Invariance of generalized wordlength patterns

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Abstract

The generalized wordlength pattern (GWLP) introduced by Xu and Wu (2001) for an arbitrary fractional factorial design allows one to extend the use of the minimum aberration criterion to such designs. Ai and Zhang (2004) defined the J-characteristics of a design and showed that they uniquely determine the design. While both the GWLP and the J-characteristics require indexing the levels of each factor by a cyclic group, we see that the definitions carry over with appropriate changes if instead one uses an arbitrary abelian group. This means that the original definitions rest on an arbitrary choice of group structure. We show that the GWLP of a design is independent of this choice, but that the J-characteristics are not. We briefly discuss some implications of these results.

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1 Introduction

In a regular fractional factorial design D, the quantities

$$A_i(D) = \text{the number of defining words of length } i$$

contain useful information about the design. In particular, the smallest index i for which $A_i(D) > 0$ is the resolution of the design. Moreover, one way of comparing two designs having k factors and equal resolution is to compare their *wordlength patterns* $(A_1, A_2, \ldots, A_k)$ [6, 7]. The better design is said to have less *aberration*.

While nonregular designs no longer have defining words as such, a *generalized wordlength pattern (GWLP)* can be defined for them combinatorially. This was done for two-level designs by Tang and Deng [13], and was generalized to arbitrary (possibly mixed-level) designs by Xu and Wu [15] using group characters.

An intermediate computation in the two-level case gives a set of values that Tang and Deng called J-characteristics (first introduced in [4]), and Tang [12] showed that these numbers completely determine the design D, somewhat analogous to the way that a defining subgroup determines a regular design. Ai and Zhang [1] generalized this to arbitrary designs by looking closely at the corresponding computation in [15].

In defining generalized wordlength patterns of arbitrary designs, Xu and Wu [15] assigned to the ith factor the cyclic group $\mathbb{Z}_{s_i}$, where $s_i =$ the number of levels of the factor. While this choice is a computational convenience, it is also arbitrary, and in fact the calculation of the GWLP can be carried through using other abelian groups as well, as we indicate below.

This, however, raises the following question for non-prime s_i. Since the (irreducible) characters of two groups of equal order will generally be different, does the choice of group affect either the J-characteristics or the GWLP of a given design? Certainly any dependence of the GWLP on an arbitrary choice would raise a serious question about its use in comparing designs using relative aberration. It will be clearly seen that the J-characteristics do depend on this choice. However, perhaps surprisingly, this does not affect the values of the GWLP. That is our main result.

There are many excellent expositions of character theory, such as [8], [9] and [10]. In general we will mention known results without citation. We will also use a number of results from multilinear algebra (the theory of tensor products). These are collected in an appendix.

Notation. We will denote the integers by $\mathbb{Z}$, and the integers modulo s by $\mathbb{Z}_s$ as above. The complex numbers will be denoted by $\mathbb{C}$ and complex Euclidean space by $\mathbb{C}^s$. Vectors in $\mathbb{C}^s$ will be viewed as columns. The conjugate of $z \in \mathbb{C}$ will be denoted by $\bar{z}$, the transpose of a vector or matrix by a prime ($'$), and the adjoint (or conjugate transpose) of a matrix or linear transformation A by A^*. The inner product of $v = [v_1, \ldots, v_s]'$ and $w = [w_1, \ldots, w_s]' \in \mathbb{C}^s$ is given by

$$\langle\, v, w \,\rangle = \sum_{i=1}^{s} v_i \overline{w}_i. \tag{1}$$

The cardinality of a set E will be written $|E|$.

The *Hamming weight* of $u = (u_1, \ldots, u_k)$, $\mathrm{wt}(u)$, is the number of nonzero components of u. (In Section 2 we will replace "nonzero" by "nonidentity" in order to deal with groups whose identity element is not 0.)

We alert the reader to the fact that we will use G (or G_i) as an index set, with elements g or h. Sometimes such sets will be groups, but often they will be viewed just as sets. We will try to make absolutely clear from context when a result requires a group structure and when it doesn't.

