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APPLICATION OF THE RESIDUE METHOD TO A MIXED PROBLEM

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Abstract

In this paper, the residue method of separation of variables is applied to a mixed problem which includes the flow of a stratified compresslble fluid and inner gravitational waves of a stratified incompressible fluid for special values of the coefficients of the equation. To apply this method one defines two auxiliary problems. The first one is a spectral problem and the second is a Cauchy problem. Using the solutions of auxiliary problems, an existence and uniqueness theorem is proved for the given mixed problem. an explicit formula for the unique solution is also given.

Introduction

In this paper the following mixed problem will be studied by the residue method of the seperation of variables:

$$M\left(\frac{\partial}{\partial t}\right)u = N\left(\frac{\partial}{\partial t}\right)P\left(\frac{\partial}{\partial x}\right)u,$$

$$(x,t)\in\Omega=\{(x,t)|0\le x\le 1,\ t\ge 0\} \tag{1}$$

$$\frac{\partial^{k-1}u}{\partial t^{k-1}}|_{t=0}=\phi_k(x)\quad (k=1,2,\dots,q),\ 0\le x\le 1, \tag{2}$$

$$U_i(u)\equiv\sum_{j=1}^{2}\left[\alpha_{ij}\frac{\partial^{j-1}u}{\partial x^{j-1}}\bigg|_{x=0}+\beta_{ij}\frac{\partial^{j-1}u}{\partial x^{j-1}}\bigg|_{x=1}\right]=0\quad (i=1,2), t\ge 0, \tag{3}$$

where

$$M(z)=\sum_{k=0}^{K_0}M_kz^k,\ N(z)=\sum_{k=0}^{2}N_kz^k,\ P(z)=\sum_{k=0}^{2}P_kz^k$$

and

$$M_k(k=0,1,\ldots,K_0),\ P_k(k=0,1,2),\ \alpha_{ij},\beta_{ij}(i,j=1,2)$$

are complex constants with $M_{K_0} \neq 0,\ N_2 \neq 0,\ P_2 = 1$. Also $K_0, 0 \leq K_0 \leq 4$ is an integer, and $q = \max(2, K_0)$. In the above, the function $u(x,t)$ is the unknown of the problem and the initial data $\phi_k(x)\,(k=1,2,\ldots,q)$ are given continuous functions.

It is easy to see that in dimensionless quantities, for

$$K_0 = 4, M_4 = \varepsilon^2, M_2 = N_2 = N_0 = 1, M_0 = M_1 = M_3 = N_1 = P_0 = P_1 = 0$$

Eq. (1) is reduced to the one dimensional equation of dynamics of an exponentialy stratified compressible fluid [2,5]:

$$\varepsilon^2 \frac{\partial^4 u}{\partial t^4} - \frac{\partial^2}{\partial t^2}(u_{xx} - u) - u_{xx} = 0. \tag{4}$$

For

$$K_0 = 2,\ M_2 = N_2 = N_0 = 1,\ M_0 = M_1 = N_1 = P_0 = P_1 = 0$$

Eq. (1) gives the equation of dynamics of one dimensional inner gravitational waves of a stratified incompressible fluid [2]:

$$-\frac{\partial^2}{\partial t^2}(u_{xx} - u) - u_{xx} = 0. \tag{5}$$

Our investigation will cover both of these problems and the results obtained for these equations will be noted in the following sections. Let us specify what we understand of a classical solution:

Definition *The classical solution of the problem (1)-(3) is a function $u(x,t)$ which has continuous derivatives*

$$\frac{\partial^{k+i} u}{\partial t^k \partial x^i} \quad (0 \leq k, i \leq 2), \frac{\partial^k u}{\partial t^k} \quad (k = 1, 2, \ldots, q)$$

in the domain Ω and satisfies the equations (1)-(3) in the ordinary sense.

1. Auxiliary Problems and Facts

To apply the residue method to our problem, following [7(see p.211, 239)], let us define the auxiliary spectral problem

$$L(\lambda, y) \equiv P\left(\frac{d}{dx}\right)y - \lambda y = 0, \tag{6}$$

$$U_i(y) = 0 \quad (i = 1, 2) \tag{7}$$

and the Cauchy problem

$$M\left(\frac{d}{dt}\right)z - \lambda N\left(\frac{d}{dt}\right)z = 0, \tag{8}$$

$$\left.\frac{\partial^{k-1}z}{\partial t^{k-1}}\right|_{t=0} = \phi_k(x) \quad (k = 1, 2, \ldots, q), \quad 0 \leq x \leq 1, \tag{9}$$

where λ is a complex parameter.

Let $D(P)$ be the subset of $C^2([0,1])$ of functions satisfying the boundary conditions (3). Apperently $D(P)$ is the domain of the operator P generated by the differential form $P\left(\frac{d}{dx}\right)$ and the boundary conditions (3). Let $\phi(x) \in D^K(P)$ whenever $P^i\left(\frac{d}{dx}\right)\phi(x) \in D(P)$ $(i = 0, 1, \ldots, K)$ where

$$P^0\left(\frac{d}{dx}\right) = 1, \quad D^0(P) = D(P), \text{ and } P^i\left(\frac{d}{dx}\right) = P^{i-1}\left(\frac{d}{dx}\right)P\left(\frac{d}{dx}\right).$$

Assume that the following condition holds:

1°. The boundary conditions (3) are regular in Birkhoff sense [6 (see pp. 120-130)]. Then it is known that [7 (see p. 249 Lemma 1.)] the eigenvalues of the auxiliary spectral problem (6)-(7) form a countable set $\{\lambda_\nu\}$ and these eigenvalues have the asymptotic representation

$$\lambda_\nu = -(2\pi\nu)^2\left[1 + O\left(\frac{1}{\nu}\right)\right]. \tag{10}$$

