

Fermions in Brans-Dicke cosmology

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Using the Brans-Dicke theory of gravitation we put under investigation a hypothetical universe filled with a fermionic field (with a self interaction potential) and a matter constituent ruled by a barotropic equation of state. It is shown that the fermionic field (in combination with the Brans-Dicke scalar field $\varphi(t)$) could be responsible for a final accelerated era, after an initial matter dominated period.

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I. INTRODUCTION

Einstein's General Relativity (GR) is the usually invoked theory for describing the evolution of the fundamental space-time variables in cosmological models[1]. As an alternative to GR we have the Brans-Dicke theory of gravitation[1, 2], a scalar-tensor formulation that rules the gravitational phenomena through the interplay between the metric tensor and a scalar field φ that controls the intensity of the gravitational constant G . On the other hand, cosmological models that include dark energy are strong candidates for explaining the present regime of positive acceleration of the universe[1, 4]. Going in that direction one possibility is to consider fermionic fields as gravitational sources for those accelerated universes; these sources has been investigated using several approaches, with results including exact solutions, anisotropy-to-isotropy scenarios and cyclic cosmologies (see, for example [3, 5–7]). Recently, these authors [7] proposed a cosmological model, based on GR, in a dissipative Universe and showed that in a young universe scenario the fermionic field produces a fast expansion where matter (included via a barotropic equation of state) is created till it starts to predominate and the initial accelerated period gives place to a decelerated era. In this case the fermionic field plays the role of the inflaton in an early period and the role of dark energy for an old universe; without the need of a cosmological constant term or a scalar field[7]. In an exclusive old universe scenario an initially matter dominated period gradually turns into a dark (fermionic) energy period when an accelerated regime rules for the rest of the system evolution. One important point is that it was considered a self-interaction term for the fermionic constituent, in the form of a function that included the Nambu-Jona-Lasinio potential as a particular case [8]). Testing those properties in a universe ruled by Brans-Dicke gravitation would be of interest and this is the main focus of this work. The manuscript is structured as follows. In section II we

make a brief review of the tetrad formalism used to include fermionic fields in a dynamical curved space-time. The Brans-Dicke field equations for an isotropic, homogeneous and spatially flat universe are derived in section III where we present the analysis of the different scenarios in which the fermionic constituent would, in principle, answer for the transition to accelerated periods. Finally we display our conclusions. The metric signature used is $(+, -, -, -)$ and units have been chosen so that $8\pi G = c = \hbar = 1$.

II. FERMIONS AND GRAVITATION

When one tries to include fermions in a gravitational model what must be taken into account is that the gauge group of general relativity does not admit a spinor representation. The tetrad formalism solves the problem[1, 2, 9, 10]. Following the general covariance principle, the metric tensor $g_{\mu\nu}$ satisfies

$$g_{\mu\gamma} = e_{\mu}^a e_{\gamma}^b \eta_{ab}, \quad a = 0, 1, 2, 3 \quad (1)$$

where e_{μ}^a denotes the tetrad or vierbein and η_{ab} is the Minkowski metric tensor; latin indices refer to a local inertial frame whereas Greek indices to the general coordinates system. The next step to couple the fermionic field to gravity is to construct an action for the model; we start reminding that the Dirac lagrangian density in Minkowski space-time is

$$L_D = \frac{i}{2}[\bar{\psi}\gamma^a\partial_a\psi - (\partial_a\bar{\psi})\gamma^a\psi] - m\bar{\psi}\psi - V, \quad (2)$$

where m is the fermionic mass, $\bar{\psi} = \psi^{\dagger}\gamma^0$ denotes the adjoint spinor field. V is a function of ψ and $\bar{\psi}$, and represents a fermionic self-interaction. The general covariance principle imposes that the Dirac-Pauli matrices γ^a must be replaced by their generalized counterparts $\Gamma^{\mu} = e_{\mu}^a\gamma^a$, where the new matrices satisfy the extended Clifford algebra, $\{\Gamma^{\mu}, \Gamma^{\nu}\} = 2g^{\mu\nu}$. In a second step we need to substitute the ordinary derivatives by their covariant versions

$$\partial_{\mu}\psi \rightarrow D_{\mu}\psi = \partial_{\mu}\psi - \Omega_{\mu}\psi, \quad \partial_{\mu}\bar{\psi} \rightarrow D_{\mu}\bar{\psi} = \partial_{\mu}\bar{\psi} + \bar{\psi}\Omega_{\mu}, \quad (3)$$