2 Definitions

A *fractional factorial design* on k factors is a multisubset D of a finite Cartesian product $G = G_1 \times \cdots \times G_k$, that is, the set G with the element g repeated $O(g)$ times, $O(g) \geq 0$. The set G_i indexes the s_i levels of factor i, and we let $s = s_1 \cdots s_k$. We will refer to O as the *counting* or *multiplicity function* of D. The elements (k-tuples) of the design are referred to as *runs*, and the number of runs in the design, counting multiplicities, is

$$N = |D| = \sum_{g \in G} O(g). \tag{2}$$

The design D may also be viewed as an *orthogonal array*, particularly if its runs are displayed in matrix form, say as columns of a $k \times N$ matrix.

In [15] Xu and Wu defined the generalized wordlength pattern $(A_1(D), \ldots, A_k(D))$ of D as follows. If G_i has s_i elements, we take $G_i = \mathbb{Z}_{s_i}$, the additive group of integers modulo s_i. This makes G an abelian group. To each $g \in \mathbb{Z}_s$ we associate a function $\chi_g : \mathbb{Z}_s \to \mathbb{C}$ such that

$$\chi_g(h) = \xi^{gh}, \tag{3}$$

where ξ is a primitive sth root of unity (say $\xi = e^{2\pi i/s}$). For elements $g = (g_1, \ldots, g_k)$ and $h = (h_1, \ldots, h_k)$ of $G = G_1 \times \cdots \times G_k$, we let

$$\chi_g(h) = \prod_i \chi_{g_i}(h_i), \tag{4}$$

and define[1]

$$\chi_g(D) = \sum_{h \in G} O(h)\chi_g(h). \tag{5}$$

Finally, the "generalized wordlengths" are given by

$$A_j(D) = N^{-2} \sum_{\mathrm{wt}(g)=j} |\chi_g(D)|^2 \qquad \text{for } j = 1, \ldots, k, \tag{6}$$

where $\mathrm{wt}(g)$ is the Hamming weight of g.

Ai and Zhang [1] note that when $s_1 = \cdots = s_k = 2$ the quantities $\chi_g(D)$ are the *J-characteristics* of Tang and Deng, and rename them so in the general case, with the notation $J_g(D)$.

We now indicate the way in which other groups may be used in (5) and (6).

The functions χ_{g_i} are the *irreducible characters* of the group $\mathbb{Z}_{s_i}$, and so the functions χ_g are the irreducible characters of G. Among these is $\chi_e \equiv 1$, the *trivial character* of G, corresponding to the identity e of G. Something similar holds for abelian groups, in particular the indexing of irreducible characters by group elements.

Specifically, the irreducible characters of an abelian group G are precisely the homomorphisms of G into the multiplicative group $\mathbb{C}^* = \mathbb{C} \setminus 0$. The indexing of these characters is based on the following result.

Theorem 2.1. *Let* $\mathrm{Irr}(G)$ *denote the set of irreducible characters of the group* G*. If* G *is abelian, then* $\mathrm{Irr}(G)$ *forms a group under pointwise multiplication, and if* G *is also finite, then* $G \cong \mathrm{Irr}(G)$*. In particular, the identity element of* G *corresponds to the trivial character of* G*.*

The isomorphism is not canonical – and, in particular, not unique – as it depends on the representation of an abelian group as a product of cyclic groups (the Fundamental Theorem of Abelian Groups), and for cyclic groups on the choice of root of unity in (3). (See, e.g., [9, Theorem 2.4]). We will assume that we have fixed an isomorphism $G_i \to \mathrm{Irr}(G_i)$ for each i, and thus an indexing of the irreducible characters of G_i by group elements. We will not need

[1] In [15] $\chi_g(D)$ is defined as $\sum_{h \in D} \chi_g(h)$, and it is to be understood that the hth term is repeated the number of times h appears in the design [14]. Equation (5), which is essentially the same as that used in [1], makes this explicit.

to know the indexing explicitly. The irreducible characters of the direct product G are still given by (4).