And it is also known that for all values $\lambda \neq \lambda_\nu, (\nu = 1, 2, \ldots)$ of the parameter λ and for the arbitrary function $h(x) \in C([0,1])$ the nonhomogeneous problem

$$L(\lambda, y) = h(x), \; U_i(y) = 0 \; (i = 1, 2)$$

has a unique solution and this solution has the integral representation

$$y(x, \lambda, h) = \int_0^1 G(x, \xi, \lambda)h(\xi)d\xi, \tag{11}$$

where $G(x,\xi,\lambda)$ is the Green function of the spectral problem (6)-(7), and is analytic in λ outside the set $\{\lambda\}$. This set is the set of poles for the Green function $G(x,\xi,\lambda)$ and for large values of $|\lambda|$ and in some δ-neighborhood of the poles λ_ν, this function has the estimation

$$\left|\frac{\partial^k G(x,\xi,\lambda)}{\partial x^k}\right| \leq C|\lambda|^{\frac{k-1}{2}} \quad (k=0,1,2), \tag{12}$$

where $C>0$ is a constant.

It is also known that [5], [7 (see p.145 Thm. 10)] the function $h(x)\in D(P)$ has the expansion

$$h(x)=\sum_{\nu=1}^{\infty} res_{\lambda_\nu}\int_0^1 G(x,\xi,\lambda)h(\xi)d\xi. \tag{13}$$

2. Discussion of the Cauchy Problem

Let $F(\omega,\lambda)$ be the characteristic polynomial of the Eq.(8):

$$F(\omega,\lambda)=M(\omega)-\lambda N(\omega),$$

and $R(F,F_\omega)$ is the resultant of the polinomials $F(\omega,\lambda)$ and $F_\omega(\omega,\lambda)$. Simple manupulations show that $R(F,F_\omega)$ is a polynomial of λ of degree $q+1$ and the coefficients of the highest degree terms are

$$\begin{cases} N_2(N_1^2-4N_0N_2), & \text{for } K_0=0,1,2, \\ -M_3N_2^2(N_1^2-4N_0N_2), & \text{for } K_0=3, \\ 4M_4^2N_2^3(N_1^2-4N_0N_2), & \text{for } K_0=4. \end{cases}$$

Let us further assume that

$2^\circ.N_1^2-4N_0N_2\neq 0.$

Then the equation $R(F,F_\omega)=0$ has $q+1$ $(q=q(K_0))$ roots, counting the multiplicities, $\lambda^{(1)},\lambda^{(2)},\dots,\lambda^{(q+1)}$ where

$$q=\begin{cases} 2, & \text{for } 0\leq K_0\leq 1, \\ K_0, & \text{for } 2\leq K_0\leq 4. \end{cases}$$

In the neighborhoods of points $\lambda\notin\{\lambda^{(0)}=\infty,\lambda^{(1)},\lambda^{(2)},\dots,\lambda^{(q+1)}\}$ there exist q different analytic functions [4] $\omega_1(\lambda),\omega_2(\lambda),\dots,\omega_q(\lambda)$ such that

$$F(\omega,\lambda)=\begin{cases} -\lambda N_2\prod_{i=1}^2(\omega-\omega_i(\lambda)), & \text{for } K_0=0,1,. \\ (M_2-\lambda N_2)\prod_{i=1}^2(\omega-\omega_i(\lambda)), & \text{for } K_0=2, \\ M_{K_0}\prod_{i=1}^{K_0}(\omega-\omega_i(\lambda)), & \text{for } K_0=3,4. \end{cases}$$

Remark 1. For Eq. (4) one obtains $K_0 = 4$,

$$\begin{aligned} R(F, F_\omega) &= -16\varepsilon^3\lambda[4\varepsilon^2\lambda + (\lambda-1)^2]^2, \\ \lambda^{(1)} &= 0, \lambda^{(2)} = \lambda^{(3)} = 2\varepsilon^2\left(-1-\sqrt{\varepsilon^2-1}\right)+1, \end{aligned}$$

and

$$\lambda^{(4)} = \lambda^{(5)} = 2\varepsilon^2\left(-1+\sqrt{\varepsilon^2-1}\right)+1,$$

whereas for Eq. (5), $K_0 = 2$ and

$$R(F, F_\omega) = -4\lambda + (1-\lambda)^2, \lambda^{(1)} = 0, \lambda^{(2)} = \lambda^{(3)} = 1.$$

It is not difficult to see that for $3 \le K_0 \le 4$, the Cauchy problem (8)-(9) has a unique solution, and the solution is analytic in λ for $\lambda \neq \infty$ according to the Poincare theorem [3(see p. 20)]. This solution has the representation

$$z(t, \lambda, \phi_1, \dots, \phi_q) = \sum_{k=1}^{q(K_0)}\left[\sum_{j=1}^{q(K_0)} \frac{W_{jk}(\lambda)}{W(\lambda)}\phi_k(x)\right] e^{\omega_k(\lambda)t}, \tag{14}$$

where $W(\lambda)$ is the Wandermond determinant of the quantities $\omega_1(\lambda), \dots, \omega_q(\lambda)$, $W_{jk}(\lambda)$ is algebraic complement of its (j,k)' th element, and $\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(q+1)}$ are the removable singularites for the function $z(t, \lambda, \phi_1, \dots, \phi_{K_0})$ in (14).

