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# ON THE $\ell_P$ NORMS OF ALMOST CAUCHY-TOEPLITZ MATRICES

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**Abstract**

In this study, we have given the definition of almost Cauchy-Toeplitz matrix. i.e. its elements are $t_{ij} = a(i = j)$ and $t_{ij} = 1/(i-j)\,(i \neq j)$ such that $a$ is a real number. We have found a lower and upper bounds for the $\ell_p$ norm of this matrix. Furthermore, we have done the proof of the conjecture that were given by myself for the spectral norm of this matrix.
**Key Words:** Cauchy, Toeplitz, norm, Cauchy-Toeplitz matrix, matrix norm.

## 1. Introduction

In this study, we have given the definition of almost Cauchy-Toeplitz matrix (i.e. its elements are $t_{ij} = a(i = j)$ and $t_{ij} = 1/(i-j)$ such that $a$ is a real number).

It is called Cauchy and Toeplitz matrices as in the following forms:

$$C_n = [1/(x_i - y_i)]_{i,j=1}^n$$

where $x_i \neq y_j$ for all $i, j$ and

$$T_n = [t_{j-i}]_{i,j=1}^n$$

respectively. It is easy to check that an arbitrary Cauchy- Toeplitz matrix of order $n$ is of the form

$$T_n = \left[\frac{1}{g + (i-j).h}\right]_{i,j=1}^n \tag{1.1}$$

where $g$ and $h$ are some numbers. We assume that $h \neq 0$ and the quotient $g/h \notin Z(Z$ is the set of integers).

In [5], E. E. Tyrtyshnikov found a lower bound for the spectral norm of Cauchy-Toeplitz matrix such that $h = 1$ and $g = 1/2$. He had computed that with $\varepsilon = 10^{-4}$ for $n = 40, 60, 100$ then numbers of singular values $\rho_{1n} \geq \rho_{2n} \geq \ldots \geq \rho_{nn}$ of the matrix $T_n$ satisfying the condition $\rho_{jn} > \pi - \varepsilon$ are equal to 6,7,8 respectively.

A few years ago these matrices attracted the attention of C. Moler, who had experimentally discovered that most of their singular values are clustered near $\pi$. Recently S. Parter gave an explanation of this phenomenon [3].

In [1], we had generalized the bounds that E. E. Tyrtyshnikov had found. i. e. we had established that

$$\frac{\sqrt{n}\pi}{|h|} - s_n \leq \| T_n \| \leq \frac{\sqrt{n}\pi}{|h|} \tag{1.2}$$

where $h \neq 0, g/h$ is not integer, $s_n > 0$ and $s_n = O(1/n)$.

In Section 2 we have given the definition of almost Cauchy-Toeplitz matrix. We have found a lower and an upper bounds for the $\ell_p$ norm of this matrix. Furthermore, we have done the proof of the conjecture that we had given in [2] for the spectral norm of this matrix.

## 2. The $\ell_p$ Norms of the almost Cauchy-Toeplitz Matrices

Let $A$ be any nxn matrix. The $\ell_p$ norm of the matrix $A$ is

$$\| A \|_p = \left[ \sum_{i,j=1}^{n} |a_{ij}|^p \right]^{1/p} \tag{2.1}$$

where $p \geq 1$. We can write the equation (2.1) in the following form:

$$\| A \|_p = \left[ \sum_{i,j=1}^{n} |a_{ij}|^p \right]^{1/p} = \left[ \sum_{i=1}^{n} (\|_{1i}\, e_i \|^p + \| a_{2i} e_i \|^p + \ldots + \| a_{ni} e_i \|^p) \right]^{1/p} \tag{2.2}$$

where $e_i$ $(1 \leq i \leq n)$ are the standard basis of $R^n$.