J-characteristics and generalized wordlength counts are still defined by (5) and (6), respectively, where we now define the *weight* of the element $g = (g_1, \ldots, g_k) \in G$ to be the number of *nonidentity* components of g. Our main result is this:

Theorem 2.2. *The quantities $A_j(D)$ in (6) are independent of the group structure of G.*

The proof of this theorem is given in Section 4. Before considering this, we take a moment to study the effect of the choice of group on the J-characteristics of a design.

3 J-characteristics. The character table.

We see that the irreducible characters of a finite abelian group G of order s may be written $\chi_{g_1}, \ldots, \chi_{g_s}$, where g_i are the elements of G in some order. The values $\chi_{g_i}(g_j)$ form the *character table* of G, the columns of which are mutually orthogonal and of norm $\sqrt{s}$ (with respect to the inner product (1)). Another way to say this is that the $s \times s$ matrix H formed by this table has the property that $H^*H = HH^* = sI$, where H^* is the adjoint of H (H is thus a complex Hadamard matrix).

Let $G = G_1 \times \cdots \times G_k$ where each G_i is an abelian group, so that G is as well, and assume that the elements of G are ordered in some fashion. (Ai and Zhang [1] use a lexicographic or *Yates* order).) If we consider the set of J-characteristics $\chi_g(D)$ and the counts $O(g)$ as $s \times 1$ vectors χ and O indexed by $g \in G$, then (5) may be written

$$\chi = HO. \tag{7}$$

Multiplying through by H^*, we see that $H^*\chi = H^*HO = sO$, so that

$$O = (1/s)H^*\chi,$$

and in particular that the J-characteristics determine the design. This is Theorem 1 of [1].

However, in general H depends on the group structure of G, and so from (7) or directly from (5) we see that *the values of the J-characteristics depend on the choice of group structure.* This is illustrated with the following example.

Example 3.1. Consider the 3-factor design

$$D = \begin{bmatrix} 0000 & aaaa & bbbb & cccc \\ 0abc & 0abc & 0abc & 0abc \\ 0abc & b0ca & ac0b & cba0 \end{bmatrix}. \tag{8}$$

Each factor has 4 levels, namely $0, a, b$, and c, and each column is a treatment combination. One can check that this is an orthogonal array of strength 2 and index 1 (it is taken from [5], where it is shown to be non-regular).

For each factor the symbol set $G_i = \{0, a, b, c\}$ may be given two group structures, namely that of the cyclic group $\mathbb{Z}_4$ and that of the "Klein 4-group" V (isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$). Table 3.1 displays the non-zero values of $\chi_g(D)$ as g runs over the 64 elements of $G = G_1 \times G_2 \times G_3$,

Table 1: J-Characteristics for Design D under two different group structures
For the design in (8), the value of $\chi_g(D)$ is given for each $g \in G$, where G = either $\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4$ or $V \times V \times V$. Those g for which both values of $\chi_g(D) = 0$ are omitted. (Computation was done in Maple.)

$g =$	000	aaa	aab	aac	aba	abb	abc	aca	acb	acc
$\mathbb{Z}_4$	16	$-6-2i$	$4i$	$6-2i$	$-4i$	$4+4i$	-4	$6-2i$	4	$6+2i$
V	16	8	8	0	-8	8	0	0	0	0

$g =$	baa	bab	bac	bba	bbb	bbc	bca	bcb	bcc
$\mathbb{Z}_4$	$4i$	$-4-4i$	4	$4+4i$	8	$4-4i$	4	$-4+4i$	$-4i$
V	8	-8	0	8	8	0	0	0	0

$g =$	caa	cab	cac	cba	cbb	cbc	cca	ccb	ccc
$\mathbb{Z}_4$	$6-2i$	4	$6+2i$	-4	$4-4i$	$4i$	$6+2i$	$-4i$	$-6+2i$
V	0	0	0	0	0	0	0	0	16

where the groups G_i are all $\mathbb{Z}_4$ or all V. For example, $\chi_{aaa}(D) = -6-2i$ using $\mathbb{Z}_4$, but $= 8$ using V. Thus we see that the values of the J-characteristics depend on the group structure. Note that for both group structures we have $\chi_g(D) = 0$ if $\text{wt}(g) = 1$ or 2. Such group elements have been omitted from the table for convenience.