It is also not difficult to show that for $K_0 = 4$, the point $\lambda^{(0)} = \infty$ is a critical singularity for the functions ω_i $(i = 1,2,3,4)$, whereas the functions $\omega_i(\mu^2)$ $(i = 1,2,3,4)$ are analytic in the neighborhood of the point $\mu = \infty$ and has the asymptotic representation

$$\omega_i(\mu^2) = \omega_i^{(1)}\mu + \omega_i^{(0)} + O\left(\frac{1}{\mu}\right) \quad (i = 1,2,3,4), \tag{15}$$

where

$$\begin{aligned} -\omega_1^{(1)} &= \omega_2^{(1)} = \sqrt{\frac{N_2}{M_4}}, \omega_1^{(0)} = \omega_2^{(0)} = \frac{N_1M_4 - N_2M_3}{2N_2N_4}, \\ \omega_3^{(1)} &= \omega_4^{(1)} = 0, \omega_{3,4}^{(0)} = \frac{1}{2N_2}\left(-N_1 \pm \sqrt{N_1^2 - 4N_0N_2}\right). \end{aligned}$$

Remark 2. For Eq. (4) one obtains

$$\omega_{1,2}(\mu^2) = \pm\frac{1}{\varepsilon}\mu + O\left(\frac{1}{\mu}\right),\ \omega_{3,4}(\mu^2) = \pm\sqrt{-1} + O\left(\frac{1}{\mu}\right).$$

If $K_0 = 3$, the functions $\omega_i(\mu^2)$ $(i = 1, 2, 3)$ are analytic in the neighborhood of the point $\mu = \infty$ and has the asymptotic representation

$$\omega_i(\lambda) = \omega_i^{(1)}\lambda + \omega_i^{(0)} + O\left(\frac{1}{\lambda}\right)\ (i = 1, 2, 3), \tag{16}$$

where

$$\omega_1^{(1)} = \frac{N_2}{M_3}, \omega_2^{(1)} = \omega_3^{(1)} = 0, \omega_1^{(0)} = \frac{N_1M_3 - N_2M_2}{N_2},$$
$$\omega_{2,3}^{(0)} = \frac{1}{2N_2}\left(-N_1 \pm \sqrt{N_1^2 - 4N_0N_2}\right).$$

Let us consider the case for $0 \le K_0 \le 2$. Away from the points $\lambda = \infty$ and $\lambda = \frac{M_2}{N_2}$ (we take $M_2 = 0$ for $K_0 < 2$), the Cauchy problem (8)-(9) has a unique solution. This solution is analytic in λ and has the representation (14) with $q(K_0) = 2$. Furthermore the points $\lambda^{(i)} = 1, 2, \ldots, q+1)$ other than $\lambda = M_2/N_2$ are the removable singularities for the function $z(t, \lambda, \phi_1, \ldots, \phi_{K_0})$ in (14).

By standard reasoning it possible to deduce that [4] if $M_1N_2 - N_1M_2 \neq 0$, the functions $\omega_i(\lambda)(i = 1, 2)$ are analytic in the neighborhood of the point $\lambda = M_2/N_2$ and this point is a simple pole for the first and a removable singularity for the second one. For $\lambda \to M_2/N_2$ these functions have the asymptotic representation

$$\omega_i(\lambda) = \frac{\omega_i^{(1)}}{M_2 - \lambda N_2} + O(1) \quad (i = 1, 2),$$

where

$$\omega_1^{(1)} = \frac{N_{(1)}M_2 - M_1N_2}{N_2} \quad \omega_2^{(1)} = 0.$$

If $M_1N_2 - N_1M_2 = 0$, which is possible when $K_0 = 2$, impossible when $K_0 = 1$, and is the case when $K_0 = 0$, then the point $\lambda = M_2/N_2$ is the branch point for the functions $\omega_i(\lambda)$ $(i = 1, 2)$ but the functions $\omega_i\left(\frac{M_2 - z^2}{N_2}\right)$ $(i = 1, 2)$ are analytic in the neighborhood of the point $z = 0$ and for $z \to 0$, has the asymptotic representation

$$\omega_i\left(\frac{M_2 - z^2}{N_2}\right) = \frac{\omega_i^{(1)}}{z} + O(1) \quad (i = 1, 2),$$

where

$$\omega_i^{(1)} = (-1)^i \sqrt{\frac{N_2 M_0 - M_2 N_0}{N_2}}, (i = 1, 2).$$

Finally the functions $\omega_i(\lambda)$ $(i = 1, 2)$ are analytic in the neighborhood of the point $\lambda = \infty$ and for $\lambda \to \infty$ they have the asymptotic representation

$$\omega_i(\lambda) = \omega_i^{(0)} + O\left(\frac{1}{\lambda}\right) \quad (i = 1, 2), \tag{17}$$

where

$$\omega_{1,2}^{(0)} = \frac{1}{2N_2}\left(-N_1 \pm \sqrt{N_1^2 - 4N_0 N_2}\right).$$

Hence the point $\lambda = \infty$ is a removable singularity for $\omega_i(\lambda)$ $(i = 1, 2)$

Remark 3. For Eq. (5) the point $\lambda = 1$ is the branceh point for the functions $\omega_i(\lambda)$ $(i = 1, 2)$ but the functions $\omega_i(1 - z^2)$ $i = 1, 2)$ are analytic in the neighborhood of the point $z = 0$ and for $z \to 0$. has the asymptotic representation

$$\omega_i(1 - z^2) = \frac{\omega_i^{(1)}}{z} + O(1), \ \omega_i^{(1)} = (-1)^i \sqrt{-1} \ (i = 1, 2).$$

However the point $\lambda = \infty$ is a removable singular point for the functions $\omega_i(\lambda)$ $(i = 1, 2)$ and for $z \to \infty$, and they have the asymptotic represntation

$$\omega_i(\lambda) = \omega_i^{(0)} + O\left(\frac{1}{\lambda}\right), \ \omega_i^{(0)} = (-1)^i \sqrt{-1}, \ (i = 1, 2).$$