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where the spin connection Ω_μ is given by

$$\Omega_\mu = -\frac{1}{4}g_{\mu\nu}[\Gamma^\nu_{\sigma\lambda} - e_b^\nu(\partial_\sigma e_\lambda^b)]\gamma^\sigma\gamma^\lambda, \quad (4)$$

with $\Gamma^\nu_{\sigma\lambda}$ denoting the Christoffel symbol; the generally covariant Dirac lagrangian then becomes

$$L_D = \frac{i}{2}[\bar{\psi}\Gamma^\mu D_\mu\psi - (D_\mu\bar{\psi})\Gamma^\mu\psi] - m\bar{\psi}\psi - V. \quad (5)$$

The field equations are then obtained from the total action

$$S(g, \psi, \bar{\psi}) = \int \sqrt{-g} L_T d^4x, \quad (6)$$

where $L_T = L_G + L_D + L_M$ is the total lagrangian density. L_M is the lagrangian density of the matter field and L_G is the free gravitational lagrangian density. In the Brans-Dicke theory[1] we have

$$L_G = \{Re^{\alpha\varphi} - \alpha^2\omega e^{\alpha\varphi}(\nabla\varphi)^2\}, \quad (7)$$

where R is the curvature scalar and φ is an scalar field that controls the gravitational constant (G_N) intensity[1]; α and ω are constants[1, 4]. L_D is the Dirac lagrangian density (5). Using the variational principle we obtain then the Dirac equations for the spinor field (and its adjoint) coupled to the gravitational field

$$i\Gamma^\mu D_\mu\psi - m\psi - \frac{dV}{d\psi} = 0, \quad iD_\mu\bar{\psi}\Gamma^\mu + m\bar{\psi} + \frac{dV}{d\bar{\psi}} = 0. \quad (8)$$

Analogously, the Brans-Dicke gravitational dynamics emerges from the total action (6)

$$G_{\mu\nu} = -T_{\mu\nu}, \quad (9)$$

where

$$G_{\mu\nu} = e^{\alpha\varphi}\{R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \alpha^2(g_{\mu\nu}\square\varphi - \nabla_\nu\nabla_\mu\varphi) - \alpha^2\omega[\frac{1}{2}g_{\mu\nu}(\nabla\varphi)^2 - \nabla_\nu\varphi\nabla_\mu\varphi]\}, \quad (10)$$

and $T_{\mu\nu}$ is the energy-momentum tensor of the sources of the gravitational field: $T^{\mu\nu} = T_D^{\mu\nu} + T_M^{\mu\nu}$. The symmetric form of the fermionic energy-momentum tensor is given by

$$T_D^{\mu\nu} = \frac{i}{4}[\bar{\psi}\Gamma^\mu D^\nu\psi + \bar{\psi}\Gamma^\nu D^\mu\psi - D^\nu\bar{\psi}\Gamma^\mu\psi - D^\mu\bar{\psi}\Gamma^\nu\psi] - g^{\mu\nu}L_D. \quad (11)$$

These equations rule the dynamics of a Brans-Dicke universe filled with a fermionic and matter sources.

III. BRANS-DICKE AND DIRAC FIELD DYNAMICS

Friedmann-Robertson-Walker (FRW) metric mirrors the homogeneity and isotropy properties of the universe. The space-time interval is usually written as

$$ds^2 = dt^2 - a(t)^2(dx^2 + dy^2 + dz^2), \quad (12)$$

where $a(t)$ is the cosmic scale factor. In the FRW metric the tetrad components (1) become

$$e_0^\mu = \delta_0^\mu, \quad e_i^\mu = \frac{1}{a(t)}\delta_i^\mu, \quad (13)$$

and the Dirac matrices turn out to be

$$\Gamma^0 = \gamma^0, \quad \Gamma^i = \frac{1}{a(t)}\gamma^i, \quad \Gamma^5 = -i\sqrt{-g}\Gamma^0\Gamma^1\Gamma^2\Gamma^3 = \gamma^5, \quad (14)$$

from which the spin connection (see eq. (4)) is obtained; the non-zero components are $\Omega_i = \frac{1}{2}\dot{a}(t)\gamma^i\gamma^0$. For this isotropic and homogeneous universe the fermionic field becomes an exclusive function of time; therefore the Dirac equations (8) for a non-massive fermionic field become

$$\dot{\psi} + \frac{3}{2}H\psi + i\gamma^0\frac{dV}{d\psi} = 0, \quad \dot{\bar{\psi}} + \frac{3}{2}H\bar{\psi} - i\frac{dV}{d\bar{\psi}}\gamma^0 = 0. \quad (15)$$

