Tr. J. of Mathematics
20 (1996) , 379 – 387.

SUBORBITAL GRAPHS FOR THE NORMALIZER OF $\Gamma_o(N)$

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Abstract

In this paper we examine some properties of suborbital graphs for the normalizer $\mathcal{N}$ of $\Gamma_0(N)$ in PSL(2,R) and show that, if $\mathcal{N}/\Gamma_0(N)$ and the set of orbit representatives are denoted by B and Ω respectively, the permutation group (B, Ω) is regular and m-regular where m is an odd natural number.

Introduction

Let PSL $(2,\mathbb{R})$ denote the group of all linear fractional transformations

T: z$\rightarrow$ $(az + b)/(cz + d)$, where, a,b,c,d are real and ad-bc=1. This is the automorphism group of the upper half plane $\mathcal{U} = \{z \in \mathbb{C} | \text{Im} z > 0\}$.
Γ, the modular group, is the subgroup of $PSL(2,\mathbb{R})$ such that a,b,c and d are rational integers. $\Gamma_0(N)$ is the subgroup of Γ with $N|c$. As a matrix representation the elements of PSL(2,R) are the pairs of matricess

$$\pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} (a, b, c, d \in \mathbb{R}, ad - bc = 1) \tag{1.1}$$

We will omit the symbol $\pm$, and identify each matrix with its negative.

Let $\mathcal{N}$ denote the normalizer of $\Gamma_0(N)$ in PSL(2,$\mathbb{R}$). The normalizer is studied by Lehner and Newman [7] in connection with the Weierstrass points of $\Gamma_0(N)$. Lehner and Newman calculated the normalizer directly. In [4] Conway and Norton gave a more elegant description derived from [7] in connection with the Monster Simple group. The normalizer consists exactly of the matrices

$$\begin{pmatrix} ae & b/h \\ cN/h & de \end{pmatrix} \tag{1.2}$$

where $e \parallel N/h^2$ and h is the largest divisor of 24 for which $h^2|N$ with the understandings that the determinant of the matrix is $e > 0$, and that $r \parallel s$ means that $r \parallel s$ and

*AMS Subject Classification 46A40

$(r, s/r) = 1$ (r is called an exact divisor of s). From now on, unless otherwise stated explicitely, N will denote a square-free integer which means that every divisor of N is exact. In this case it is seen that h=1.

2. The Action of N on $\hat{\mathbb{Q}}$

Every element of $\hat{\mathbb{Q}}$ can be represented as a reduced fraction x/y, with $x, y \in \mathbb{Z}$ and $(x, y) = 1$. Since $x/y = -x/-y$ this representation is not unique. We represent ∞ as 1/0=-1/0. As in §1 the action of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ on x/y is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : x/y \rightarrow \frac{ax+by}{cx+dy}.$$

It is easily seen that if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and if $x/y \in \hat{\mathbb{Q}}$ is a reduced fraction then, since $c(ax+by) - a(cx+dy) = -y$ and $d(ax+by) - b(cx+dy) = x$,

$$(ax+by,\ cx+dy) = 1 \tag{2.1}$$

The action of a matrix on x/y and on $-x/-y$ is identical.

Lemma 2.1 *(i) The action of the normalizer $\mathcal{N}$ on $\hat{Q}$ is transitive.*
(ii) The stabilizer of a point is an infinite cyclic group

Proof. Before we prove this let us give the following theorem from [2]. □

Theorem 2.1 *Let N be any integer and $N = 2^{\alpha_1} \cdot 3^{\alpha_2} \cdot p_3^{\alpha_3} \dots p_n^{\alpha_n}$, the prime power decomposition of N. Then $\mathcal{N}$ is transitive on $\hat{\mathbb{Q}}$ if and only if $\alpha_1 \le 7, \alpha_2 \le 3$ and $\alpha_i \le 1$, where $i = 3, \dots, r$.*

The proof of the Lemma 2.1 (i) Since N is square-free, the $\alpha_i \le 1,\ i = 1, 1, \dots, r$. so we conclude that the action is transitive.
(ii) Since the action is transitive, the stabilizer of any two points in $\hat{\mathbb{Q}}$ are conjugate in $\mathcal{N}$. So it is sufficient to consider the stabilizer $\mathcal{N}_\infty$ of ∞. This consists of the elements of the form

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \text{ with } b \in \mathbb{Z}.$$

