

## AN EXTENSION OF THE BINOMIAL THEOREM WITH APPLICATION TO STABILITY THEORY

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### **Abstract**

We show how it is possible to put different stability types such as Routh- Hurwitz and Schur-Cohn on common grounds by establishing direct links between them. In the process, we obtain natural and elegant extensions of both Pascal's rule and the binomial theorem, which prove useful in establishing our main results.

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### **1. Introduction**

A linear system of differential equations is said to be Routh-Hurwitz(Schur- Cohn) stable if and only if all of its eigenvalues lie in the left half of the complex plane (within the unit circle). The problem of locating the eigenvalues of a system of differential equations has fascinated mathematicians for decades, and the literature is full of ingenious methods, analyses of these methods, and discussions of their merits. Over the last Forty years or so they had tremendous impact on various areas of control theory. In case of real systems, the theory of stability is well developed. These are results which mathematicians and engineers are familiar with and they can be readily applied to theoretical problems in differential equations and linear algebra as much as to practical problems in electrical engineering and electronics, see for example [4], [6], [11] and [13] to mention just a few. The case of complex coefficients has received much less attention in the past, but recently a flurry of results has been reported, among many others see [1], [2], [5], [7], [9] and [14]. Many fresh attempts were made to put stability criteria of different natures such as Routh-Hurwitz and Schur-Cohn on common grounds, by invoking the intimate relationships that might prevail between these various stability types, for some very recent works in this direction see [3], [ 8], [10] and [12].

This paper is basically a contribution to the mainstream of bringing together these two important types of stability. It is structured as follows: In section 2 we give the necessary definitions and notations. In section 3 natural extensions of Pascal's rule and

the binomial theorem are obtained which are then applied in section 4 to prove the main results.

## 2. Definitions and Notations

By induction, define the following sequence of sets:

$$Z^{(n)} = \{z_1, z_2, \dots, z_n\}$$

for all  $n \geq 1$ , where for any positive integer  $j$ ,  $z_j$  is a real or complex number.

A  $j$ -subset of  $Z^{(n)}$  is a set consisting of  $j$  elements of  $Z^{(n)}$  having different subscripts.  $C_j^{(n)}$  denotes the set of all  $j$ -subsets of  $Z^{(n)}$ .

If  $1 \leq k \leq \binom{n}{j}$  where  $\binom{n}{j}$  is the binomial coefficient, let  $P_{jk}^{(n)}$  be the product of all  $j$  elements of the  $k^{th}$  subset of  $C_j^{(n)}$

$$\text{Let } S_j^{(n)} = \sum_{k=1}^{\binom{n}{j}} P_{jk}^{(n)} \text{ for } j = 1, \dots, n, \text{ and } S_0^{(n)} = 1 \text{ for all } n \geq 1$$

For any  $j = 1, \dots, n$ , let  $w_j = \frac{z_j-1}{z_j+1}$  which is equivalent to  $z_j = \frac{1+w_j}{1-w_j}$ . Similarly, let  $W^{(n)} = \{w_1, w_2, \dots, w_n\}$ , and let  $D_j^{(n)}$  be the set of all  $j$ -subsets of  $W^{(n)}$ . If  $1 \leq k \leq \binom{n}{j}$ ,  $Q_{jk}^{(n)}$  denotes the product of all  $j$  elements of the  $k^{th}$  subset of  $D_j^{(n)}$

$$\text{Let } T_j^{(n)} = \sum_{k=1}^{\binom{n}{j}} Q_{jk}^{(n)} \text{ for all } j = 1, \dots, n, \text{ and } T_0^{(n)} = 1 \text{ for all } n \geq 1.$$

A given linear system of differential equations is said to be Routh- Hurwitz (Schur-Cohn) stable if and only if all its eigenvalues lie in the left-half plane (inside the unit circle).

