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WEAK⋆- INVARIANTLY COMPLEMENTED SUBSPACES OF $L^\infty(1/\omega)$ AND IDEALS OF $L^1(\omega)$ WITH A BOUNDED APPROXIMATE IDENTITY

Z. Argün & C. Tonyalı

Abstract

For a locally compact group G let $L^1(\omega)$ be the weighted group algebra and let X be a weak ⋆-closed translation invariant subspace of $L^\infty(1/\omega)$. In this paper for a certain class of functions we show that the following conditions are equivalent: (i) X is topological invariantly complemented in $L^\infty(1/\omega)$; (ii) X is invariantly complemented in $L^\infty(1/\omega)$; (iii) The left ideal $X_\perp$ has a bounded right approximate identity.

Introduction

Let G be a locally compact group with fixed left Haar measure dx. By a weighted function on G we mean a positive and locally bounded measurable function ω on G such that $\omega(st) \le \omega(s)\omega(t)$ and inf $\omega(s) = \gamma > 0 (s, t \in G)$. We may assume that ω is upper semi-continuous on G (see, for example, ([7], page 83)). If we set

$$L^1(\omega) = \left\{ f :\| f \|_\omega^1 = \int_G |f(t)| \, \omega(t) \, dt < \infty \right\}$$

then, $L^1(\omega)$ is a Banach Space: as usual, we equate functions equal dx almost everywhere. under convolution product defined by the equation

$$f * g(x) = \int_G f(y) g(y^{-1}x) dy \quad (f, g \in L^1(\omega))$$

$L^1(\omega)$ becomes a Banach algebra. Since ω is locally bounded, $L^1(\omega)$ contains the space of functions of compact supports. We also indicate that it has a bounded approximate identity (in the sequel we abbreviated it as a.i.) (see, for example ([1], Lemma 1.4.1)).

Recall that the isometric involution "$*$" on $L^1(G)$ defined by $f^*(x) = \Delta(x^{-1})\overline{f(x^1)}$, does not in general, apply to $L^1(\omega)$. In fact, f^* need not even be in $L^1(\omega)$. This is shown by examples in ([7], chapter 1, § 6.1). In this study we therefore assume the weighted function ω satisfies that ess sup $\{\omega(x^{-1})/\omega(x) : x \in G\} = \delta < \infty$. This situation does not imply that ω is bounded. For example, the weighted function ω defined by, $\omega(x) = 1 + \frac{1}{2}\left(\frac{x}{2} + |x|\right)$ is not bounded on $(\mathbb{R}, +)$, but sup $\omega(-x)/\omega(x) = 3$ and $1 \leq w(x)$ for all $x \in \mathbb{R}$. Note also that the involution "$*$" is continuous with $\| f^* \| \leq \delta \| f \| \leq \delta^2 \| f^* \|$. In this case the involution "$*$" becomes a topological algebraic anti-isomorphism of $L^1(\omega)$.

Using the notation of [5] and [6], for $f \in L^1(\omega)$ and $x \in G$ ${}_xf(y) = f(xy), f_x(y) = f(yx)(y \in G)$. Then ${}_xf, f_x$ belong to $L^1(\omega)$ and

$$\|_x f \|_\omega^1 \leq \omega(x^{-1}) \| f \|_\omega^1, \| f_x \| \leq \Delta(x^{-1})\omega(x^{-1}) \| f_x \|_\omega^1 .$$

The dual space of $L^1(\omega)$ is $L^\infty(\omega)$, Banach space of complex valued measurable functions φ on G for which φ/ω is essantially bounded; that is $\varphi/\omega \in L^\infty(G)$. The norm in $L^\infty(1/\omega)$ is;

$$\| \varphi \|_\omega^\infty = ess\, sup\{|\varphi(t))\, /\omega(t) : t \in G\} \quad (\varphi \in L^\infty(1/\omega))$$

and the duality is implemented by

$$\langle f, \varphi \rangle = \int_G f(x)\overline{\varphi(x)}dx \quad \left(f \in L^1(\omega), \varphi \in L^\infty(1/\omega)\right).$$