So $\mathcal{N}_\infty$ is the infinite cyclic group generated by the element $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

We now consider the imprimitivity of the action of $\mathcal{N}$ on $\hat{\mathbb{Q}}$. This will be a special case of the following:

Let (G,Ω) be a transitive permutation group, consisting of a group G acting on a set Ω transitively. An equivalence relation $\approx$ on Ω is called G-invariant if, whenever $\alpha,\beta \in \Omega$ satisfy $\alpha \approx \beta$ then $g(\alpha) \approx g(\beta)$ for all $g\ inG$. The equivalence classes are called blocks. We cal (G,Ω) imprimitive if Ω admits some G-invariant equivalence relation different from

(i) the identity relation, $\alpha \approx \beta$ if and only if $\alpha = \beta$;

(ii) the universal relation, $\alpha \approx \beta$ for all $\alpha, \beta \in \Omega$.

Othervise (G,Ω) is called primitive.

We give the above notion in a different way as follows.

The set Δ of Ω is called a set imprimitivity of (G,Ω) if for every $g \in G$ either $g(\Delta) = \Delta$ or $g(\Delta) \cup \Delta = \varnothing$.

Therefore the empty set, the one point subsets and Ω itself are sets of imprimitivity, called the trivial sets of imprimitivity. If (G,Ω) has a non-trivial set of imprimitivity, the (G,Ω) is called imprimitive, otherwise primitive.

In fact the above defined blocks are sets of imprimitivity. Conversly if $\{\Delta_i\}_{i\in I}$, where I is an indexing set, denote the defferent elements of the set $\{g(\Delta) | g \in G\}$, where Δ is a non-empty set of imprimitivity. Then Ω can be written as a direct sum: $\Omega = \underset{i\in I}{\cup} \Delta_i \cdot \{\Delta_i\}_{i\in I}$ is called a system of sets of impimitivity of (G,Ω). Therefore if we are given a system $\{\Delta_i\}_{i\in I}$. of course, we can define a G- invariant equivalance relation on Ω.

Lemma 2.2.[3] *Let (G,Ω) be transitive. The (G,Ω) is primitive if and only if G_α, the stabilizer of apoint $\alpha \in \Omega$, is a maximal subgroup of G for each $\alpha \in \Omega$.*

What the lemma is saying is whenever $G_\alpha < H < G$, then Ω admits some G-invariant equivalence relation other than the trivial cases. In fact, since G acts transitively, every element of Ω has the form $g(\alpha)$ for some $g \in G$. If we define the relation $\approx$ on Ω as

$$g(\alpha) \approx g'(\alpha) \text{ if and only if } g' \in gH.$$

Then it is easily seen that it is non-trivial G-invariant equivalence relation. That is (G,Ω) is imprimitive.

From the above we see that the number of blocks is equal to the index $|G : H|$ [6].

We now apply these ideas to the case where G is the normalizer $\mathcal{N}$, and Ω is $\hat{\mathbb{Q}}$. An obvious choice for H is Γ_0 (N).

Clearly $\Gamma_\infty < \Gamma_\circ(N) < \mathcal{N}$, if $N > 1$.

So, from the above discussion, the normalizer $\mathcal{N}$ acts imprimitively on $\hat{\mathbb{Q}}$.

Let $\approx$ denote the $\mathcal{N}$-invariant equivalence relation induced an $\hat{Q}$ by $\Gamma_\circ$(N). And let $v = r/s$ and $w = x/y$ be elements of $\hat{\mathbb{Q}}$ such that $(s,N) = e_1, (y,N) = e_1'$ and $s = s_1 e_1$, $y = y_1 e_1'$. If $e_2 = N/e_1$ and $e_2' = N/e_1'$ then it is easily verified that there exist elements

$$g = \begin{pmatrix} re_2 & \star \\ s_1 N & d_1 e_2 \end{pmatrix}, det = e_2 \text{ and } g' = \begin{pmatrix} ye'_2 & \star \\ y_1 N & d_2 \cdot e'_2 \end{pmatrix}, det = e'_2.$$

belanging to $\mathcal{N}$ and send ∞ to v and to w, respectively.
If v and w are of the above form then we get that

$$v_e \approx v_f \text{ if and only if } e = f.$$

By our general discussion of imprimitivity, the number $\Psi(N)$ of blocks (equivalence classes) under $\approx$ is given by $\Psi(N) = |\mathcal{N} : \Gamma_\circ(N)|$.

