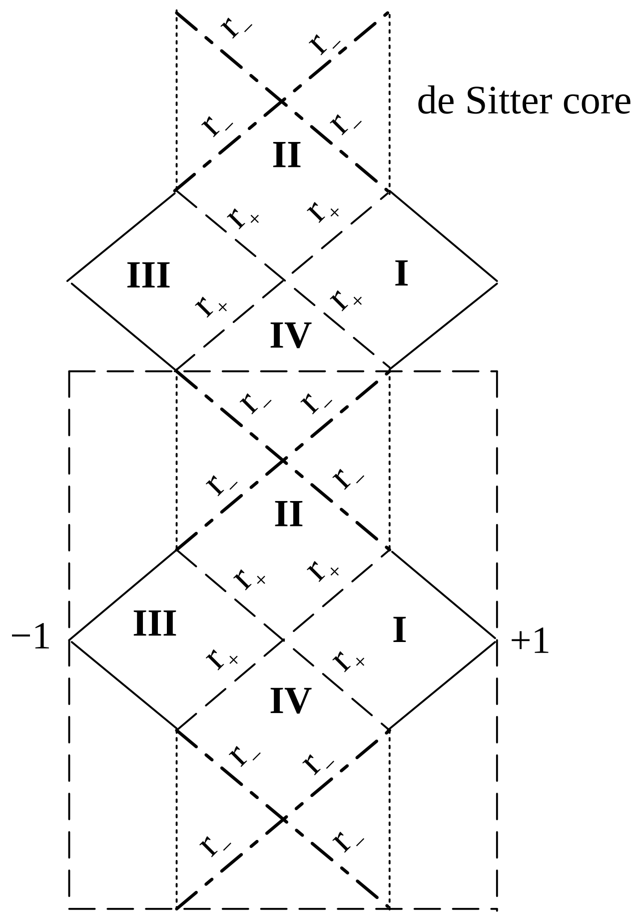

FIG. 1: The conformal diagram of the maximally extended noncommutative inspired Schwarzschild spacetime. r_+ and r_- represent the event and Cauchy horizons, respectively. The central singularity appearing in the Reissner-Nordström metric is now replaced by a regular deSitter core (dotted line). The upper and lower part of the box indicated by the dashed line can be identified to make the manifold cyclic in the time coordinate.

and $u = e^{\gamma_- r_*} \cosh(\gamma_- t)$. In this way we can patch together conformal diagrams of different parts of the original manifold and we end up with the Penrose diagram shown in the rectangle in the centre of Fig.1. Radial null geodesics would be represented by straight lines parallel to the horizons $r_\pm$. Since no future-directed null geodesic can go from region II to region I or III we conclude that $r = r_+$ is an event horizon. Note that the present situation is very different from the classical Schwarzschild or the Reissner-Nordström case since there is no central singularity at all. This point allows to interpret the noncommutative inspired Schwarzschild solution as a series of open tunnels connecting infinitely many asymptotically flat universes. However, there is also an alternative interpretation when we restrict our attention to the thick-line rectangle of Fig.1. In fact, we realize that the upper tunnel (the strip where in the Reissner-Nordström case we would expect to have the central singularity) is a copy of the lower tunnel. This suggests that we might identify the two tunnels. If we do that, the manifold becomes finite and cyclic in the timelike coordinate.

VI. THE EXTREME CASE

In the extreme case $M = M_0$ the metric (2) becomes

$$ds^2 = (r - r_0)^2 \phi(r) dt^2 - \frac{dr^2}{(r - r_0)^2 \phi(r)} - r^2 d\vartheta^2 - r^2 \sin^2 \vartheta \; d\varphi^2 \tag{28}$$

where ϕ is a differentiable and not vanishing function in the interval $[0, \infty)$. In order to derive the maximal extension we shall follow the procedure adopted in [20]. To this purpose we consider the surface $\{\vartheta = \text{const}, \; \varphi = \text{const}\}$ and we write (28) as follows:

$$ds^2 = (r - r_0)^2 \phi(r) \left[dt - \frac{dr}{(r - r_0)^2 \phi(r)} \right] \left[dt + \frac{dr}{(r - r_0)^2 \phi(r)} \right] \tag{29}$$

By introducing null coordinates p and q defined as

$$p := t + r^*, \qquad q := t - r^*,$$

$$r^* := \int \frac{dr}{(r - r_0)^2 \phi(r)} = -\frac{1}{(r - r_0)\phi(r_0)} - \frac{\phi^{'}(r_0)}{\phi^2(r_0)} \ln{(r - r_0)} + \mathcal{O}(r - r_0) \tag{30}$$

our metric becomes

$$ds^2 = (r - r_0)^2 \phi(r) dp \; dq - r^2 d\vartheta^2 - r^2 \sin^2 \vartheta \; d\varphi^2 \tag{31}$$

where now r is a function of p and q. It is worth mentioning that the surface $\{r = r_0, \; \vartheta = \text{const}, \; \varphi = \text{const}\}$ is made of radial null geodesics corresponding to lines parallel to $p = \text{const}$ and $q = \text{const}$ in the (p, q)-plane.