All of the reasonings above reveal that, in all cases $(0 \leq K_0 \leq 4))$ the solution $z(t, \lambda)$ of the auxiliary Cauchy problem (8)-(9) has a finite number of singular points $\lambda_\infty^{(0)}, \lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(q+1)}$ and away of these points the solution can be represented by the formula (14). Now let us prove that the points $\lambda^{(i)}, (i = 0, 1, 2, \ldots, q + 1)$ are all singular points of univalent character for the solution $z(t, \lambda)$. The above assertion is evident for the points except the point $\lambda = \infty$ when $K_0 = 4$, the point $\lambda = \frac{M_2}{N_2}$ when $K_0 = 2$ and $N_1 M_2 - M_1 N_2 = 0$, and the point $\lambda = 0$ when $K_0 = 0$. Only these points are brach type (more precisely algebraic) singular points, and all others are singularities of univalent character for the functions $\omega_i(\lambda)$. According to the cyclic property of the roots of the algebraic equations [1], the value of the function $\omega_k(\lambda)$, completing one revolution on a closed path surrounding a brach point, takes on the value of another root $\omega_{s_k}(\lambda)$ $(1 \leq s_k \leq q)$. Hence rewriting the solution $z(t, \lambda)$ in (14) as

$$z(t, \lambda) = \left(e^{\omega_1(\lambda)t} \ldots, e^{\omega_q(\lambda)t}\right) \cdot V^{-1}(\lambda)[\phi_1(x), \ldots, \phi_q(x)]^T$$

where

$$V(\lambda)=\left|\left|\omega_k^{(j-1)}(\lambda)\right|\right|_{k,j=1}^{q}$$

is the Wandermond matrix and $[\ ]^T$ is for the transpose, after a complete rotation around a brach point, the matrix $V(\lambda)$ transforms into a new matrix $V(\lambda)\cdot J$. Here J is a matrix whose (s_k, k) $(k=1,2,\ldots,q)$ elements are one, and all other elements are zero. Since

$$\left(e^{\omega_{s_1}(\lambda)t},\ldots,e^{\omega_{s_q}(\lambda)t}\right)\cdot J^{-1}V^{-1}(\lambda)=\left(e^{\omega_1(\lambda)t},\ldots,e^{\omega_q(\lambda)t}\right)V^{-1}(\lambda),$$

then in the domain of definition, the function $z(t,\lambda)$ is a univalent analytic function of λ.

Le us summarize the results of this section.

Lemma 1. *Let the condition 2° holds and $3\le K_0\le 4$. Then for all values of the spectral parameter λ, except $\lambda=\infty$, the auxiliary Cauchy problem (8)-(9) has a uniqe solution $z(t,\lambda)$. This solution is analytic every where except at the finite number of roots $\lambda^{(i)}$ $(i=0,1,2,\ldots,q)$ of the resultant $R(F,F_\omega)$, and has the integral representation (14). The points $\lambda^{(0)}=\infty$, and $\lambda^{(i)}$ $(i=1,2,\ldots,q)$ are isolated singularities of univalent character of the solution $z(t,\lambda)$. While at the points $\lambda^{(i)}(i=1,2,\ldots,q)$, the solution has removable singularities, the point $\lambda^{(0)}=\infty$ is an essential singular point.*

Lemma 2. *Let the condition 2° holds and $0\le K_0\le 2$. Then for all values of the spectral parameter λ, except $\lambda=\infty$, and $\lambda=\frac{M_2}{N_2}$, ($M_2=0$ when $K_0<2$), the auxiliary Cauchy problem (8)-(9) has a uniqe solution. This solution is analytic everywhere except at the finite number of roots $\lambda^{(i)}$ $(i=0,1,2,\ldots,q)$ of the resultant $R(F,F_\omega$, and has the integral representation (14). The points $\lambda^{(0)}=\infty$, and $\lambda^{(i)}$ $(i=1,2,\ldots,q)$ are isolated singularities of univalent character of the solution $z(t,\lambda)$. Furthermore, all of the singular points, except the one at $\lambda=\frac{M_2}{N_2}\in\{\lambda^{(i)}\}$ are removable singularities. If $M_1N_2-N_1M_2\neq 0$ then the point $\lambda=\frac{M_2}{N_2}$ is an essential singular point. On the other hand if $M_1N_2-N_1M_2=0$ then the point $\lambda=\frac{M_2}{N_2}$ is an essential singular point if $M_0N_2-N_0M_2\neq 0$, and a removable singularity if $M_0N_2-N_0M_2=0$ for the solution $z(t,\lambda)$.*

Remark 4. As it is seen from Lemma 2, the solution $z(t,\lambda)$ of the auxiliary Cauchy problem associated to Eq. (4) is analytic everywhere except points $\lambda^{(0)},\ldots,\lambda^{(5)}$ and has the representation given in Eq. (14). At the points,

$$\lambda^{(1)}=0,\ \lambda^{(2)}=\lambda^{(3)}=1-2\varepsilon^2\left(1+\sqrt{\varepsilon^2-1}\right),$$
$$\lambda^{(4)}=\lambda^{(5)}=1-2\varepsilon^2\left(1-\sqrt{\varepsilon^2-1}\right)$$

the solution $z(t,\lambda)$ has removable singularities, while at the point $\lambda^{(0)}=\infty$ it has an essential singularity.

Remark 5. As it is seen in Lemma 2, the solution $z(t,\lambda)$ of the auxiliary Cauchy problem associated to Eq. (5) is analytic everywhere except points $\lambda^{(0)},\lambda^{(1)},\lambda^{(2)}$ and has the representation given in Eq. (14). At the points, $\lambda^{(0)}=\infty$ and $\lambda^{(1)}=0$ the solution $z(t,\lambda)$ has removable singularities, while at the point $\lambda^{(2)}=1$ it has an essential singularity.