If  $A$  is an  $n \times n$  real or complex matrix, and  $X(t)$  is an  $n$ - dimensional column vector function of  $t$ , let  $X' = A \cdot X$  be a system of differential equations, with eigenvalues  $z_1, z_2, \dots, z_n$ . Then the characteristic polynomial of this system may be written in both factored and expanded forms as follows:

$f(z) = \prod_{j=1}^n (z - z_j) = \sum_{j=0}^n a_j z^{n-j}$  where  $a_0 = 1$  by definition. Similarly if  $X' = B \cdot X$  is a system with eigenvalues  $w_1, w_2, \dots, w_n$  (where  $w_j$  is related to  $z_j$  of the previous system by  $w_j = \frac{z_j-1}{z_j+1}$ ), then its characteristic polynomial is

$$g(w) = \prod_{j=1}^n (w-w_j) = \sum_{j=0}^n b_j w^{n-j}, \text{ with } b_0 = 1.$$

**3. Basic Results**

The following results are needed later.

**Lemma 3.1**  $S_j^{(n)} + S_{j-1}^{(n)} \cdot z_{n+1} = S_j^{(n+1)}$  for all  $j = 1, \dots, n$ .

**Proof.** If  $1 \leq j \leq n$ , let  $C = C_{j-1}^{(n)} \times \{z_{n+1}\}$  be the cartesian product of the two sets  $C_{j-1}^{(n)}$  and  $\{z_{n+1}\}$ .

If  $\{\phi_i\}_{i=1}^{\binom{n}{j-1}}$  is the family of all subsets forming  $C_{j-1}^{(n)}$ , then it is clear that  $C$  is in a one-to-one correspondence with the set  $\Psi = \left\{ \phi_i \cup \{z_{n+1}\}, i = 1, \dots, \binom{n}{j-1} \right\}$ , and  $\text{card } C = \text{card } \Psi = \binom{n}{j-1}$  where  $\text{card } X$  denotes the number of elements of a set  $X$ .

Now  $(C_j^{(n)} \cup \Phi) \subset C_j^{(n+1)}$  and  $C_j^{(n)} \cap \Phi = \emptyset$ , since no  $j$ -subsets of  $Z^{(n)}$  contain  $z_{n+1}$ . Hence

$$\text{card}(C_j^{(n)} \cup \Phi) = \text{card}C_j^{(n)} + \text{card}\Phi = \binom{n}{j} + \binom{n}{j-1} = \binom{n+1}{j} = \text{card}C_j^{(n+1)}.$$

Therefore  $C_j^{(n)} \cup \Phi = C_j^{(n+1)}$ , from which it follows automatically that  $S_j^{(n)} + S_{j-1}^{(n)} \cdot z_{n+1} = S_j^{(n+1)}$  for all  $j = 1, \dots, n$ . □

Lemma 3.1 is an extension of the famous Pascal's rule.

**Theorem 3.1**

$$f(z) = \sum_{j=0}^n (-1)^j S_j^{(n)} z^{n-j}$$

**Proof.** We proceed by induction on  $n$ .  $z - z_1 = z - S_1^{(1)}$ , hence our proposition is true for  $n = 1$ .

Suppose  $f(z) = \sum_{j=0}^n (-1)^j S_j^{(n)} z^{n-j}$  then

$$\begin{aligned}
 f(z) \cdot (z - z_{n+1}) &= \sum_{j=0}^n (-1)^j S_j^{(n)} z^{n-j+1} - \sum_{j=0}^n (-1)^j S_j^{(n)} z^{n-j} z_{n+1}, \text{ but} \\
 \sum_{j=0}^n (-1)^j S_j^{(n)} z^{n-j} z_{n+1} &= \sum_{j=1}^n (-1)^{j-1} S_{j-1}^{(n)} z^{n-j+1} z_{n+1} + (-1)^n S_{n+1}^{(n+1)} \text{ since} \\
 S_{n+1}^{(n+1)} &= S_n^{(n)} \cdot z_{n+1}. \text{ Hence} \\
 f(z) \cdot (z - z_{n+1}) &= z^{n+1} + \sum_{j=1}^n (-1)^j z^{n-j+1} (s_j^{(n)} + S_{j-1}^{(n)} z_{n+1}) + (-1)^{n+1} S_{n+1}^{(n+1)} \\
 &= \sum_{j=0}^{n+1} (-1)^j S_j^{(n+1)} z^{n-j+1}, \text{ by lemma 3.1 and the proof is complete.}
 \end{aligned}$$