Furthermore for each $\varphi \in L^\infty(1/\omega)$ and $x \in G$ ${}_x\varphi(y) = \varphi(xy), \varphi_x(y) = \varphi(yx), \tilde{\varphi}(x) = \overline{\varphi(x^{-1})}\, y \in G$, and ${}_x\varphi, \tilde{\varphi}$ belong to $L^\infty(1/\omega)$ with $\|_x \varphi \|_\omega^\infty \leq \omega(x) \| \varphi \|_\omega^\infty, \| \tilde{\varphi} \|_\infty^\omega \leq \delta \| \varphi \|_\omega^\infty \leq \delta \| \varphi \|_\omega^\infty$. Moreover, for $f, g \in L^1(\omega)$ and $\varphi \in L^\infty(1/\omega)$ we have

$$< g, f^* * \varphi > = < f * g, \varphi >$$

Where $f * \varphi(x) = \int_G f(y)\varphi\left(y^{-1}x\right)$ dy and $f^* * \varphi(x) = \int_G \overline{f(y)}\varphi(yx)dy$. Let RUC $(1/\omega)$ be the closed subspace of $L^\infty(1/\omega)$ consisting of continuous functions $\varphi \in L^\infty(1/\omega)$ for which the mapping

$$G \longrightarrow L^\infty(x/\omega), \; x \longrightarrow {}_x\varphi \quad (x \in G)$$

is continuous.

H. Rosenthal classified in [8] the closed ideals in $L^1(G)$ for which $I^\perp$ is complemented in $L^\infty(G)$ in the case of an abelian group G. By Theorem 4 in [6] it turns out that the closed ideals $X_\perp$ of $L^1(G)$ has a bounded a.i. when X is invariantly complemented in $L^\infty(G)$, where

$$X_\perp = \{f \in L^1(G) :< f, \varphi >= 0, \; \varphi \in X\}$$

Recently Bekka in [2] has proved that a weak $*$-closed left translation invariant subspace X of $L^\infty(G)$ is invariantly complemented if and only if $X_\perp$ has a bounded right a.i. when G is locally compact group.

In this work, our main purpose is to generalize Bekka's results in [2] for a certain class of weighted functions.

Definition 1. *A weak $*$-closed left translation invariant subspace X of $L^{\infty}(1/\omega)$ is said to be invariantly complemented in $L^{\infty}(1/\omega)$ if X has a closed left translation invariant compmlemented in $L^{\infty}(1/\omega)$ or equivalently, if X is the range of a bounded projection on $L^{\infty}(1/\omega)$ or equivalently, if X is the range of a bounded projection on $L^{\infty}(1/\omega)$ commuting with left translation. Indeed, if X has a closed left translation invariant complement in $L^{\infty}(1/\omega)$ then there exists a closed subspace Y of $L^{\infty}(1/\omega)$ such that if $\varphi \in Y$, then ${}_x\varphi \in Y$ $(x \in G)$ and $L^{\infty}(1/\omega) \cong X \oplus Y$. Define a mapping P on $L^{\infty}(1/\omega)$ such that $P(\varphi_1 + \varphi_2) = \varphi_1$ where $\varphi = \varphi_1 + \varphi_2 \in L^{\infty}(1/\omega)$. It's routine to show that P satisfies the conditions of a bounded projetcion commuting with left translations.*

On the other hand if P is a bounded projection commuting with left translations on $L^{\infty}(1/\omega)$ with its range X, then the kernel of P is a left translation invariant subspace and complement of X in $L^{\infty}(1/\omega)$.