The following formula for $\Psi(N)$ is knomwn [1], but for completeness we will sketch a proof here.

Lemma 2.3. $\Psi(N) = 2^r$, *where* r *is the number of prime factors of N.*

Proof. We will count equivalence classes under $\approx$. From the above we know that $v_e \approx v_f$ if and only if $e = f$. So counting the blocks is equivalent to counting the number of divisors of N. This means that the number of blocks is just 2^r, where r the number of primes dividing N. □

3. Suborbital Graphs For $\mathcal{N}$ on $\hat{\mathbb{Q}}$

Let (G, Ω) denote a transitive permutation group. For $(\alpha, \beta) \in \Omega^2$ and $g \in G$, we define $g(\alpha, \beta) = (g(\alpha), g(\beta))$. Therefore (G, Ω^2) becomes a permutation group. The orbits of this action are called suborbitals of G, that containing (α, β) being denoted by $0(\alpha, \beta)$. From $0(\alpha, \beta)$ we form a suborbital graph $\Delta(\alpha, \beta)$: its vertices are the elements of Ω and there is a directed edge from γ to δ if $(\gamma, \delta) \in 0(\alpha, \beta)$.
$0(\alpha, \beta)$ is also a suborbital, and it is either equal to or disjoint from $0(\alpha, \beta)$. In the latter case $\Delta(\beta, \alpha)$ is just $\Delta(\alpha, \beta)$ with the arrows reserved, and we call, in this case, $\Delta(\alpha, \beta)$ and $\Delta(\beta, \alpha)$ paired suborbital graphs.
In the former case, $\delta(\alpha, \beta) = \Delta(\beta, \alpha)$ and the graph consists of pairs of oppositly directed edges; it is convenient to replace each such pair by a single undirected edge, so that we have an undirected graph which we call self-paired.

The above dieas were first introduced by Sims [11], and are also described in a paper by Neumann [9] and in books by Tsuzuku [13] and by Biggs and white [3], the emphasis being on applications to finite groups.

We now apply the above to the normalizer $\mathcal{N}$ an $\hat{\mathbb{Q}}$. Since $\mathcal{N}$ acts transitively on $\hat{\mathbb{Q}}$, each suborbital contains a pair (∞, v) for some $v \in \hat{\mathbb{Q}}$; writting $v = u/n$, wih $n \geq 0$ and $(u, n) = 1$, we denote this suborbital by $O_{u,n}$, and corresponding suborbital graph by $\Delta_{u,n}$.

If $v = \infty = 1/10 = -1/0$, then this is the trivial suborbital graph $\Delta_{1,0} = \Delta_{-1,0}$, so assume that $v \in \hat{\mathbb{Q}}$ (we are not interested in trivial suborbital graphs). If $v' \in \hat{\mathbb{Q}}$, then $0(\infty, v) = 0(\infty, v')$ if and only if v and v' are in the same orbit of $\mathcal{N}_\infty$; since $\mathcal{N}_\infty$ is generated by $z : v \to v+1$, this is equivalent to $v' = u'/n$ where $u = u'$ modn. Therefore

$\Delta_{u,n} = \Delta_{u'}, n'$ if and only if $n = n'$ and $u = u'$ modn.

We will write $r/s \to x/y$ in $\Delta_{u,n}$ if $(r/s, x/y) \in O_{u,n}$.

Theorem 3.1 *$r/s \to x/y$ in $\Delta_{u,n}$ if and only it $\exists e \in \mathbb{Z}$ with $e|N$, $N/e|s$ and if $(n,e) = e_n, n = n_1 e_n, e = e_1 e_n$ then either*

a) $ry - sx = n_1$ and $x = re_1 u$ mod$n_1, y = e_1$ su mod $e_1 n$ or

b) $ry - sx = n_1$ and $x = -re_1 u$ mod$n_1, y = -e_1$ su mod $e_1 n$.