Besides the Dirac field, we have considered as gravitational source a matter constituent that is ruled by a barotropic equation of state, namely $\rho_m = b_m p_m$, where b_m is the barotropic coefficient ($0 \leq b_m \leq 1$); then the total energy density ρ is given as a sum of these individual contributions, i.e., $\rho = \rho_D + \rho_m = T_{00}$, where $T_{\mu\nu}$ is the total energy-momentum tensor of sources. The energy density ρ satisfies the conservation law $\dot{\rho} + 3\dot{a}/a(\rho + p) = 0$. Besides, the non-vanishing components of the fermionic energy-momentum tensor follow from (12) yielding

$$(T_D)^0_0 = V = \rho_D, \quad (16)$$

$$(T_D)^1_1 = V - \frac{dV}{d\psi}\frac{\psi}{2} - \frac{\bar{\psi}}{2}\frac{dV}{d\bar{\psi}} = -p_D, \quad (17)$$

which are exclusive functions of ψ and $\bar{\psi}$. Combining the Dirac and Einstein equations one can obtain an independent conservation law for the energy density of the fermionic field[7]. This implies into a decoupled conservation equation for the energy density of the matter constituent, i.e.,

$$\dot{\rho}_m + 3H(\rho_m + p_m) = 0, \quad (18)$$

where $H = \dot{a}/a$ is the Hubble parameter.

In order to analyze the cosmological solutions of our model we have to define first some sources properties.

For the fermionic field we consider a self-interaction potential V ; this can be modeled as an exclusive function of the scalar invariant $(\bar{\psi}\psi)^2$; following the Pauli-Fierz theorem[7, 8] we have

$$V = [(\bar{\psi}\psi)^2]^n, \quad (19)$$

where n is a constant real number. A particular case of the Nambu-Jona-Lasinio potential [8] is obtained when $n = 1$. In fact, the fermionic field behavior can be classified according to the value of the exponent n . In particular, for $n < 1/2$ the pressure of the fermion field is negative and it could represent (in a universe ruled by Einstein gravity) either the inflaton or a dark energy constituent[7]. The fermionic energy density is given by $\rho_D = [(\bar{\psi}\psi)^2]^n$. On the other hand the second constituent, the matter field, is described, as mentioned in the precedent section, following a barotropic equation of state[7]. After some algebraic manipulation we can put the model dynamics in the following form:

$$2\frac{\ddot{a}}{a} + H^2 + \alpha^2\ddot{\varphi} + 2\alpha H\dot{\varphi} + \frac{\alpha^2\omega}{2}\dot{\varphi}^2 = e^{-\alpha\varphi} \left[(2n-1)(\bar{\psi}\psi)^{2n} + p_m \right], \quad (20)$$

$$\alpha^2\ddot{\varphi} + 3\alpha H\dot{\varphi} = \frac{e^{-\alpha\varphi}}{3+2\omega} \left[V + \rho_m - 3(2n-1)(\bar{\psi}\psi)^{2n} - 3p_m \right], \quad (21)$$

$$\dot{\rho}_m + 3H(\rho_m + p_m) = 0, \quad (22)$$

$$\dot{\psi} + \frac{3}{2}H\psi + v\gamma^0 \frac{dV}{d\psi} = 0, \quad \dot{\bar{\psi}} + \frac{3}{2}H\bar{\psi} - v \frac{dV}{d\bar{\psi}} \gamma^0 = 0 \quad (23)$$

We consider a pressureless matter field ($p_m = 0$), so that one can obtain from equation (22) that $\rho_m(t) = \rho_m(0)/a(t)^3$. Furthermore, the equations (23) can also be integrated for the potential (19) resulting that the fermionic bilinear evolution is governed by $[\bar{\psi}\psi](t) = [\bar{\psi}\psi](0)/a(t)^3$. The remaining equations (20) and (21) constitute a highly non-linear system of differential equations and we proceed to solve it numerically. We analyze the time evolution of our model choosing first the conditions for $t = 0$:

$$a(0) = 1, \quad \dot{a}(0) = 1, \quad [\bar{\psi}\psi](0) = 0.001, \quad \rho_m(0) = 1, \\ \varphi(0) = 1, \quad \dot{\varphi}(0) = 0.001.$$

These conditions characterize qualitatively an initial proportion between the constituents; an era when matter predominates over the fermionic density. Besides that, we have to specify the magnitude of the remaining parameters: we suppose initially that $\alpha = 1.0$, $\omega = 4 \times 10^4$ [11] and a value for the potential power n , $n = 0.2$ [7].