□

Theorem 3.1 is an extension to the well-known binomial theorem. The following is now clear,

**Corollary 3.1**  $S_j^{(n)} = (-1)^j a_j$  and  $T_j^{(n)} = (-1)^j b_j$  for all  $j = 0, 1, \dots, n$ .

The intimate relationship between Routh-Hurwitz and Schur-Cohn types of stability could best be expressed by the following:

**Theorem 3.2** *The system  $X' = A \cdot X$  is Schur-Cohn stable if and only if  $X' = B \cdot X$  is Routh-Hurwitz stable.*

**Proof.** Suppose  $z = \frac{1+w}{1-w}$  or equivalently  $w = \frac{z-1}{z+1}$  where  $z$  and  $w$  are complex numbers. The following relationships can easily be established

$$w + \bar{w} = \frac{2(z\bar{z}-1)}{|z+1|^2} \text{ and } z \cdot \bar{z} - 1 = \frac{2(w+\bar{w})}{|1-w|^2}, \text{ from either of which it follows that } |z| < 1 \text{ if and only if } \operatorname{Re} w < 0. \quad \square$$

#### 4. Routh-Hurwitz in Terms of Schur-Cohn

If  $r$  and  $s$  are non-negative integers, define:

$$\binom{s}{r} = \begin{cases} \frac{s!}{r!(s-r)!} & \text{if } s \geq r \\ 0 & \text{if } s < r \end{cases}$$

For technical purposes we also define:

$$\binom{-1}{-1} = 1 \text{ and } \binom{s}{-1} = 0 \text{ for any integer } s \geq 0.$$

If  $X' = A \cdot X$  and  $X' = B \cdot X$  are the two systems defined in section 2 with their corresponding characteristic polynomials, the

**Theorem 4.1**

$$b_p = \frac{\sum_{t=0}^n \sum_{s=0}^{t'} \sum_{r=0}^s (-1)^{\min(p,t)+t+r} \binom{s-1}{r-1} \binom{n-t'}{|p-t|+r} a_t}{\sum_{r=0}^n (-1)^r a_r}$$

for all  $p = 1, \dots, n$  and where  $t' = \begin{cases} t & \text{if } t \leq p \\ n-t & \text{if } t > p \end{cases}$

**Proof.** Let  $1 \leq p \leq n$ . By corollary 3.1  $(-1)^p b_p = \sum_{k=1}^{\binom{n}{p}} Q_{pk}^{(n)}$ .

We bring all terms  $Q_{pk}^{(n)}$  for  $1 \leq k \leq \binom{n}{p}$  to a common denominator  $D_p = \prod_{r=1}^n (z_r + 1)$ .

Call  $N_p$  the numerator, hence  $(-1)^p b_p = \frac{N_p}{D_p}$ . A typical element in the sum appearing in  $N_p$  is

$$t_p = \begin{cases} \prod_{r=1}^p (z_r - 1) \cdot \prod_{s=p+1}^n (z_s + 1) & \text{if } p < n \\ \prod_{r=1}^n (z_r - 1) & \text{if } p = n \end{cases}$$