It is not hard to calculate the values $A_j(D)$ given by equation (6), where we have $N = 16$. (The computation is shortened by the fact that $|\pm a \pm bi|^2 = a^2 + b^2$.) We find that under both group structures we have $A_1(D) = A_2(D) = 0$ and $A_3(D) = 3$, as guaranteed by Theorem 2.2.

Before we leave this topic, we develop the properties of the character table a little further.

In enumerating the elements of a group G one typically chooses g_1 = the identity. With this convention, which we shall adopt, χ_{g_1} is the trivial character of G, so that $\chi_{g_1}(h) = 1$ for all $h \in G$. On the other hand, since G is abelian, $\chi_g(g_1) = 1$ for every $g \in G$, and so we see that H must have the form

$$H = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & * & \cdots & * \\ \vdots & \vdots & & \vdots \\ 1 & * & \cdots & * \end{pmatrix}.$$

The matrix $U = (1/\sqrt{s})H$ is said to be the *normalized character table* of G. We list its important properties here, which follow from the preceding.

Lemma 3.2. *U unitary ($U^*U = UU^* = I$), and in particular defines an isometry on $\mathbb{C}^s$ ($\langle Uv, Uw \rangle = \langle v, w \rangle$). If $e = [1, 0, \ldots, 0]'$ and $b = (1/\sqrt{s})[1, \ldots, 1]'$ then*

$$Ue = b = U^*e.$$

4 Independence of group structure

By imposing a group structure on the set $G = G_1 \times \cdots \times G_k$, we define the irreducible characters χ_g. We want to show that the numbers

$$\sum_{\mathrm{wt}(g)=j} |\chi_g(D)|^2, \quad j = 1, \ldots, k,$$

appearing in (6) are independent of the group structure chosen. This sum is somewhat unwieldy, and so we will break it into smaller sums over elements g which are not only of weight j but also differ from the identity in exactly the same components.

To begin with, we fix an order of the elements in each set G_i, with the understanding that *whenever we impose a group structure, the first element will be the identity of the group.* We may denote by 1_i the chosen element of G_i.

Now, for each $J \subset \{1, \ldots, k\}$ with $|J| = j$, let

$$S_J = \{g = (g_1, \cdots, g_k) \in G : g_i \neq 1_i \text{ iff } i \in J\}. \tag{9}$$

(Here J is merely an index set and has no relation to the J-characteristics mentioned earlier.) Clearly the sets S_J are disjoint and their union is the set of elements of G of weight j. Then

$$\sum_{\mathrm{wt}(g)=j} |\chi_g(D)|^2 = \sum_{|J|=j} \sum_{g \in S_J} |\chi_g(D)|^2.$$

We will show that for each J the inner sum

$$\sum_{g \in S_J} |\chi_g(D)|^2 \tag{10}$$

is independent of the group structure chosen.

To do this, we will write these sums as squared norms of elements in an appropriate subspace V_J of $\mathbb{C}^s$. Assuming a fixed ordering of the elements of G, the components of a vector $v \in \mathbb{C}^s$ are complex numbers indexed by the elements of G, something like

$$v = [\cdots, v_g, \cdots]'. \tag{11}$$

The standard basis elements are of form

$$e_g = [0, \ldots, 0, 1, 0 \ldots, 0]', \tag{12}$$

where 1 occurs in just the g-th coordinate. Then

$$v = \sum_{g \in G} v_g e_g. \tag{13}$$

Let

$$V_J = \{v = [\cdots, v_g, \cdots]' \in \mathbb{C}^s : v_g = 0 \text{ if } g \notin S_J\}.$$

It is clear that $\dim V_J = |S_J| = \prod_{i \in J}(s_i - 1)$, and that the sum (10) is $\|M_J(\chi)\|^2$ where M_J is the orthogonal projection of $\mathbb{C}^s$ onto V_J. Now the next result follows immediately from (7) and the fact that $H = \sqrt{s}\,U$.