**Definition** *The matrix $A_n$ is called almost Cauchy-Toeplitz matrix if*

$$A_n = \begin{cases} a, i = j \\ \frac{1}{i-j}, i \neq j \end{cases} \tag{2.3}$$

*where $a \in R$ ($R$ is the set of the real numbers) and $1 \leq i, j \leq n$.*

*In [2] we had shown that the following theorem is valid $p > 2$.*

**Theorem 2.1.[2]** *Let the matrix $A_n$ be as in (2.3). Then*

$$\sqrt[p]{|a|^p - \alpha_n} \leq n^{-1/p} \| A_n \|_p \leq \sqrt[p]{|a|^p + 2\zeta(p)}$$

*is valid where $\zeta(p)$ is Riemann's zeta function, $a_n > 0, \alpha_n = O(1/n)$ and $p > 2$.*

**Proof.** From (2.2), we have

$$\| A \|_p = \left[ \sum_{i,j=1}^{n} |a_{ij}|^p \right]^{1/p} = \left[ \sum_{i=1}^{n} (|a_{1i}|^p \| e_i \|^p + |a_{2i}|^p \| e_i \|^p + \ldots + |a_{ni}|^p \| e_i \|^p) \right]^{1/p}.$$

Since $e_i$ are the standard basis of $R^n$ for all $i\,(1 \leq i \leq n)$, its norms are equal to 1. Then, we obtain

$$\begin{aligned} \| A_n \|_p &= \left\{ n|a|^p + 2\left[ (n-1) + (n-2)\left(\frac{1}{2}\right)^p + (n-3)\left(\frac{1}{3}\right)^p + \ldots + \left(\frac{1}{n-1}\right)^p \right] \right\}^{1/p} \\ &= \left( n|a|^p + 2\sum_{k=1}^{n-1} \frac{n-k}{k^p} \right)^{1/p} \end{aligned}$$

Hence

$$\| A_n \|_p = \left( n|a|^p + 2n\sum_{k=1}^{n-1} \frac{1}{k^p} - 2\sum_{k=1}^{n-1} \frac{1}{k^{p-1}} \right)^{1/p}. \tag{2.4}$$

Multiplying the equation (2.4) with $n^{-1/p}$, then

$$n^{-1/p} \| A_n \|_p = \left( |a|^p + 2\sum_{k=1}^{n-1} \frac{1}{k^p} - \frac{2}{n}\sum_{k=1}^{n-1} \frac{1}{k^{p-1}} \right)^{1/p}. \tag{2.5}$$

From (2.5)

$$\lim_{n\to\infty} n^{-1/p} \| A_n \|_p = \lim_{n\to\infty} \left( |a|^p + 2\sum_{k=1}^{n-1} \frac{1}{k^p} - \frac{2}{n}\sum_{k=1}^{n-1} \frac{1}{k^{p-1}} \right)^{1/p}.$$

Since

$$\sum_{k=1}^{\infty} \frac{1}{k^p} = \zeta(p) \text{ ve } \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n-1} \frac{1}{k^{p-1}} = 0,$$

then we obtain

$$n^{-1/p} \| A_n \|_p \leq \sqrt[p]{|a|^p + 2\zeta(p)}. \tag{2.6}$$

from (2.5). Thus , we have found upper bound for $A_n$.

From (2.5) we have

$$n^{-1/p} \parallel A_n \parallel_p \geq \left( |a|^p - \frac{2}{n} \sum_{k=1}^{n-1} \frac{1}{k^{p-1}} \right)^{1/p}. \tag{2.7}$$