Definition 2. *We say that X is topologically invariantly complemented in $L^{\infty}(1/\omega)$ if X is the range of a bounded projection P on $L^{\infty}(1/\omega)$ such that $P(f * \varphi) = f * P(\varphi)$ for all $f \in L^1(\omega)$ and $\varphi \in L^{\infty}(1/\omega)$ where X is a weak $*$-closed left translation invariant subspace of $L^{\infty}(1/\omega)$.*

A close examination of the proof of Theorem 4.1 in [9] shows that a bounded projection on $L^{\infty}(1/\omega)$ commuting with left translations need not commute with left convolution by functions from $L^1(\omega)$. The notion of (topologically) invariant complemented subspace of RUC $(1/\omega)$ is defined in a similar way.

Now, we need the following lemmas.

Lemma 1. *If $\varphi \in RUC$ $(1/\omega)$, the ${}_x\varphi \in RUC$ $(1/\omega)$ for all x in G.*

Proof. Let $\varphi \in$ RUC$(1/\omega)$ and $x \in G$. Then for given any $\varepsilon > 0$ there exists a relatively compact neighborhood U of the unit e_G of G such that;

$$\|_s \varphi -_t \varphi \|_{\omega}^{\infty} \langle \varepsilon$$

when $st^{-1} \in U$. let W be an open subset of U which contains e_G. If $uv^{-1} \in x^{-1}Wx$, then we have $(xu)(xv)^{-1} \in W \subseteq U$ and so,

$$\|_{xu} \varphi -_{xv} \varphi \|_{\omega}^{\infty} = \|_u ({}_x\varphi) -_v ({}_x\varphi) \|_{\omega}^{\infty} \langle \varepsilon;$$

that is, we have found an open neighborhood $x^{-1}Wx$ of e_G such that;

$$\|_u ({}_x\varphi) -_v ({}_x\varphi) \|_{\omega}^{\infty} \langle \varepsilon$$

when $uv^{-1} \in x^{-1}Wx$. By definition of the right uniform continuity, ${}_x\varphi \in RUC(1/\omega)$, as required (see, for example ([7], ch.3, § 1.8) for the definition of the right and left uniform continuity). □

Lemma 2. *If $g \in L^1(\omega)$, then for each $\varepsilon > 0$ there exists a symmetric neighborhood V of e_G (then unit of G) such that;*

$$\|_x g -_y g\|_\omega^\infty \langle \varepsilon \quad \textit{when } xy^{-1} \in V.$$

Proof. $f \in L^1(\omega)$ with compact support K. By Lemma 20.4 (i) in [5] and ([7], p. 85, § 7.2) for given any $\varepsilon' > 0$, there exists a relatively compact neighborhood U of e_G such that $\|_u f - f\|_1 \langle \varepsilon'$ when $u \in U$. Let V be a symmetric neighborhood of e_G such that $V \subseteq U$. Then for all $v \in V, \|_v f - f\|_1 < \varepsilon'$.
On the other hand, for all $v \in V$ we have;

$$supp(_v f - f) \subseteq V^{-1}K \cup K = VK \cup K \subseteq \overline{U}K \cup K$$

and $\overline{U}K$ is compact (hence $\overline{U}K \cup K$ is compact).

Now set $Q = sup\{\omega(z) : z \in \overline{U}K \cup K\} < \infty$. If $v \in V$, then we obtain

$$\begin{aligned}
\|_v f - f\|_\omega^1 &= \int_G |(_v f - f)(y)|\,\omega(y)dy \\
&= \int_{supp(_v f - f)} |(_v f - f)(y)|\,\omega(y)dy \\
&\leq \int_{VK \cup K} |(_v f - f)(y)|\,\omega(y)dy \\
&\leq \int_{\overline{U}K \cup K} |(_v f - f)(y)|\,\omega(y)dy \\
&\leq Q\,\|_v f - f\|_1 < Q\varepsilon' = \varepsilon
\end{aligned}$$

which completes the proof of this Lemma. □

Lemma 3. *If $f \in L^1(\omega)$ and $\varphi \in L^\infty(1/\omega)$, then $f * \varphi \in RUC(1/\omega)$.*