Proposition 4.1. *Suppose that $G = G_1 \times \cdots \times G_k$ where G_i is an abelian group. Then the sum in (10) is equal to*

$$s\|M_J U O\|^2, \tag{14}$$

where M_J is the orthogonal projection of $\mathbb{C}^s$ on the subspace V_J, U is the normalized character table of G, and O is the vector of multiplicities of the design D.

Our goal is now to show that the quantity (14) is independent of the group structure of G.

A very useful way to describe V_J is as follows. Associate to G_i the Euclidean space $\mathbb{C}^{s_i}$, where the components of a vector v are indexed by the elements of G_i. Let $e_g^{(i)}$ be the unit vector in $\mathbb{C}^{s_i}$ having a 1 in the gth place and zeros elsewhere, so that $e_1^{(i)} = [1, 0, \cdots, 0]'$. Define the subspaces $V_i \subset \mathbb{C}^{s_i}$, $i = 1, \ldots, k$, by setting

$$\begin{aligned} V_i &= (e_1^{(i)})^{\perp}, \quad & i \in J, \\ &= \mathrm{span}(e_1^{(i)}), \quad & i \notin J, \end{aligned}$$

where orthocomplement ($^{\perp}$) and span are within $\mathbb{C}^{s_i}$. Thus the vectors of V_i have a zero in the first position if $i \in J$ and zeros in all the other positions if $i \notin J$.

Let P_i be the projection of $\mathbb{C}^{s_i}$ onto $\mathrm{span}(e_1^{(i)})$, I_i the identity matrix, and $Q_i = I_i - P_i$.

Proposition 4.2. *With the above definitions, we have*

$$V_J = V_1 \otimes \cdots \otimes V_k. \tag{15}$$

The orthogonal projection M_J of $\mathbb{C}^s$ on V_J is given by

$$M_J = M_1 \otimes \cdots \otimes M_k, \tag{16}$$

where M_i is the orthogonal projection of C^{s_i} on V_i. We have

$$\begin{aligned} M_i &= Q_i, \quad & i \in J, \\ &= P_i, \quad & i \notin J. \end{aligned}$$

Proof. The vectors in this tensor product are sums of vectors of the form

$$v_1 \otimes \cdots \otimes v_k, \quad v_i \in V_i.$$

It is not hard to see that a vector of this form has zeros in exactly the positions indexed by $g \notin S_J$, so that $V_1 \otimes \cdots \otimes V_k \subset V_J$. However,

$$\begin{aligned} \dim V_i &= s_i - 1, \quad & i \in J, \\ &= 1 & \text{otherwise}, \end{aligned}$$

so $\dim V_1 \otimes \cdots \otimes V_k = \dim V_1 \cdots \dim V_k = \prod_{i \in J}(s_i - 1) = \dim V_J$. Thus (15) holds, and (16) follows immediately. The formula for M_i is obvious. □

We also note the following, which is implicit in equation (4).

Proposition 4.3. *If G_i is a finite group having character table H_i and normalized table U_i, then $G = G_1 \times \cdots \times G_k$ has character table $H = H_1 \otimes \cdots \otimes H_k$ and normalized character table $U = U_1 \otimes \cdots \otimes U_k$.*

We use this to evaluate the vector $M_J UO$ appearing in (14). As in (13), the vector O of multiplicities may be written

$$O = \sum_{g \in G} O(g) e_g.$$

But if $g = (g_1, \dots, g_k) \in G_1 \times \cdots \times G_k$, then

$$e_g = e^{(1)}_{g_1} \otimes \cdots \otimes e^{(k)}_{g_k}$$

where $e^{(i)}_j$ is the unit vector in $\mathbb{C}^{s_i}$ having a 1 in the jth place and zeros elsewhere. Then

$$O = \sum_{g=(g_1,\dots,g_k)} O(g)\, e^{(1)}_{g_1} \otimes \cdots \otimes e^{(k)}_{g_k}.$$

Thus

$$\begin{aligned} M_J UO &= \sum_{g=(g_1,\dots,g_k)} O(g)\, M_J U(e^{(1)}_{g_1} \otimes \cdots \otimes e^{(k)}_{g_k}) \\ &= \sum_{g=(g_1,\dots,g_k)} O(g)\, M_1 U_1(e^{(1)}_{g_1}) \otimes \cdots \otimes M_k U_k(e^{(k)}_{g_k}). \end{aligned} \tag{17}$$

To analyze the (squared) norm of this, we need to analyze the terms in such sums. This leads to evaluating $M_i U_i$ on the basis elements $e^{(i)}_{g_i}$. We will just need to do this when $M_i = P_i$.