Proof. let $r/s \to x/y$ in $\Delta_{u,n}$. Then there is an element $\begin{pmatrix} ae & b \\ cN & de \end{pmatrix} \in \mathcal{N}$ sending ∞ to r/s, and u/n to x/y and therefore $ae/cN = r/s$ and $(aeu + bn)/(cNu + dn) = x/y$. Since the determinant $ade^2 \cdot bcN = e$, we get $(a, cN/e) = 1$. So $a = r$ and $s = cN/e$, that is $N/e|s$. Let $(n, e) = e_n, n = n_1 e_n$ and $e = e_1 e_n$. Since $\begin{pmatrix} ae & b \\ cN/e & d \end{pmatrix}$ has determinant 1, then using (2.1) we see that $(aeu + bn,\ cNu/e + dn) = 1$. Hence we will have the following matrix equation:

$$\begin{pmatrix} ae & b \\ cN & de \end{pmatrix}\begin{pmatrix} 1 & u \\ 0 & n \end{pmatrix} = \begin{pmatrix} ae & aeu + bn \\ cN & cNu + den \end{pmatrix} =$$

$$\begin{bmatrix} ae & e_n(ae_1 u + bn_1) \\ cN & e_n e_1 (cNu/e + dn) \end{bmatrix} = \begin{bmatrix} (-1)^i er & (-1)^j e_n x \\ (-1)^i es & (-1)^j e_n y \end{bmatrix}, \tag{3.1}$$

where $i, j = 0, 1$. If $i = j = 0$ then ae=er, $en(ae_1 u + bn_1) = e_n x$, $cN = es$, $e_n e_1(cNu/e + dn) = e_n y$. That is, $x = ae_1 u + bn_{x1}$ and $y = cNue_1/e + dne_1$. So $x = re_1 u$ modn$_1$ and $y = e_1$ su mode$_1 n$ and taking determinants in (3.1) we see that $ry - sx = n_1$, and so (a) holds. Similarly if i=1 and j=0 we obtain (b). If $i = j = -1$, then again (a) holds. If, finally, i=0 and j=1, then (b) holds.

Conversly, if (a) holds, then there exist integers b,d such that $x = re_1 u + bn_1$ and $y = e_1 su + de_1 n$. We now show that the element $\begin{pmatrix} re & b \\ se & de \end{pmatrix}$ belongs to $\mathcal{N}$ and sends ∞ to r/s, and u/n to x/y.

In fact, using $ry - sx = n_1$ and $N/e|s$ we get $rde^2 - sbe = e$, that is, the above element is in $\mathcal{N}$. Finally re/se=r/s and $(reu + bn)/(seu + dne) = e_n(re_1 u + bn_1)/e_n(se_1 u + de_1 n) = x/y$. As above $(re_1 u + bn_1, se_1 u + de_1 n) = 1$. If (b) holds the proof follows similarly. □

Notation *Let "$r/s \to x/y$ in $\Delta_{u,n}$" be denoted by "$r/s \xrightarrow{e_1} x/y$ in $\Delta_{u,n}$", where e_1 is as in Theorem 3.1. The set of e_1 's occured in $\Delta_{u,n}$ will be denoted by $E_{u,n}$.*

Corollary 3.2 *Let* $E_{u,n} = E_{v,n} = \{1\}$ *and let* $uv = -1$ *modn, then the suborbital graph* $\Delta_{u,n}$ *is paired with* $\Delta_{v,n}$.

Proof. We will observe that $r/s \rightarrow x/y$ in $\Delta_{u,n}$ if and only $x/y \rightarrow r/s$ in $\Delta_{v,n}$. Since $r/s \rightarrow x/y$ in $\Delta_{u,n}$, using the hypothesis and Theorem 3.2, we have that $\exists e|N, N/e|s$, $(n,e) = e, n = n_1 e$ such that either $x = ru$ modn$_1$, $y =$su modn and $ry - sx = n_1$, or

$$x = -ru \text{ modn}_1,\ y = - \text{ su modn and } ry - sx = -n_1.$$

Suppose that the former holds. Then $xs - yr = -n_1$ and $vx = ruv$ modn$_1$, $vy = suv$ modn. Since vy=-1 modn, we have $xs - yr = -n_1$ and $r = -vx$ modn$_1$, $s = -vy$ modn, that is, $x/s \rightarrow r/s$ in $\Delta_{v,n}$. □

Corollary 3.3 $\Delta_{u,n}$ *is self-paired if and only if* $\exists e|N$ *such that N—ne and* $u^2 e = -1$ *modn.*

Proof. Suppose $\Delta_{u,n}$ is self-paired. So the pair $(\infty, u/n)$ is sent to $(u/n, \infty)$ by $\mathcal{N}$. It is easily seen that such elements of $\mathcal{N}$ must be of the form $\begin{pmatrix} ue & b \\ ne & -ue \end{pmatrix}$, where determinant is e. Therefore $e|N$ and $N|ne$ and $u^2 e = -1$ modn.