Finally from the explicit equations for $W(\lambda)$ and $W_{jk}(\lambda)$ and expansions (15)-(17) let us state the estimates for large $|\lambda|$ of the ratio $W_{jk}(\lambda)/W(\lambda)$ in the formula (14): for $K_0=4$.

$$\begin{aligned}&\left|\frac{W_{1k}(\lambda)}{W(\lambda)}\right|\le C;\left|\frac{W_{jk}(\lambda)}{W(\lambda)}\right|\le\frac{C}{\sqrt{|\lambda|}}\ (j=2,3);\\&\left|\frac{W_{4k}(\lambda)}{W(\lambda)}\right|\le\frac{C}{|\lambda|}\ (k=3,4);\ \left|\frac{W_{jk}(\lambda)}{W(\lambda)}\right|\le\frac{C}{|\lambda|}\quad (j=1,2,3);\\&\left|\frac{W_{4k}(\lambda)}{W(\lambda)}\right|\le\frac{C}{\sqrt{|\lambda|^3}}\ (k=1,2),\end{aligned}\tag{18}$$

for $K_0=3$,

$$\begin{aligned}\left|\frac{W_{j1}(\lambda)}{W(\lambda)}\right|\le\frac{C}{|\lambda|^2}\ (j=1,2,3),\ \left|\frac{W_{jk}(\lambda)}{W(\lambda)}\right|\le C\ (j=1,2;\ k=2,3);\\ \left|\frac{W_{3k}(\lambda)}{W(\lambda)}\right|\le\frac{C}{|\lambda|}\ (k=2,3),\end{aligned}\tag{19}$$

and finally for $0\le K_0\le 2$,

$$\left|\frac{W_{jk}(\lambda)}{W(\lambda)}\right|\le C\ (j,k=1,2).\tag{20}$$

3. The Method of Residue

In this section, along the general scheme given in [7 (see chapter 5.)], first a formal solution to the problem (1)-(3) is constructed and the uniqueness theorem is proved, then in the basis of the investigation of the appropriate formal series for the derivatives of the solution we are seeking for, the existence of the clasical solution is proved.

Let us consider the following linear operators from C([0,1]) to $D(P)$;

$$\mathcal{H}_{vs}(h)\equiv h_{vs}(x)\equiv$$

$$\equiv res_{\lambda_\nu}\lambda^s \int_0^1 G(x,\xi,\lambda)h(\xi)d\xi \ (s=0,1,2,\dots;\nu=1,2\dots).$$

According to the expansion formula (13), if $h(x)\in D(P)$ then

$$\sum_{\nu=1}^{\infty} h_{\nu 0}(x) = h(x). \tag{21}$$

Let χ_ν be the order of the pole λ_ν of the Green function $G(x,\xi,\lambda)$. Let us assume that $\phi_k(x)\in C$ ([0,1]) $(k=1,2,\dots,q)$, and the problem (1)-(3) has the classical solution $u(x,t)$. applying operators $\mathcal{H}_{\nu s}$ $(s=0,1,\dots,\chi_\nu-1)$ on the both sides of the identities

$$M\left(\frac{\partial}{\partial t}\right)u(\xi,t)\equiv N\left(\frac{\partial}{\partial t}\right)P\left(\frac{\partial}{\partial \xi}\right)u(\xi,t),$$
$$\left.\frac{\partial^{k-1}u(\xi,t)}{\partial t^{k-1}}\right|_{t=0}\equiv \phi_k(\xi)\ (k=1,2,\dots,q)$$

one obtains the identities

$$M\left(\frac{d}{dt}\right)u_{\nu s}(x,t)\equiv N\left(\frac{d}{dt}\right)res_{\lambda_\nu}\lambda^s\int_0^1 G(x,\xi,\lambda)P\left(\frac{d}{d\xi}\right)u(\xi,t)d\xi\equiv$$
$$\equiv N\left(\frac{d}{dt}\right)res_{\lambda_\nu}\lambda^s\int_0^1 G(x,\xi,\lambda)\left[P\left(\frac{d}{d\xi}\right)u(\xi,t)-\lambda u(\xi,t)+\lambda u(\xi,t)\right]d\xi\equiv$$
$$\equiv N\left(\frac{d}{dt}\right)res_{\lambda_\nu}\lambda^s\left[u(x,t)+\lambda\int_0^1 G(x,\xi,\lambda)u(\xi,t)d\xi\right]\equiv$$
$$\equiv N\left(\frac{d}{dt}\right)u_{\nu\, s+1}(x,t)\quad (s=0,1,\dots,\chi_\nu-1), \tag{22}$$
$$\left.\frac{d^{k-1}u_{\nu\, s}(x,t)}{dt^{k-1}}\right|_{t=0}\equiv \phi_{k,\nu s}(x)\quad (k=1,2,\dots,q). \tag{23}$$

It is not difficult to see that

$$res_{\lambda\nu}(\lambda-\lambda_\nu)^{\chi_\nu}\int_0^1 G(x,\xi,\lambda)u(\xi,t)d\xi\equiv 0.$$

Hence

$$\sum_{j=0}^{\chi_\nu} C^j_{\chi_\nu}(-\lambda\nu)^{\chi_\nu-j}u_{\nu j}(x,t)\equiv 0, \tag{24}$$

where $C^j_{\chi_\nu} = \frac{\chi_\nu}{j(\chi_\nu - j)}$ From Eq. (24) one obtains

$$U_{\nu\chi_\nu}(x,t) \equiv -\sum_{j=0}^{\chi_\nu - 1} C^j_{\chi_\nu}(-\lambda_\nu)^{\chi_\nu - j} u_{\nu j}(x,t).$$