Lemma 4.4. *For each i and for every $g \in G_i$ we have*

$$P_i U_i(e^{(i)}_g) = \frac{1}{\sqrt{s_i}}\, e^{(i)}_1.$$

Proof. For simplicity, suppress the index i. Now for any $w \in \mathbb{C}^s$ we have

$$Pw = \langle\, w, e_1 \,\rangle e_1,$$

so from Lemma 3.2 we have

$$PUv = \langle\, Uv, e_1 \,\rangle e_1 = \langle\, v, U^* e_1 \,\rangle e_1 = \langle\, v, b \,\rangle e_1$$

for any v, with $b = (1/\sqrt{s})[1, \dots, 1]'$. In particular,

$$PU(e_g) = \langle\, e_g, b \,\rangle e_1 = \frac{1}{\sqrt{s}}\, e_1,$$

as claimed. □

We now evaluate the squared norm of sums of form (17). This will rest on the following calculation.

Lemma 4.5. *Let I_j be identity matrix of order s_j, and let*

$$c_{g_1\cdots g_i} = \sum_{g_{i+1},\dots,g_k} O(g),$$

the sum of the numbers $O(g)$ over those $g \in G$ with the first i values fixed at $(g_1,\dots,g_i)$. Then for every $0 \le i \le k$,

$$\|(I_1 \otimes \cdots \otimes I_i \otimes P_{i+1} \otimes \cdots \otimes P_k)UO\|^2 = \frac{1}{s_{i+1}\cdots s_k} \sum_{g_1,\dots,g_i} c^2_{g_1\cdots g_i}. \tag{18}$$

In particular, this quantity is independent of the group structure on G.

Proof. We see that $(I_1 \otimes \cdots \otimes I_i \otimes P_{i+1} \otimes \cdots \otimes P_k)UO$

$$\begin{aligned}
&= \sum_{g=(g_1,\dots,g_k)} O(g)\; I_1U_1(e^{(1)}_{g_1}) \otimes \cdots \otimes I_iU_i(e^{(i)}_{g_i}) \otimes P_{i+1}U_{i+1}(e^{(i+1)}_{g_{i+1}}) \otimes \cdots \otimes P_kU_k(e^{(k)}_{g_k}) \\
&= \sum_{g=(g_1,\dots,g_k)} O(g) \quad U_1(e^{(1)}_{g_1}) \otimes \cdots \otimes U_i(e^{(i)}_{g_i}) \quad \otimes \quad \frac{1}{\sqrt{s_{i+1}}} e^{(i+1)}_1 \otimes \cdots \otimes \frac{1}{\sqrt{s_k}} e^{(k)}_1 \\
&= \frac{1}{\sqrt{s_{i+1}\cdots s_k}} \sum_{(g_1,\dots,g_i)} \left(\sum_{(g_{i+1},\dots,g_k)} O(g) \right) U_1(e^{(1)}_{g_1}) \otimes \cdots \otimes U_i(e^{(i)}_{g_i}) \otimes e^{(i+1)}_1 \otimes \cdots \otimes e^{(k)}_1 \\
&= \frac{1}{\sqrt{s_{i+1}\cdots s_k}} \sum_{(g_1,\dots,g_i)} c_{g_1\cdots g_i}\; U_1(e^{(1)}_{g_1}) \otimes \cdots \otimes U_i(e^{(i)}_{g_i}) \otimes e^{(i+1)}_1 \otimes \cdots \otimes e^{(k)}_1.
\end{aligned}$$