These choices are reference values that permit final adjustments to follow several cosmological constraints, like the spectrum of the Brans-Dicke coupling ω [11] and the

present value of the Brans-Dicke scalar field φ . These parameters can be in fact adjusted due to invariance properties of the Brans-Dicke gravitational field equations, by using the following change of variables: $\alpha \rightarrow \bar{\alpha}$, $\varphi \rightarrow \bar{\varphi}$, $\bar{\alpha}\bar{\varphi} = \gamma/\varphi + \alpha$ where $\exp(-\gamma) = x_0$ is associated to the asymptotic value of $\bar{\varphi}_0 = (2\omega + 4)/(2\omega + 3)$. These transformations show how a value of the Brans-Dicke parameter ω is linked to the definition of a new Brans-Dicke scalar field $\bar{\varphi}$. In fact, after numerical integration, it is possible to verify that these features are included in the φ evolution (see figure 1) that imply into the evolution of the gravitational “constant” as $G(t) = (2\omega + 4)/[(2\omega + 3)\bar{\varphi}(t)]$.

In figure 2 it is plotted the acceleration field \ddot{a} as function of time t for two different values of n . The results show that initially the universe is expanding with negative acceleration, a period where the matter constituent predominates over the fermionic field. With the evolution of time we have increasing values in energy transference to the fermionic field (this is not happening directly but via the gravitational field $a(t)$ and the scalar field $\varphi(t)$, as the equations of motion (20-23) show). The negative pressure of $\psi(t)$ help in fact to promote a final accelerated period (also shown in figure 1) indicating that in this model the dark energy role would be played by a *combination of the fermionic constituent with the scalar $\varphi(t)$* . Other interesting results appear when we choose the power n to be in the neighborhood of values $n \approx 0.33$. In fact, for a fixed value of $\dot{\varphi}(0)$ and increasing values of n what emerges is a universe that fails to show a final accelerated period, even when the fermionic field still exhibits a negative pressure ($p_D = (2n-1)[(\bar{\psi}\psi)^2]^n$). On the other hand, for values $n \leq 0.33$ we will find a universe that is permanently in accelerated expansion. Another important remark here is that this general qualitative behavior depends strongly on the initial value of the time derivative of the scalar field $\varphi(t)$: what we verify is that increasing values of $\dot{\varphi}(0)$ promote an earlier entrance on the accelerated period.

In figure 3 it is shown the behavior of the energy densities of the fermionic field ρ_D and matter ρ_m as functions of time t . The numerical results show that eventually the energy density of the fermionic field overcomes the energy density of the matter field, although this does not coincide with the instant when the universe goes into an accelerated period (opposed to what occurs in some Einstein gravity based models[12]). Another feature showed by the numerical results is that for larger values of n it follows that: (a) the energy density of fermionic field grows more slowly causing a larger decelerated period and (b) the energy density of the matter field has a less significant decay and also promotes a larger decelerated period. On the other hand, as figure 3 shows, both densities have decreasing values in time due to the permanent expansion of the universe (this was verified with plots of the scale factor against time). Again, all cases are strongly dependent on the exponent n of the self-interacting potential and one can obtain different behaviors, that are

in tune with the acceleration patterns presented above. Finally we verify that the scalar field φ approaches a final constant value that validates the accord between Brans-Dicke and Einstein gravitation for large t .

As concluding remarks we stress that we have investigated the role of a fermionic field – with a self-interacting potential – in an old Brans-Dicke universe. We have

shown that the fermionic field, in combination with the scalar φ behave like a dark energy constituent, promoting a final accelerated period after an initial era dominated by matter. These results depend strongly on the initial values of $\dot{\varphi}$ and on the power n present in the self-interactive fermionic potential.

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