Conversly, let $e|N$ Such that $N|ne$ and $u^2 e = -1$ modn. since $u^2 e = -1$ modn, then there exists an integer b such that $-u^2 e - bn = 1$, that is, $-u^2 e^2 - bne = e$. Therefore the element $\begin{pmatrix} ue & b \\ ne & -ue \end{pmatrix}$ is in $\mathcal{N}$ and satisfies the required properties. □

4. The Quotient Group $B = \mathcal{N}/\Gamma_\circ(N)$

In this final section we do some calculations about the representatives of orbits of $\Gamma_\circ(N)$. Then we show that the permutation group (B, Ω) is regular and m-regular where m is an odd natural number.

Theorem 4.1 *Given an arbitrary rational number k/s with (k, s)=1, then there exists an element* $A \in \Gamma_\circ(N)$ *such that* $A(k/s) = (k_1/s_1)$ *with* $s_1|N$.

Proof.

$$\begin{pmatrix} a & b \\ cN & d \end{pmatrix} \begin{pmatrix} k \\ s \end{pmatrix} \begin{pmatrix} ak + bs \\ Nck + ds \end{pmatrix}$$

we find some pairs $\{c, d\}$ for which the equation

$$Nck + ds = (N, s) \tag{4.1}$$

holds, for $(N,s)|N$, so $s_1 = (N,s)$ works.
Since $(Nk/(N,s),\ s/(N,s)) = 1$ there exists a pair $\{c_\circ, d_\circ\}$ so that the equation (4.1) is satisfied. Therefore, as we know, the general solution of (4.1) is

$$
\begin{aligned}
c &= c_\circ + sn/(N,s) \\
d &= d_\circ \cdot + Nkn/(N,s), \text{ where } n \in \mathbb{Z}
\end{aligned}
\qquad (4.2)
$$

Let $N = q_\circ^{\alpha_\circ} q_1^{\alpha_1} \ldots q_{k_\circ}^{\alpha_{k_\circ}}$ be the prime power decomposition of N. We must show that there exists a pair $\{c_\star, d_\star\}$ obeying (4.2) such that

$$(Nc_\star, d_\star) = 1.$$

If $(d_\circ, N) = 1$, there is nothing to prove. If $(d_\circ, N) > 1$ then $d_\circ$ does have a commen factor with $N, q_\circ$ say. using (4.1), $(q_\circ, Nk/(N,s)) = 1$ therefore taking n=1 in (4.2) we get an integer d_1 such that $q_\circ | d_1$.
If $(d_1, N) > 1$ then d_1 has a common factor with N, q_1 say. Let $d_2 = d_1 - q_\circ Nk/(N,s)$ then d_2 does not have q_1 as a factor. If $(d_2, N) > 1, d_2$ has a common factor with N, q_2 say. eventually we arrive at

$$d_3 = d_2 - q_\circ q_1 Nk/(N,s), \text{ and so } d_3 \text{ has no } q_\circ, q_1, q_2 \text{ as factors}$$
$$d_{k_\circ+1} = d_{k_\circ} - q_\circ q_\circ \ldots q_{k_\circ - 1} Nk/(N,s), \text{ and so } d_{k_\circ+1} \text{ has no } q_\circ, q_1, \ldots - q_{k_\circ}$$

as factors. Hence $(d_{k_\circ+1}, N) = 1$. Let $d_\star = d_{k_\circ+1}$ and the corresponding c, $c_\star$ say, and so $(Nc_\star, d_\star) = 1$. This implies that there exists an element $A \in \Gamma_\circ(N)$ such that $A(k/s) = k_1/s_1$ with $s_1 | N$. □

Therefore we have

Corollary 4.2. *Let $d_1|N$ and for some $A \in \Gamma_\circ(N)$ $A(a_1/d_1) = (a_2/d_1)$ with $(a_1, d_1) = (a_2, d_1) = 1$. Then $a_1 = a_2$ mod t, where $t = (d_1, N/d_1)$.*

Corollary 4.3. *Let $d|N$ and let $(a_1, d) = (a_2, d) = 1$. Then $\begin{pmatrix} a_1 \\ d \end{pmatrix}$ and $\begin{pmatrix} a_2 \\ d \end{pmatrix}$ are conjugate under $\Gamma_\circ(N)$ if and only if $a_1 = a_2$ modt, where $t = (d, N/d)$.*
Proof. Using the above and a theorem from [10] the result follows. □

From the above lemma and corollaries we can write down the set of orbits of $\Gamma_\circ(N)$ as $O = \left\{ \left[\frac{1}{d}\right] |d|N \right\}$ and it can be easily seen that the number of them is just 2^r, where r is the number of primes dividing N. So we take Ω as the set $\left\{\frac{1}{d} : d|N\right\}$, as the set of representatives of O.