All elements of  $N_p$  can be produced from  $t_p$  by considering all possible positions of the  $p$  minus signs of  $t_p$  into the  $n$  factors of  $t_p$ . It is clear that the constant term in  $N_p$  is  $(-1)^p \binom{n}{p}$  and

$$D_p = \sum_{r=0}^n (-1)^r a_r.$$

Let  $1 \leq t \leq p$ . We propose to calculate the coefficient of

$$S_t^{(n)} = \sum_{k=1}^{\binom{n}{t}} P_{tk}^{(n)}$$

appearing in  $N_p$ . But first we note the following:

If we consider the product of any  $t$  factors chosen from the set  $\{z_r - 1, 1 \leq r \leq p\} \cup \{z_s + 1, p + 1 \leq s \leq n\}$  if  $p < n$  and from the set  $\{z_r - 1, 1 \leq r \leq n\}$  if  $p = n$ , this product clearly shows up in exactly  $\binom{n-t}{p-t}$  of the elements forming  $N_p$ . This leads to the fact that the arrangement of the factors in such products is not significant. Therefore

all  $P_{tk}^{(n)}$  for  $1 \leq k \leq \binom{n}{t}$  have the same coefficient, which is that of  $S_t^{(n)}$ . Hence, it suffices to calculate the coefficient of  $P_{t1}^{(n)}$  where naturally  $P_{t1}^{(n)} = \prod_{r=1}^t z_r$ .

Next we explain our strategy in producing all terms of  $N_p$  from  $t_p$ : As a first step, consider the terms of  $N_p$  corresponding to all possible positions of the  $(p-t)$  minus signs into the  $(n-t)$  different positions indicated in

$$t_p = \prod_{j=1}^t (z_j - 1) \cdot \underbrace{\prod_{k=t+1}^p (z_k - 1)}_{p-t} \cdot \underbrace{\prod_{m=p+1}^n (z_m + 1)}_{n-t},$$

where we suppose  $t < p < n$ . If  $c_0$  is the coefficient of  $P_{t1}^{(n)}$  calculated among these terms, then  $c_0 = (-1)^{p-t} \binom{n-t}{p-t}$ . The cases  $t < p = n$ ,  $t = p < n$  and  $t = p = n$  lead to the same conclusion.

Next we go back to  $t_p$  and consider the block of  $(p-t+1)$  minus signs appearing in the product  $\prod_{k=t}^p (z_k - 1)$  of  $t_p$  which we shift one step to the right to get to the position:

$$\prod_{j=1}^{t-1} (z_j - 1) \cdot (z_t + 1) \cdot \underbrace{\prod_{k=t+1}^{p+1} (z_k - 1)}_{p-t+1} \cdot \underbrace{\prod_{m=p+2}^n (z_m + 1)}_{n-t}. \tag{1}$$

Then consider the terms which arise from all possible positions of the  $(p-t+1)$  minus signs into the  $(n-t)$  factors shown in (1). If  $c_1$  is the coefficient of  $P_{t1}^{(n)}$  calculated among these terms, then  $c_1 = (-1)^{p-t+1} \binom{n-t}{p-t+1}$ .

In general if  $1 \leq s \leq t$ , shift the block of  $(p-t+s)$  minus signs of the product  $\prod_{k=t-s+1}^p (z_k - 1)$  one step to the right to obtain the position

$$\prod_{j=1}^{t-s} (z_j - 1) \cdot (z_{t-s+1} + 1) \cdot \underbrace{\prod_{k=t-s+2}^{p+1} (z_k - 1)}_{p-t+s} \cdot \underbrace{\prod_{m=p+2}^n (z_m + 1)}_{n-t+s-1}. \tag{2}$$

Let  $c_s$  be the coefficient of  $P_{1t}^{(n)}$  calculated among the terms of  $N_p$  which correspond to all possible positions of the  $(p-t+s)$  minus signs into the  $(n-t+s-1)$  different positions shown in (2).

We claim that  $c_s = \sum_{r=1}^s (-1)^{p-t+r} \binom{s-1}{r-1} \binom{n-t}{p-t+r}$  for all  $s, 1 \leq s \leq t$ .