If we get

$$\alpha_n = \frac{2}{n} \sum_{k=1}^{n-1} \frac{1}{k^{p-1}}$$

in (2.7), $\alpha_n > 0$ and

$$\lim_{n \to \infty} \frac{\alpha_n}{1/n} \leq 2\zeta(p-1).$$

Then $\alpha_n = O(1/n)$. Hence

$$n^{-1/p} \parallel A_n \parallel_p \geq \sqrt[p]{|a|^p - \alpha_n} \tag{2.8}$$

Thus, the proof of the theorem is completed from (2.6) and (2.8). □

**Theorem 2.2.** *Let the matrix $A_n$ be as in (2.3). Then*

$$\begin{gathered} |a| + o(n) \leq n^{-1} \parallel A_n \parallel_p \leq |a| + \gamma + \log n + O(1/n), \ (p = 1) \\ |a| \leq n^{-1/p} \parallel A_n \parallel_p \leq \sqrt[p]{|a|^p + \frac{2^p}{2^{p-1} - 1}}, \ (1 < p \leq 2) \\ \sqrt[p]{|a|^p + \zeta(p)} \leq n^{-1/p} \parallel A_n \parallel_p \leq \sqrt[p]{|a|^p + 2\zeta(p)} \end{gathered} \tag{2.9}$$

*are valid where $\zeta(p)$ is Riemann's zeta-function and $\gamma$ is Euler's constant.*

The lower bound in (2.9) is better than the lower bound in Theorem 2.1.

**Proof.** Let us do three steps the proof of theorem.

**Step 1.** Let $p = 1$. From (2.5) we obtained

$$\left. \begin{aligned} n^{-1} \parallel A_n \parallel_1 &= |a| + 2\textstyle\sum_{k=1}^{n-1} \frac{1}{k} - \frac{2(n-1)}{n} \\ &= |a| + 2\textstyle\sum_{k=1}^{n-1} \frac{1}{k} - 2 + \frac{2}{n} \\ &\leq |a| + 2\textstyle\sum_{k=1}^{n-1} \frac{1}{k} + \frac{2}{n} \end{aligned} \right\}. \tag{2.10}$$

Since $\frac{2}{n} \to 0$ and $\sum_{k=1}^{n} \frac{1}{k} = \gamma + \log n + O(1/n)$ for the sufficient large n, then

$$n^{-1} \parallel A_n \parallel_1 \leq |a| + \gamma + \log n + O(1/n). \tag{2.11}$$

We have

$$n^{-1} \parallel A_n \parallel_1 \geq |a| + \sum_{k=2}^{n} \frac{1}{k}$$

from the first equation of (2.10). Taking

$$\alpha_n = \sum_{k=2}^{n} \frac{1}{k},$$

then $\alpha_n = o(n)$ (i.e. $\lim_{n\to\infty} \alpha_n/n = 0$). We have

$$n^{-1} \parallel A_n \parallel_1 \geq |a| + o(n) \tag{2.12}$$

**Step 2.** Let $1 < p \leq 2$. We obtain

$$n^{-1/p} \parallel A_n \parallel_p \leq \left( |a|^p + 2 \sum_{k=1}^{n-1} \frac{1}{k^p} \right)^{1/p}$$

from (2.5). Since

$$\sum_{k=1}^{\infty} \frac{1}{k^p} \leq \frac{2^{p-1}}{2^{p-1} - 1}$$

according to p-test for the series, we find

$$n^{-1/p} \parallel A_n \parallel_p \leq \sqrt[p]{|a|^p + \frac{2^p}{2^{p-1} - 1}}. \tag{2.13}$$

From (2.5) we have

$$n^{-1/p} \parallel A_n \parallel_p \geq \left( |a|^p - \frac{2}{n} \sum_{k=1}^{n-1} \frac{1}{k^{p-1}} \right)^{1/p}.$$

Then since

$$\lim_{n\to\infty} \frac{2}{n} \sum_{k=1}^{n-1} \frac{1}{k^{p-1}} = 0,$$

we obtain

$$n^{-1/p} \parallel A_n \parallel_p \geq |a|. \tag{2.14}$$

**Step 3.** Let $p > 2$. Since the proof of the right side of the inequality (2.9) was done Theorem 2.1, here we will do the proof of the left side of the inequality (2.9). We have

$$2\sum_{k=1}^{n-1}\frac{1}{k^p} - \frac{2}{n}\sum_{k=1}^{n-1}\frac{1}{k^{p-1}} \geq \sum_{k=1}^{n-1}\frac{1}{k^p}$$

from the equation (2.5) as $n \to \infty$. Then

$$n^{-1/p} \parallel A_n \parallel_p \geq \left(|a|^p + \sum_{k=1}^{n-1}\frac{1}{k^p}\right)^{1/p}.$$