Substituting this result in the right hand side of (22) for $s = \chi_\nu - 1$ and regarding the obtained equations together with equations (23), as the Cauchy problem for the system of linear differential equation for the unknown functions $u_{\nu 0}(x,t), u_{\nu 1}(x,t), \ldots, u_{\nu\chi_\nu - 1}(x,t)$ one obtains

$$M\left(\frac{d}{dt}\right) u_{\nu s}(x,t) - N\left(\frac{d}{dt}\right) u_{\nu\, s+1}(x,t) = 0 \quad (s = 0,1,\ldots \chi_\nu - 2),$$

$$M\left(\frac{d}{dt}\right) u_{\nu\chi_\nu - 1}(x,t) + \sum_{j=0}^{\chi_\nu - 1} C^j_{\chi_\nu}(-\lambda_\nu)^{\chi_\nu - j} N\left(\frac{d}{dt}\right) u_{\nu j}(x,t) = 0 \tag{25}$$

$$\left.\frac{d^{k-1} u_{\nu s}(x,t)}{dt^{k-1}}\right|_{t=0} = \phi_{k,\nu s}(x) \quad (k = 1,2,\ldots,q;\ s = 0,1,\ldots,\chi_\nu - 1). \tag{26}$$

It is not difficult to see that the solution of the above Cauchy problem is unique for $3 \le K_0 \le 4$. When $0 \le K_0 \le 2$, for arbitrary functions $\phi_{k,\nu s}(x)$, the necessary and sufficient condition for th existence and uniqueness of solution is that the number M_2/N_2 $(M_2 = 0$ when $K_0 < 2)$ is not an eigenvalue of the matrix

$$\begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -C^0_{\chi_\nu}(-\lambda_\nu)^{\chi_\nu} & -C^1_{\chi_\nu}(-\lambda_\nu)^{\chi_\nu - 1} & -C^2_{\chi_\nu}(-\lambda_\nu)^{\chi_\nu - 2} & \ldots & -C^{\chi_\nu - 1}_{\chi_\nu}(-\lambda_\nu) \end{pmatrix}$$

It can be shown that this latter condition is equivalent to $M_2/N_2 \notin \{\lambda_\nu\}$.

For the above reasons if $3 \le K_0 \le 4$ or if $0 \le K_0 \le 2$ and $M_2/N_2 \notin \{\lambda_\nu\}$, the problem (25), (26) has a unique solution. We are going to prove that this solution can be given by the formula

$$u_{\nu\, s}(x,t) = res_{\lambda_\nu} \lambda^s \int_0^1 G(x,\xi,\lambda) z(t,\lambda,\phi_1,\ldots \phi_q) d\xi, \tag{27}$$

where the function $z(t,\lambda,\phi_q)$ is the solution of the auxiliary Cauchy problem (8),(9) given by the formula (14).

By direct calculation one obtains

$$M\left(\frac{d}{dt}\right)u_{\nu s}(x,t)-N\left(\frac{d}{dt}\right)u_{s\,s+1}(x,t)\equiv res_{\lambda_\nu}\lambda^s\int_0^1 G(x,\xi,\lambda)\times$$
$$\times\left[M\left(\frac{d}{dt}\right)z(t,\lambda,\phi_1,\dots,\phi_q)-\lambda N\left(\frac{d}{dt}\right)z(t,\lambda,\phi_1,\dots,\phi_q)\right]d\xi\equiv 0$$
$$(s=0,1,\dots,\chi_\nu-2)$$
$$M\left(\frac{d}{dt}\right)u_{\nu\chi_{\nu-1}}(x,t)+\sum_{j=0}^{\chi_\nu-1}C^j_{\chi_\nu}(-\lambda_\nu)^{\chi_\nu-j}N\left(\frac{d}{dt}\right)u_{\nu j}(x,t)\equiv$$
$$\equiv res_{\lambda_\nu}\int_0^1 G(x,\xi,\lambda)\left[\lambda^{\chi_\nu-1}M\left(\frac{d}{dt}\right)z(t,\lambda,\phi_1,\dots,\phi_q)+\right]$$
$$\left[+\sum_{j=0}^{\chi_\nu-1}C^j_{\chi_\nu}(-\lambda_\nu)^{\chi_\nu-j}\lambda^jN\left(\frac{d}{dt}\right)z(t,\lambda,\phi_1,\dots,\phi_q)\right]d\xi\equiv$$
$$\equiv N\left(\frac{d}{dt}\right)res_{\lambda_\nu}\int_0^1 G(x,\xi,\lambda)(\lambda-\lambda_\nu)^{\chi_\nu}z(t,\lambda,\phi_1,\dots,\phi_q)d\xi\equiv 0.$$

According to the Lemma 1 and Lemma 2, the singular points of the functions $z(t,\lambda,\phi_1,\dots,\phi_q)$ are all removable when $3\le K_0\le 4$, when $0\le K_0\le 2$ all of the singularities with the exception $\lambda=M_2/N_2$ are removable and the essential singular point $\lambda=M_2/N_2$ is not in $\{\lambda_\nu\}$. Hence

$$\left.\frac{d^{k-1}u_{\nu\,s}(x,t)}{dt^{k-1}}\right|_{t=0}\equiv res_{\lambda_\nu}\lambda^s\int_0^1 G(x,\xi,\lambda)\times$$
$$\times\left.\frac{d^{k-1}}{dt^{k-1}}z(t,\lambda,\phi_1,\dots,\phi_q)\right|_{t=0}d\xi=\phi_{k,\nu s}\quad(k=1,\dots,q).$$

By assumption $u(x,t)$ is a classical solution, hece for all $t, u(x,t)\in D(P)$ in x. Therefore from equations (21), (27) one obtains

$$u(x,t)=\sum_{\nu=1}^{\infty}res_{\lambda_\nu}\int_0^1 G(x,\xi,\lambda)z(t,\lambda,\phi_1,\dots,\phi_q)d\xi. \tag{28}$$

Hence we proved the following theorem:

Theorem 1. Let $\phi_k(x)\in C$ ([0,1]) $(k=1,2,\dots,q)$, and the conditions 1° and 2° hold and for $0\le K_0\le 2$ the point $M_2/N_2\notin\{\lambda_\nu\}$. Then if the problem (1)-(3) has a classical solution, this solution is defined by the formula (28).