We proceed by induction on  $s$ . Our claim is true when  $s = 1$ . Suppose it is true for  $s$  where  $1 \leq s < t$ . In  $t_p$ , we move the  $(p-t+s+1)$  minus signs appearing in the product  $\prod_{k=t-s}^p (z_k - 1)$  one step to the right. So we are in the position:

$$\prod_{j=1}^{t-s-1} (z_j - 1) \cdot (z_{t-s} + 1) \cdot \underbrace{\prod_{k=t-s+1}^{p+1} (z_k - 1)}_{p-t+s+1} \cdot \prod_{m=p+2}^n (z_m + 1). \quad (3)$$

Since  $\prod_{k=t-s+1}^{p+1} (z_k - 1) = (z_{t-s+1} - 1) \cdot \prod_{k=t-s+2}^{p+1} (z_{k-1})$ , the product

$\prod_{k=t-s+2}^{p+1} (z_{k-1})$  corresponds to shifting in  $t_p$  the  $(p-t+s)$  minus signs showing

up in the product  $\prod_{k=t-s+1}^p (z_k - 1)$  one step to the right to get the position already shown

in (2). If  $c_s$  as defined above, then by our induction assumption

$$c_s = \sum_{r=1}^s (-1)^{p-t+r} \binom{s-1}{r-1} \binom{n-t}{p-t+r}.$$

Once this done, we go back to  $\prod_{k=t-s+1}^{p+1} (z_k - 1)$  in (3) and shift the  $(p-t+s+1)$  minus signs

one further step to the right to get the position:

$$\prod_{j=1}^{t-s-1} (z_j - 1) \cdot (z_{t-s} + 1) \cdot (z_{t-s+1} + 1) \cdot \underbrace{\prod_{k=t-s+2}^{p+2} (z_k - 1)}_{p-t+s+1} \cdot \prod_{m=p+3}^n (z_m + 1). \quad (4)$$

Call  $c'_s$  the coefficient of  $P_{t1}^{(n)}$  calculated among the terms of  $N_p$  corresponding to all possible positions of the  $(p-t+s+1)$  minus signs into the  $(n-t+s-1)$  different

positions shown in (4). If we compare (4) to (2), we realize that in (4) we are dealing with the product  $\prod_{k=t-s+2}^{p+2} (z_k - 1)$  whereas in (2) we dealt with  $\prod_{k=t-s+2}^{p+1} (z_k - 1)$ . therefore we may obtain  $c'_s$  by replacing  $p+1$  by  $p+2$  or equivalently  $p$  by  $p+1$  in  $c_s$ . Therefore

$$c'_s = \sum_{r=1}^s (-1)^{p-t+r+1} \binom{s-1}{r-1} \binom{n-t}{p-t+r+1}$$

It is clear that  $c_{s+1} = c_s + c'_s$ . Hence

$$\begin{aligned} c_{s+1} &= \sum_{r=1}^s (-1)^{p-t+r} \binom{s-1}{r-1} \binom{n-t}{p-t+r} + \sum_{r=1}^s (-1)^{p-t+r+1} \binom{s-1}{r-1} \binom{n-t}{p-t+r+1} \\ &= (-1)^{p-t+1} \binom{s-1}{0} \binom{n-t}{p-t+r} + \sum_{r=1}^{s-1} (-1)^{p-t+r+1} \binom{s-1}{r} \binom{n-t}{p-t+r+1} \\ &\quad + \sum_{r=1}^{s-1} (-1)^{p-t+r+1} \binom{s-1}{r-1} \binom{n-t}{p-t+r+1} + (-1)^{p-t+s+1} \binom{s-1}{s-1} \binom{n-t}{p-t+s+1} \\ &= (-1)^{p-t+1} \binom{s}{0} \binom{n-t}{p-t+1} + \sum_{r=1}^{s-1} (-1)^{p-t+r+1} \left[ \binom{s-1}{r} + \binom{s-1}{r-1} \right] \binom{n-t}{p-t+r+1} \\ &\quad + (-1)^{p-t+s+1} \binom{s}{s} \binom{n-t}{p-t+s+1} \end{aligned}$$