Proof. Let $f \in L^1(\omega)$ and $\varphi \in L^\infty(1/\omega)$. Thus

$$\begin{aligned}
|f * \varphi(vx) - f * \varphi(x)| &= \left| \int_G f(t)\varphi(t^{-1}vx)dt - \int_G f(t)\varphi(t^{-1}x)dt \right| \\
&= \left| \int_G f(vxz^{-1})\varphi(z)\Delta(z^{-1})dz - \int_G f(xz^{-1})\varphi(z)\Delta(z^{-1})dz \right|
\end{aligned}$$

$$
\begin{aligned}
&= \left| \int_G (f(vxz^{-1}) - f(xz^{-1}))\overline{\tilde{\varphi}(z^{-1})}\Delta(z^{-1})dz \right| \\
&= \left| \int_G ({}_{vx}f -{}_x f)(z)\overline{\tilde{\varphi}(z)}dz \right| \\
&\leq \|{}_{vx} f -{}_x f \|_\omega^1 \| \tilde{\varphi} \|_\omega^\infty \\
&\leq \delta \, \|{}_{vx} f -{}_x f \|_\omega^1 \| \varphi \|_\omega^\infty
\end{aligned}
$$

where for the first term $z = t^{-1}vx$ and for the second term $z = t^{-1}x$.

Now by the Lemma 2, for given any $\varepsilon > 0$ there exists symmetric neighborhood V of e_G such that; $\|{}_{vx} f -{}_x f \|_\omega^1 \langle \varepsilon$ when $v \in V$. This inequality and boundedness from below of ω imply that $f * \varphi \in RUC(1/\omega)$ for each $f \in L^1(\omega)$ and $\varphi \in L^\infty(1/\omega)$, as required. □

Lemma 4. *let G be a locally compact group and let $L^1(\omega)$ be its weighted group algebra. If I is a closed left ideal in $L^1(\omega)$ then, its annihilator*

$$I^\perp = \{\varphi \in L^\infty(1/\omega) :< f, \varphi >= 0 \; for \; all \; f \in I\}$$

is a weak $*$-closed left translation invariant subspaces of $L^\infty(1/\omega)$. Conversely, if X is a weak $*$-closed left translation invariant subspaces of $L^\infty(1/\omega)$ then, its annihilator $X_\perp$ is a closed left ideal in $L^1(\omega)$.

Proof. Recall that $< g, f * \varphi >=< f^* * g, \varphi >$ for all $f, g \in L^1(\omega)$ where; $f^*(x) = \Delta(x^{-1})\overline{f(x^{-1})}$ and Δ denotes the modular function of G. If I is a left ideal in $L^1(\omega)$ then obviously, $I^\perp$ is a weak $*$-closed subspace of $L^\infty(1/\omega)$ and if $\{u_\alpha\}$ is a bounded a.i. of $L^1(\omega)$ then, we have

$$
\begin{aligned}
\langle f, {}_x \varphi\rangle = \lim_\alpha \langle u_\alpha * f, {}_x \varphi > &= \lim_\alpha \langle f, u_\alpha^* *_x \varphi\rangle \\
&= \lim_\alpha \langle \Delta(x) f, (u_\alpha^*)_x * \varphi\rangle \\
&= \lim_\alpha \langle \Delta(x) \left((u_\alpha^*)_x\right)^* * f, \varphi\rangle \\
&= 0 \, (since f \in I)
\end{aligned}
$$

for all $x \in G$ and $f \in I$ which implies that $I^\perp$ is left tanslation invariant.