We see that W_e of all matrices of the form $\begin{pmatrix} ae & b \\ cN & de \end{pmatrix}$ is a single coset of $\Gamma_\circ(N)$, where e$\|$N and the determinant is e. We have the relation $W_e^2 = 1, W_e W_f = W_f W_e =$

$W_g(mod\Gamma_\circ(N))$, where $g = \frac{e}{(e,f)} \cdot \frac{f}{(e,f)}$. This means that any element (except the identity) of B has order 2. since $\mathcal{N}$ acts transitively on $\hat{Q}$, then B acts transitively on Ω. Therefore (B, Ω) is a transitive permutation group.

Furthermore, we have the following results

Corollary 4.4. *(B, Ω) is a regular permutation group.*

Proof. As we see above the number $|\Omega|$ is equal to 2^r, and on the other hand $|B| = 2^r$. So the stabilizer B_x of any element x is just the identity. Hence the action is regular. □

Corollary 4.5 *Let m be an odd natural number. Then the group B is m-regular.*

Proof. Since the abelian group B is finite then is it a torsion group. On the other hand, the order of any element of B is relatively prime to m. so B is m-regular. □

References

[1] M. Akbaş and D. Singerman, The normalizer of $\Gamma_\circ$(N) in PSL(2$\mathbb{R}$). Glasgow Math. J. 32 (1990) 317-327.

[2] M. Akbaş and D. Singerman. The signature of the normalizer of $\Gamma_\circ$(N). London Math. Soc. Lecture Notes 165 Cambridge University Press, Cambridge, 1992.

[3] N. L. Bigg and A. T. White. Permutation group and combinatorial structures. London Math. Soc. Lecture Notes 33, Cambridge University Press, Cambridge, 1979.

[4] J. H. Conway, and S. P. Norton, Montrous Moonshine. Bull. London Math. Soc. 11, 308-339 (1979).

[5] G. A. Jones, and D. Singerman, Complex funtions: an algebraic ang geometric viewpoint. Cambridge university Press. Cambridge, 1987.

[6] A. Jones, D. Singerman and K. Wicks, The modular group and generalized Farey graphs, London Math. Soc. Lecture Notes 160, Cambridge university Press, Cambridge 1991, 316-338.

[7] J. Lehner and M. Newman, Weierstrass points of $\Gamma_\circ$(n). annals of Mathematics Vo: 79. No:2, March, 1964.

[8] W. J. LeVeque, Fundamentals of number theory. Addison-Qesley, Reading, Mass., 1977.

[9] P. M. Neumann, Finite permutation groups, edge-coloured graphs and matrices. Topics in group theory and computation, Ed. M. P. J. Curran, Academic Press, London, New York, San Francisco, 1977.

[10] B. Schoeneberg, Elliptic modular functions. Springer-Verlag, Berlin, Heidelberg, New York, 1974.

[11] C. C. Sims, graphs and finite permutation groups. Math. Z. 95 (1967), 76-86.

[12] M. Suzuki, Group Theory I. Springer-Verlag, Berlin, Heidelberg, New York, 1982.

[13] T. Tsuzuku, Finite groups and funite geometries. Cambridge University Press, Cambridge, 1982.

$\Gamma_\circ$(N) nin NORMALLEŞTİRİCİSİ İÇİN ALT ÇEVRESEL GRAFİKLER

Özet

Bu çalışmada $\Gamma_\circ$(N) nin PSL (2, $\mathbb{R}$) deki η normalleştiricisi için altyörüngesel grafiklerin bazı özellikleri belirtildi ve eğer $\mathcal{N}/\Gamma_o(N)$ ve yörünge temsilciler kümesi sırası ile B ve Ω ile gösterilirse, (B, Ω) parmütasyon grubunun regular ve ayrıca m bir tek doğal sayı ise m-reguler olduğu gösterildi.

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Received 9.11.1995