Since  $\binom{s-1}{r} + \binom{s-1}{r-1} = \binom{s}{r}$  and by shifting indices, we get

$$C_{s+1} = \sum_{r=1}^{s+1} (-1)^{p-t+r} \binom{s}{r-1} \binom{n-t}{p-t+r} \text{ proving our claim.}$$

The coefficient of  $S_t^{(n)}$  is therefore

$$\begin{aligned} \sum_{s=0}^t c_s &= (-1)^{p-t} \binom{n-t}{p-t} + \sum_{s=1}^t \sum_{r=1}^s (-1)^{p-t+r} \binom{s-1}{r-1} \binom{n-t}{p-t+r} \\ &= \sum_{s=0}^t \sum_{r=0}^s (-1)^{p-t+r} \binom{s-1}{r-1} \binom{n-t}{p-t+r} \end{aligned}$$

Now let  $p+1 \leq t \leq n$  and reconsider



$$t_p = \underbrace{\prod_{j=1}^p (z_j - 1)}_t \cdot \underbrace{\prod_{k=p+1}^t (z_k + 1)}_{t-p} \cdot \prod_{m=t+1}^n (z_m + 1), \tag{5}$$

where we suppose  $t < n$ .

Again we propose to calculate the coefficient of  $S_t^{(n)}$ . First consider all terms of  $N_p$  arising from all possible positions of the  $(t - p)$  plus signs into the  $t$  different positions shown in (5). Let  $c'_0$  be the coefficient of  $P_{t1}^{(n)}$  calculated among these terms, then  $c'_0 = \binom{t}{t-p}$ . If  $t = n$ , we are clearly lead to the same conclusion.

In general if  $1 \leq s \leq n - t$ , in (5) above we shift the block of  $(t - p + s)$  plus signs of the product  $\prod_{k=p+1}^{t+s} (z_k + 1)$  one step to the left to get:

$$\underbrace{\prod_{j=1}^{p-1} (z_j - 1)}_{t+s-1} \cdot \underbrace{\prod_{k=p}^{t+s-1} (z_k + 1)}_{t-p+s} \cdot (z_{t+s} - 1) \cdot \prod_{m=t+s+1}^n (z_m + 1). \tag{6}$$

If  $c'_s$  is the coefficient of  $P_{t1}^{(n)}$  calculated among all terms of  $N_p$  corresponding to all possible positions of the  $(t - p + s)$  plus signs into the  $(t + s - 1)$  different positions shown in (6). By an induction similar to the previous one, we show that  $c'_s = \sum_{r=1}^s (-1)^r \binom{s-1}{r-1} \binom{t}{t-p+r}$  for all  $s, 1 \leq s \leq n - t$ .

The coefficient of  $S_t^{(n)}$  is therefore

$$\sum_{s=0}^{n-t} c'_s = \sum_{s=0}^{n-t} \sum_{r=0}^s (-1)^r \binom{s-1}{r-1} \binom{t}{t-p+r}.$$

So if  $(-1)^p b_p = \frac{N_p}{D_p}$ , then

$$\begin{aligned} N_p &= \sum_{t=0}^p \sum_{s=0}^t \sum_{r=0}^s (-1)^{p-t+r} \binom{s-1}{r-1} \binom{n-t}{p-t+r} S_t^{(n)} \\ &+ \sum_{t=p+1}^n \sum_{s=0}^{n-t} \sum_{r=0}^s (-1)^r \binom{s-1}{r-1} \binom{t}{t-p+r} S_t^{(n)}. \end{aligned}$$

Easy to see how  $b_p$  can be brought to the form stated in the theorem. □

**5. Schur-cohn in Terms of Routh-Hurwitz**

The converse of theorem 4.1 states the following:

**Theorem 5.1** *If  $t'$  as defined before, then*

$$a_{n-p} = \frac{\sum_{t=0}^n \sum_{s=0}^{t'} \sum_{r=0}^s (-1)^{\max(p,t)+n+r} \binom{s-1}{r-1} \binom{n-t'}{|p-t|+r} b_t}{\sum_{r=0}^n b_r}$$

for all  $p = 0, 1, \dots, n-1$

**Proof.** Suppose  $1 \leq p \leq n-1$ , then

$$(-1)^{n-p} a_{n-p} = S_{n-p}^{(n)} = \sum_{k=1}^{\binom{n}{n-p}} P_{(n-p)k}^{(n)}$$

From both sides of this relation we cancel out the factor  $(-1)^{n-p}$  and we bring all terms in the right-hand side to a common denominator  $D'_p = \prod_{r=1}^n (w_r - 1)$ . cal  $N'_p$  the numerator. Hence  $a_{n-p} = \frac{N'_p}{D'_p}$ . It is clear that  $D'_p = (-1)^n \sum_{r=0}^n b_r$ . A typical element in the sum appearing in  $N'_p$  is

$$\underbrace{\prod_{r=1}^{n-p} (w_r + 1)}_{n-p} \cdot \underbrace{\prod_{s=n-p+1}^n (w_s - 1)}_p$$

This element is entirely similar to  $t_p$  of theorem 4.1 except that the  $w$ 's replace the  $z$ 's. Therefore,

$$\begin{aligned} N'_p &= \sum_{t=0}^p \sum_{s=0}^t \sum_{r=0}^s (-1)^{p-t+r} \binom{s-1}{r-1} \binom{n-t}{p-t+r} T_t^{(n)} \\ &+ \sum_{t=p+1}^p \sum_{s=0}^{n-t} \sum_{r=0}^s (-1)^r \binom{s-1}{r-1} \binom{t}{t-p+r} T_t^{(n)}, \text{ and} \\ a_{n-p} &= \frac{\sum_{t=0}^n \sum_{s=0}^{t'} \sum_{r=0}^s (-1)^{\max(p,t)+n+r} \binom{s-1}{r-1} \binom{n-t'}{|p-t|+r} b_t}{\sum_{r=0}^n b_r} \end{aligned} \tag{7}$$

for  $1 \leq p \leq n - 1$ .

Let  $N'_0$  be the numerator of the right-hand side of (7) corresponding to  $p = 0$ . Then

$$N'_0 = (-1)^n + \sum_{t=1}^n \sum_{s=0}^{n-t} \sum_{r=0}^s (-1)^{t+n+r} \binom{s-1}{r-1} \binom{t}{t+r} b_t, \text{ which reduces to}$$

$$N'_0 = (-1)^n + \sum_{t=1}^n (-1)^{t+n} b_t = \sum_{t=0}^n (-1)^{t+n} b_t, \text{ and}$$

$$(-1)^n a_n = S_n^{(n)} = \prod_{t=1}^n z_t = (-1)^n \prod_{t=1}^n \left( \frac{w_t + 1}{w_t - 1} \right). \text{ Therefore}$$

$$a_n = \frac{\sum_{t=0}^n (-1)^t b_t}{(-1)^n \sum_{r=0}^n b_r} = \frac{\sum_{t=0}^n (-1)^{t+n} b_t}{\sum_{r=0}^n b_r}.$$

We conclude that relation (7) covers the case  $p = 0$  and the proof is complete □

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## BİNOM THEOREMİNİN BİR GENELLEŞTİRMESİ VE STABİLİTE TEORİSİNE UYGULAMALARI

### Özet

Bu makalede Routh-Hurwitz ve Schur-Cohn gibi değişik stabilite tipleri daha geniş bir çerçevede ele alınıp aralarındaki ilişkiler ortaya konmuştur.

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