Since

$$\sum_{k=1}^{\infty}\frac{1}{k^p} = \zeta(p),$$

we have

$$n^{-1/p} \parallel A_n \parallel_p \geq \sqrt[p]{|a|^p + \zeta(p)}. \tag{2.15}$$

The proof of the Theorem 2.2 is completed from (2.11), (2.12), (2.13), (2.14) and (2.15). □

**Conjecture 2.1**[2] *Let the matrix $A_n$ be as in (2.3). Then*

$$\parallel A_n \parallel_2 \leq \sqrt{a^2 + \pi^2}$$

*where $\parallel \cdot \parallel_2$ is the spectral norm of the matrix $A_n$.*

**Proof.** Let $H_n = H_n^\star = i(A_n - aI)$ with spectral radius $r(H_n)$ where I is the unit matrix of order n. Then $A_n$ is normal matrix with spectrum

$$\sigma(A_n) = \{a + i\lambda : \lambda \in \sigma(H_n)\},$$

and hence

$$\parallel A_n \parallel_2 = r(A_n) = \sqrt{a^2 + r(H_n)^2}.$$

Let us consider the symmetric matrix

$$\tilde{H}_n = \frac{1}{i}\begin{bmatrix} 0 & H_n^T \\ H_n & 0 \end{bmatrix}$$

The matrix $\tilde{H}_n$ is permutation-similar to the block Toeplitz matrix $\hat{H}_n$ as in the following form:

$$\tilde{H}_n = \begin{bmatrix} 0 & 0 & -1 & 0 & -\frac{1}{2} & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & \frac{1}{2} & \dots \\ 1 & 0 & 0 & 0 & -1 & 0 & \dots \\ 0 & -1 & 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

After all the odd rows have been multiplied by the imaginary unit $i$ and the even ones by $-i$, the result is Hermitian Toeplitz matrix $T_n = [t_{i-j}]_{i,j=0}^{2n-1}$. Let $f(x)$ be a real- valued function and $u = [u_0, u_1, \dots, u_{2n-1}]^T$. Then we have

$$\begin{aligned}(T_n u, u) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|u_0 + u_1 e^{ix} + \dots + u_{2n-1}e^{i(2n-1)x}\right|^2 f(x)dx, \\ (u,u) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|u_0 + u_1 e^{ix} + \dots + u_{2n-1}e^{i(2n-1)x}\right|^2 dx\end{aligned}$$

[4, pp. 63] and

$$\| H_n \|_2 = \| T_n \|_2 = \max_{u\neq 0}\left|\frac{(T_n u, u)}{(u,u)}\right| \leq \pi \tag{2.16}$$

[5, pp. 9 and 3, pp. 120]. Let $T_n(f)$ be a Hermitian Toeplitz form. If $a$ and $b$ are arbitrary real numbers, the eigenvalue associated with the function $a + bf(x)$ will be $a + b\lambda$ [4, pp. 64]. So, we have

$$\| A_n \|_2 = r(A_n) \leq \sqrt{a^2 + \pi^2}.$$

Thus, the proof of the conjecture is completed. □

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## ALMOST CAUCHY-TOEPLITZ MATRİSLERİNİN $\ell_p$ NORMLARI

**Özet**

Bu çalışmada, Almost Cauchy-Toeplitz matris tanımını yaptık (yani $a$ herhangi bir reel sayı olmak üzere elemanları $t_{ij} = a(i = j)$ ve $t_{ij} = 1/(i-j)(i \neq j)$ şeklinde olan matris). Bu matrisin $\ell_p$ normu için bir alt ve üst sınır bulduk. Ayrıca bu matrisin spektral normuyla ilgili bir konjektürün ispatını yaptık.

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