Corollary 1. *Let the hyphoteses of Theorem 1. be satisfied, then the problem (1)-(3) has at most one solution.*

Using the asymptotic representations (10), (15)-(17), estimations (18)-(20), and the uniform convergence of the series (28), its first two derivatives in x, and first $q = \max(2, K_0)$ derivatives in t, it is also possible to prove the following theorem about the existence and uniqueness of the classical solution of the problem (1)-(3):

Theorem 2. Let the conditions 1° and 2° hold and

$$\begin{cases} \frac{N_2}{M_4} > 0, \phi_1(x) \in D^3(P), \phi_k(x) \in D^2(P) \ (k = 2,3,4) \text{ for } K_0 = 4, \\ \mathcal{R}\frac{N_2}{M_3} \geq 0, \phi_k(x) \in D^3(P) \ (k = 1,2,3) \text{ for } K_0 = 3, \\ M_2/N_2 \notin \{\lambda_\nu\}, \ \phi_k(x) \in D^2(P) \ (k = 1,2) \text{ for } 0 \leq K_0 \leq 2 \end{cases} \tag{29}$$

Then the problem (1)-(3) has a unique classical solution given by (28).

Proof. We are going to give the proof for $0 \leq K_0 \leq 2$ (in this case $q = 2$). Applying the operator

$$M\left(\frac{\partial}{\partial t}\right) - N\left(\frac{\partial}{\partial t}\right) P\left(\frac{\partial}{\partial x}\right) \tag{30}$$

to the both sides of (28) and interchanging the operator by the summation on the right hand side formally one obtains

$$\begin{aligned} & M\left(\frac{\partial}{\partial t}\right) u(x,t) - N\left(\frac{\partial}{\partial t}\right) P\left(\frac{\partial}{\partial x}\right) u(x,t) = \\ & = \sum_{\nu=1}^{\infty} res_{\lambda_\nu} \int_0^1 G(x,\xi,\lambda) M\left(\frac{\partial}{\partial t}\right) z(t,\lambda,\phi_1,\phi_2) d\xi \\ & -P\left(\frac{\partial}{\partial x}\right) \int_0^1 G(x,\xi,\lambda) N\left(\frac{\partial}{\partial t}\right) z(t,\lambda,\phi_1,\phi_2) d\xi. \end{aligned} \tag{31}$$

Using the property

$$\begin{aligned} & P\left(\frac{\partial}{\partial x}\right) \int_0^1 G(x,\xi,\lambda) N\left(\frac{\partial}{\partial t}\right) z(t,\lambda,\phi_1,\phi_2) d\xi \\ & = \lambda \int_0^1 G(x,\xi,\lambda) N\left(\frac{\partial}{\partial t}\right) z(t,\lambda,\phi_1,\phi_2) d\xi \end{aligned}$$

of the Green's function, and the relation (8) we can write equation (31) as

$$M\left(\frac{\partial}{\partial t}\right)u(x,t) - N\left(\frac{\partial}{\partial t}\right)P\left(\frac{\partial}{\partial x}\right)u(x,t) =$$
$$= \sum_{\nu=1}^{\infty} res_{\lambda_\nu} \int_0^1 G(x,\xi,\lambda)\left[M\left(\frac{\partial}{\partial t}\right)z - \lambda N\left(\frac{\partial}{\partial t}\right)z\right]d\xi = 0. \tag{32}$$

Applying the operators

$$U_i(i=1,2), \quad \left.\frac{\partial^k}{\partial t^k}\right|_{t=0} \quad (k=0,1) \tag{33}$$

formally to the both sides of (28) and interchanging the operator by the summation on the right hand side one also obtains

$$U_i(u) = \sum_{\nu=1}^{\infty} res_{\lambda_\nu} \int_0^1 U_i(G)z(t,\lambda,\phi_1,\ldots,\phi_q)d\xi,$$
$$\left.\frac{\partial^k u}{\partial t^k}\right|_{t=0} = \sum_{\nu=1}^{\infty}\int_0^1 G(x,\xi,\lambda)\left.\frac{\partial^k z}{\partial t^k}\right|_{t=0} d\xi =$$
$$= \sum_{\nu=1}^{\infty} res_{\lambda_\nu} \int_0^1\int_0^1 G(x,\xi,\lambda)\phi_k(\xi)d\xi = \phi_k(x)$$

by (13).

Hence to prove that the function given by (28) is a solution of the problem (1)-(3), one needs to prove that the operators (30) and (33) can be interchanged by the summation. For this, the uniform convergence of the series

$$\sum_{\nu=1}^{\infty} res_{\lambda_\nu} \frac{\partial^i u}{\partial x^i}\int_0^1 G(x,\xi,\lambda)\frac{\partial^i u}{\partial t^j}z(t,\lambda,\phi_1,\phi_2)d\xi, \; (0 \le i \le 2,\, 0 \le j \le 2) \tag{34}$$

is sufficient, and according to the asymptotic representations in (10), the uniform convergence of these series is equivalent to the uniform convergence of the sequences

$$\int_{\Lambda_\nu} d\lambda \frac{\partial^i u}{\partial x^i}\int_0^1 G(x,\xi,\lambda)\frac{\partial^j u}{\partial t^j}z(t,\lambda,\phi_1(\xi),\phi_2(\xi))d\xi, \; (0 \le i \le 2, \; 0 \le j \le 2) \tag{35}$$

for $\nu \to \infty$. Since $\phi_k(x) \in D^2(P)$, according to the formulae (5), (11) and (14) one has

$$\int_0^1 G(x,\xi,\lambda)\left[P\left(\frac{\partial}{\partial\xi}\right)\frac{\partial^j}{\partial t^j}z-\frac{\partial^j z}{\partial t^j}\right]d\xi=\frac{\partial^j z(t,\lambda,\phi_1(x),\phi_2(x))}{\partial t^j}. \tag{36}$$