Conversely, let X be a weak $*$-closed left translation invariant subspace of $L^\infty(1/\omega)$. Then for each $x \in G$ and $\varphi \in X$, we have ${}_x\varphi \in X$. Let $g \in X_\perp$ and $f \in L^1(\omega)$. Then

$$\langle f * g, \varphi\rangle = \int_G (f * g)(x)\overline{\varphi(x)}dx$$

$$
\begin{aligned}
&= \int_G \left\{ \int_G \Delta\left(t^{-1}\right) f\left(t^{-1}\right) g(tx)dt \right\} \overline{\varphi(x)}dx \\
&= \int_G \left\{ \int_G {}_t g(x)\overline{\varphi(x)}dx \right\} \Delta\left(t^{-1}\right) f(t^{-1})dt \\
&= \int_G \langle {}_t g, \varphi \rangle \Delta\left(t^{-1}\right) f\left(t^{-1}\right) dt \\
&= \int_G \langle g, {}_{t^{-1}} \varphi \rangle \Delta\left(t^{-1}\right) f\left(t^{-1}\right) dt = 0.
\end{aligned}
$$

It follows that $X_\perp$ is a left ideal in $L^1(\omega)$. Obviously, is closed. Hence $X_\perp$ is a closed left ideal of $L^1(\omega)$, as required. □

Theorem 5. *Let G be a locally compact group, X be a weak $*$-closed left translation invariant subspace of $L^\infty(1/\omega)$ and let $X_\perp$ be the annihilator of X in $L^1(\omega)$. Then the following statements are equivalent:*

(i) X is topologically invariantly complemented in $L^\infty(1/\omega)$;

(ii) X is invariantly complemented in $L^\infty(1/\omega)$;

(iii) $X \cap RUC(1/\omega)$ is invariantly complemented in $RUC(1/\omega)$;

(iv) $X \cap RUC(1/\omega)$ is topologically invariantly complemented in $RUC(1/\omega)$;

(v) The closed left ideal $X_\perp$ has a bounded right a.i..

Proof. *(i)* $\Longrightarrow$ *(ii)*; Let $P : L^\infty(1/\omega) \to X$ be a projection with $p(f * \varphi) = f * P_\varphi$ for all $f \in L^1(\omega), \varphi \in L^\infty(1/\omega)$. Let $\{e_\alpha\}$ be an a.i. for $L^1(\omega)$. By the equation $< g, f * \varphi >=< f^* * g, \varphi >$ and lemma 3 we have

$$
\begin{aligned}
\langle f, P({}_x\varphi)\rangle &= \lim_\alpha \langle e_\alpha * f, P\left({}_x\varphi\right)\rangle \\
&= \lim_\alpha \langle f, e_\alpha^* * P\left({}_x\varphi\right)\rangle \\
&= \lim_\alpha \langle f, \Delta(x) P\left((e_\alpha^*)_x * \varphi\right)\rangle \\
&= \lim_\alpha \langle f, \Delta(x)(e_\alpha^*)_x * P\varphi\rangle \\
&= \lim_\alpha \langle f, e_\alpha^* *_x (P\varphi)\rangle \\
&= \langle f, {}_x P\varphi\rangle
\end{aligned}
$$

which implies that ${}_x(P\varphi) = P({}_x\varphi)$ and so, x is invariantly complemented in $L^\infty(1/\omega)$.

To show $(ii) \Longrightarrow (iii)$; let $P : L^{\infty}(1/\omega) \to X$ be a bounded projection on X commuting with left translations. If $\varphi \in RUC(1/\omega)$ and $\varepsilon > 0$

$$\begin{aligned} \|_x P\varphi -_y P(\varphi) \|_\omega^\infty &= \| P(_x\varphi) - P(_y\varphi) \|_\omega^\infty = \| P(_x\varphi -_y \varphi) \|_\omega^\infty \\ &\leq \| P \| \, \|_x \varphi -_y \varphi \|_\omega^\infty \\ &< \| P \| \, \varepsilon \end{aligned}$$

whenever $xy^{-1} \in U$ where U is a neighborhood of e_G the unit of G. Hence $P\varphi \in RUC(1/\omega)$; that is, $P\varphi \in X \cap RUC(1/\omega)$. On the other hand, if $\varphi \in X \cap RUC(1/\omega)$ then $\varphi \in X$ and so, $P\varphi = \varphi$. Moreover, φ being an element of $RUC(1/\omega)$ the restriction $P\big|_{RUC(1/\omega)}$ of P to $RUC(1/\omega)$ is onto. By the definition of invariantly complemented of $X \cap RUC(1/\omega)$, it is an invariantly complemented subspace of $RUC(1/\omega)$, as required.