The metric (31) is regular for all real values of p and q. In order to understand where the coordinate singularity r_0 lies in the (p, q)-plane we need to study the signs of $\phi(r_0)$ and $\phi^{'}(r_0)$. First of all, notice that

$$\phi(r_0) = \frac{1}{2} g_{00}^{''}(r_0), \quad \phi^{'}(r_0) = \frac{1}{6} g_{00}^{'''}(r_0)$$

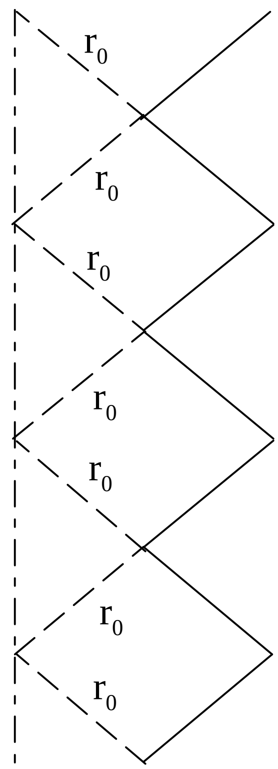

FIG. 2: The conformal diagram of the maximal extension of the extreme noncommutative inspired Schwarzschild spacetime where thin straight segments are the images of the null infinities as $r \to \infty$, dashed segments denote the event horizon at r_0 and hatched segments represent the deSitter core. As in the non extreme case we can identify the square at the bottom with the next one up.

Thus, the problem reduces to finding the signs of $g_{00}^{''}(r_0)$ and $g_{00}^{'''}(r_0)$. Using the software Maple we find the following numerical values $g_{00}^{''}(r_0) \approx 0.287$ and $g_{00}^{'''}(r_0) \approx -0.277$. This implies that the coordinate singularity r_0 is at $p = -\infty$ and $q = \infty$ when we move toward it from $r > r_0$ and at $p = \infty$ and $q = -\infty$ when we approach it from $r < r_0$, keeping in mind that these two regions will be covered by two different coordinate patches. With respect to the first coordinate patch the spatial infinity is at $p = \infty$ and $q = -\infty$. Again we can make the infinities finite by means of the transformation $p = \tan P$ and $q = \tan Q$, the nice feature of which is that we can lay the images of $r = r_0$ side by side in the two patches. With respect to the coordinate patch in the region $r > r_0$ the image of $r = r_0$ is the point with coordinates $P = -\pi/2$ and $Q = \pi/2$ whereas the choice of the coordinate patch relative to the region $r < r_0$ sends r_0 to $P = \pi/2$ and $Q = -\pi/2$. in passing we note that spacelike infinity is mapped also to the point $(\pi/2, -\pi/2)$ in the (P, Q)-plane. Putting all this information together we can construct an infinite chain of conformal diagrams as shown in Fig.2. In the present case the spacelike variable r does not become timelike when we cross the event horizon. As we did in the non extreme case with respect to the thick-line rectangle we can identify the square at the bottom with the square at the top. Proceeding like that we obtain a manifold which is cyclic in the timelike coordinate.

VII. CONCLUSIONS

If we divide the quantum black hole physics into the three categories: (i) problems zeroing around the central singularity (final stage of a black hole) [3, 5], (ii) those zeroing around the horizon (Hawking evaporation) [21] and (iii) those around the question regarding the existence of bound orbits in the outer regions [22], this article addresses the first issue in the framework of noncommutative geometry. We found a maximal singularity-free extension of the noncommutative geometry inspired Schwarzschild metric. The new coordinate chart we derived in the text has the advantage that we can illustrate more clearly the overall topology of the non extreme and extreme noncommutative geometry inspired Schwarzschild manifold. The Penrose diagrams are shown in Figs.1 and 2, respectively. The most striking feature of the manifold is black hole tunnels connecting different universes and/or a cyclic structure of the manifold in the time coordinate after identification of two parts of the Penrose diagram (see Fig. 1) is performed. This together with the absence of a central singularity reveals the main difference as compared to the classical Schwarzschild black hole structure. By some minor modification the method we used can be applied to derive the maximal extension of the noncommutative geometry inspired Reissner-Nordström black hole [9]. Finally, the stability issue of the tunnels has been examined in [23], where it has been shown that the Cauchy horizon of the noncommutative geometry inspired Schwarzschild black hole is stable under massless scalar perturbations governed by a wave equation modified accordingly to noncommutative geometry.

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