Hence

$$\begin{aligned}
&\int_0^1 G(x,\xi,\lambda)\frac{\partial^j z(t,\lambda,\phi_1(\xi),\phi_2(\xi))}{\partial t^j}d\xi\\
&=-\frac{1}{\lambda}\frac{\partial_j z(t,\lambda,\phi_1(x),\phi_2(x))}{\partial t^j}+\frac{1}{\lambda}\int_0^1 G(x,\xi,\lambda)P\left(\frac{\partial}{\partial\xi}\right)\frac{\partial^j z}{\partial t^j}d\xi=\\
&=-\frac{1}{\lambda}\frac{\partial_j z(t,\lambda,\phi_1(x),\phi_2(x))}{\partial t^j}-\frac{1}{\lambda^2}P\left(\frac{\partial}{\partial x}\right)\frac{\partial^j z}{\partial t^j}d\xi+\\
&+\frac{1}{\lambda^2}\int_0^1 G(x,\xi,\lambda)P^2\left(\frac{\partial}{\partial\xi}\right)\frac{\partial^j z}{\partial t^j}d\xi.
\end{aligned}\tag{37}$$

Therefore, to prove the uniform convergences of the sequences (35), it is sufficient to prove the uniform convergences of the sequences

$$\int_{\Lambda_\nu}\frac{d\lambda}{\lambda^2}\frac{\partial^i}{\partial x^i}\int_0^1 G(x,\xi,\lambda)P^2\left(\frac{\partial}{\partial\xi}\right)\frac{\partial^j z}{\partial t^j}d\xi,\ (0\le i\le 2,\ 0\le j\le 2), \tag{38}$$

where

$$\Lambda_\nu=\{\lambda|\ |\lambda|=\tau_\nu=(|\lambda_\nu|+|\lambda_{\nu+},1)/2.$$

According to the inequalities (12), (20) and formulae (14), (17) one has

$$\left|\frac{\partial^i}{\partial x^i}\int_0^1 G(x,\xi,\lambda)P^2\left(\frac{\partial}{\partial\xi}\right)\frac{\partial^j z}{\partial t^j}\right|\le C|\lambda|^{\frac{i-1}{2}}\le C|\lambda|^{\frac{1}{2}}. \tag{39}$$

Hence from (38) one obtains

$$\left|\int_{\Lambda_\nu}\frac{d\lambda}{\lambda^2}\frac{\partial^i}{\partial x^i}\int_0^1 G(x,\xi,\lambda)P^2\left(\frac{\partial}{\partial\xi}\right)\frac{\partial^j z}{\partial t^j}d\xi\right|\le C\int_{\Lambda_\nu}\frac{|d\lambda|}{|\lambda|^{3/2}}. \tag{40}$$

Since

$$|\lambda|=\tau_\nu,\ |d\lambda|=|\tau_\nu e^{i\phi}d\phi|=\tau_\nu d\phi\ (0\le\phi\le 2\pi)\text{ on }\Lambda_\nu$$

we have

$$\int_{\Lambda_\nu}\frac{|d\lambda|}{|\lambda|^{3/2}}=\int_0^{2\pi}\frac{\tau_\nu}{\tau_\nu^{3/2}}d\phi=\frac{1}{\tau_\nu^{1/2}}\cdot 2\pi\to 0(\text{ as }\nu\to\infty).$$

Hence the sequences (38) are uniform convergent.
Since

$$\begin{cases} K_0 = 4, \frac{N_2}{M_4} = \frac{1}{\varepsilon^2} > 0, \ N_1^2 - 4N_0N_2 = -4 \neq 0; \ \text{for Eq. (4)}, \\ K_0 = 2, \frac{M_2}{N_2} = 1, \ N_1^2 - 4N_0N_2 = -4 \neq 0; \ \text{for Eq. (5)}, \end{cases}$$

then for these equations we can formulate a final theorem as follows: □

Theorem 3. Let the condition 1° holds and

$$\phi_1(x) \in D^3(P), \ \phi_k(x) \in D^2(P) \ (k = 2, 3, 4),$$

then the mixed problem (4), (2), (3) has a unique classical solution. If

$$\lambda_\nu \neq 1 \ (\nu = 1, 2, \ldots), \ \phi_k(x) \in D^2(P) \ (k = 1, 2,),$$

then the mixed problem (5), (2), (3) has a unique classical solution. In both cases this solution is given by (28).

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REDİDÜ YÖNTEMİNİN BİR KARIŞIK PROBLEME UYGULANIŞI

Özet

Bu makalede değişkenlere ayırma rezidü yöntemi, denklemin katsayılarının özel değerleri için sıkışabilir tabakalı sıvının akışını ve sıkışmaz tabakalı sıvının iç gravite dalgalarını da modelleyen bir karışık probleme uygulanmaktadır. Yöntemin uygulanması için önce iki tane yardımcı problem tanımlanıyor. Bunlardan birincisi bir spektral problem, ikincisi de bir Cauchy problemidir. Yardımcı problemlerin çözümleri kullanılarak daha sonra karışık problem için bir varlık ve teklik teoremi ispat edilmiş, ayrıca tek olan çözüm için açık bir formül verilmiştir.

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