But for each j, the set $\{U_j(e^{(j)}_g),\ g \in G_j\}$ is orthonormal in $\mathbb{C}^{s_j}$ as the unit vectors $e^{(j)}_g, g \in G_j$, are orthonormal and U_j is an isometry. Hence the elements $U_1(e^{(1)}_{g_1}) \otimes \cdots \otimes U_i(e^{(i)}_{g_i}) \otimes e^{(i+1)}_1 \otimes \cdots \otimes e^{(k)}_1$ are orthonormal in $\mathbb{C}^s$, and so $\|(I_1 \otimes \cdots \otimes I_i \otimes P_{i+1} \otimes \cdots \otimes P_k)UO\|^2$

$$= \frac{1}{s_{i+1}\cdots s_k} \sum_{(g_1,\dots,g_i)} c^2_{g_1\cdots g_i}\; \|U_1(e^{(1)}_{g_1}) \otimes \cdots \otimes U_i(e^{(i)}_{g_i}) \otimes e^{(i+1)}_1 \otimes \cdots \otimes e^{(k)}_1\|^2.$$

But this

$$\begin{aligned}
&= \frac{1}{s_{i+1}\cdots s_k} \sum_{(g_1,\dots,g_i)} c^2_{g_1\cdots g_i}\; \|U_1(e^{(1)}_{g_1})\|^2 \cdots \|U_i(e^{(i)}_{g_i})\|^2\; \|e^{(i+1)}_1\|^2 \cdots \|e^{(k)}_1\|^2 \\
&= \frac{1}{s_{i+1}\cdots s_k} \sum_{(g_1,\dots,g_i)} c^2_{g_1\cdots g_i}
\end{aligned}$$

as all the norms in the next-to-last line are 1. This is formula (18). □

Now fix $J \subset \{1,\dots,k\}$.

Proposition 4.6. *$\|M_JUO\|^2$ is independent of the group structure of G.*

Proof. Actually, we will prove something more general, namely that the proposition holds for projections M made up of a tensor product of P_i's, Q_i's and I_i's, where $Q_i = I_i - P_i$. Letting $q =$ the number of factors Q_i in the projection, we prove this by induction on q.

We simplify matters by proving our result for projections of form

$$M = Q_1 \otimes \cdots \otimes Q_q \otimes I_{q+1} \otimes \cdots \otimes I_{q+i} \otimes P_{q+i+1} \otimes \cdots \otimes P_k. \tag{19}$$

The proof is the same for projections with other ordering of the tensor factors.

The base case ($q = 0$) is precisely Lemma 4.5. For the induction step, assume that the result holds for projections having $q-1$ factors Q (not necessarily the first $q-1$ factors). Now $Q_q = I_q - P_q$, so the projection (19) is

$$\begin{aligned} M &= Q_1 \otimes \cdots \otimes Q_{q-1} \otimes I_q \otimes I_{q+1} \otimes \cdots \otimes I_{q+i} \otimes P_{q+i+1} \otimes \cdots \otimes P_k \\ &\quad - Q_1 \otimes \cdots \otimes Q_{q-1} \otimes P_q \otimes I_{q+1} \otimes \cdots \otimes I_{q+i} \otimes P_{q+i+1} \otimes \cdots \otimes P_k \\ &= T_1 - T_2, \end{aligned}$$

say. Since $T_1 = M + T_2$ and M and T_2 are orthogonal, the Pythagorean Theorem gives

$$\|MUO\|^2 = \|T_1UO\|^2 - \|T_2UO\|^2. \tag{20}$$

But since T_1 and T_2 contain $q-1$ factors Q_i, the induction hypothesis applies to both terms on the right-hand-side of (20), and therefore to the left-hand-side, as desired. □

By Proposition 4.1 this shows that the sum (10), and therefore the quantities $A_j(D)$, are independent of the group structure of G. Theorem 2.2 is now proved.

5 Conclusion

The definition of the generalized wordlength pattern (GWLP) given in [15] makes sense if one chooses abelian rather than cyclic groups to index the levels of each factor. The choice to use cyclic groups in [15] is arbitrary, and we have shown that while it does affect the so-called J-characteristics of a design, it does not affect the GWLP. This removes a possible ambiguity in the definition of the GWLP, and therefore in the use of minimum aberration as an optimality criteria for nonregular designs. The choice of cyclic groups may be useful computationally as the irreducible characters are then especially simple.

A special case of the invariance with respect to group structure is already implicit in the coding literature [3]. (The connection with regular designs is given in [15].) However, this covers designs in which (a) the index sets G_i are the same (the alphabet) and (b) the design is actually a subset of G (so that the counting function O is simply an indicator function). Our Theorem 2.2 is quite general, and makes no use of concepts borrowed from coding theory.

The wordlength pattern of a regular design does not determine the design, and in particular does not tell us its alias structure. For that, one needs the defining words. We have seen that an analog of the set of defining words of a nonregular design is the set of J-characteristics, at least in respect of determining the design. However, as we noted in Section 3, the J-characteristics vary with the choice of group structure assigned to factors. Certainly the aliasing structure of a design does not depend on this arbitrary choice. The GWLP is independent of this choice, and one may therefore ask just what statistical information it carries. This is a question worthy of further investigation.

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A Multilinear background

In this section we briefly review some results on tensor products that we have used in this paper. We only deal with Euclidean spaces (specifically $\mathbb{C}^k$) since that is all we need here. For simplicity we concentrate on the bilinear case (two tensor factors).

There are many expositions of multilinear algebra, such as that in [2]. An interesting exposition with some statistical applications is given in [11].

As is well-known, the *Kronecker* or *tensor product* of the matrices A $(m \times n)$ and B is

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}.$$

For vectors $v \in \mathbb{C}^a$ and $w \in \mathbb{C}^b$ we thus have

$$v \otimes w = \begin{bmatrix} v_1 w \\ \vdots \\ v_a w \end{bmatrix} \in \mathbb{C}^{ab},$$

where $v = [v_1, \ldots, v_a]'$. This product satisfies the usual bilinear properties, for example, $cv \otimes w = c(v \otimes w) = v \otimes cw$ (c a scalar) and $A \otimes (B + C) = A \otimes B + A \otimes C$.

If $V \subset \mathbb{C}^a$ and $W \subset \mathbb{C}^b$ are subspaces, then we define their tensor product to be the subspace of $\mathbb{C}^{ab}$ given by

$$V \otimes W = \operatorname{span}\{v \otimes w : \ v \in V, w \in W\}.$$

(Technically, $V \otimes W$ is constructed as a free vector space modulo bilinear relations, and is only isomorphic to a subspace of $\mathbb{C}^{ab}$, but we will identify it with that subspace.) If $\{e_1, \cdots, e_k\}$ is a basis of V and $\{f_1, \cdots, f_\ell\}$ is a basis of W, then

$$\{e_i \otimes f_j : \ i = 1, \ldots k, \ j = 1, \ldots \ell\}$$

is a basis of $V \otimes W$. Thus in particular

$$\dim(V \otimes W) = \dim(V) \cdot \dim(W).$$

If we use $\langle v_1, v_2 \rangle$ to denote the inner (or dot) product and $\|v\| = \sqrt{\langle v, v \rangle}$ the norm, then we have

$$\langle v_1 \otimes w_1, \, v_2 \otimes w_2 \rangle = \langle v_1, v_2 \rangle \langle w_1, w_2 \rangle,$$

and in particular

$$\|v \otimes w\| = \|v\| \|w\|.$$

The norm and inner product of vectors of the form $\sum v_i \otimes w_i$ are calculated by expanding the inner product in the usual way.

If T_i is a linear transformation on V_i, then $T = T_1 \otimes T_2$ is a linear transformation on $V_1 \otimes V_2$ such that

$$T(v_1 \otimes v_2) = T_1(v_1) \otimes T_2(v_2).$$

T is evaluated on sums of such terms by linearity. The matrix of T is given by the Kronecker product of the matrices T_i. Finally, if $S = S_1 \otimes S_2$ is a linear transformation such that $S_i T_i$ is defined for each i, then

$$ST = S_1 T_1 \otimes S_2 T_2.$$

All of the preceding extends in the obvious way to more than two tensor factors.