For the proof of implication $(iii) \Longrightarrow (iv)$ it is enough to show that the mapping P which is taken in the implication $(ii) \Longrightarrow (iii)$ satisfies the equality $P(f * \varphi) = f * P\varphi$ for all f in $L^1(\omega)$ and φ in $RUC(1/\omega)$. To this end, let $\varphi \in RUC(1/\omega)$ and, set $\Psi(y) = \int_G f(x)_{x^{-1}}\varphi(y)dx$.

Then we have $\Psi \in L^{\infty}(1/\omega)$ and

$$\begin{aligned} \|_y \Psi -_z \Psi \|_\infty^\infty &= \underset{u\in G}{ess\ sup} |_y\Psi(u) -_z \Psi(u)/\omega(u)| \\ &= \underset{u\in G}{ess\ sup} |\Psi(yu) - \Psi(zu)/\omega(u)| \\ &= \underset{u\in G}{ess\ sup} \left| \int_G f(x) \left\{ \varphi(x^{-1}yu) - \varphi(x^{-1}zu) \right\} dx \right| /\omega(u) \\ &= \underset{u\in G}{ess\ sup} \left| \int_G \left\{ f(yuv^{-1}) = f(zuv^{-1}) \right\} \varphi(v)\Delta(v^{-1})dv \right| /\omega(u) \\ &= \underset{u\in G}{ess\ sup} \left| \int_G (_{yu}f -_{zu} f)(v^{-1})\overline{\tilde{\varphi}(v^{-1})}\Delta(v^{-1})dv \right| /\omega(u) \\ &\leq \underset{u\in G}{ess\ sup} \int_G |_u(_yf -_z f)(v)| \, |\varphi(\tilde{v})| \, dv/\omega(u) \\ &\leq ess\ sup\, \omega(u^{-1})/\omega(u). \, \|_y f -_z f \|_\omega^1 \cdot \| \tilde{\varphi} \|_\omega^\infty \\ &\leq \delta^2 \, \|_y f -_z f \|_\omega^1 \| \varphi \|_\omega^\infty \end{aligned}$$

□

By Lemma 2 for give any $\varepsilon > 0$, there exists a symmetric neighborhood U of e_G such that $\|_y f -_z f \| < \varepsilon$ $(yz^{-1} \in U)$, and so, if $yz^{-1} \in U$, then

$$\|_y \Psi -_z \Psi \|_\omega^\infty < \varepsilon\delta^2 \| \varphi \|_\omega^\infty$$

which implies that Ψ belongs to $RUC(1/\omega)$. By the definition of the convolution product $\int_G f(x)_{x^{-1}}\varphi dx$ represents $f*g$. Thus it is not difficult to see that $P(f*\varphi)=f*P(\varphi)$.

To prove $(iv) \implies (v)$, let $P: RUC(1/\omega) \to X \cap RUC(1/\omega)$ be a bounded projection onto $X \cap RUC(1/\omega)$ with $P(f*\varphi)=f*P\varphi$ for all $f \in L^1(\omega), \varphi \in RUC(1/\omega)$. Define $P': L^\infty(1/\omega) \to L^\infty(1/\omega)$ by

$$< p, P'\varphi >= \overline{P(f^* * \varphi)(e_G)} \quad (\varphi \in L^1(1/\omega), f \in L^\infty(\omega))$$

To see that the range of P' is X, first observe that when $\varphi \in RUC(1/\omega)$ $\overline{P(f*\varphi)(e_G)} =< f, P\varphi >$; that is, $P'\big|_{RUC(1/\omega)} = P$. Let $\{e_\alpha\}$ be a bounded a.i. in $L^1(\omega)$. By Lemma 3, we obtain

$$\begin{aligned} < f, P\varphi >= \lim_\alpha < e_\alpha * f, P\varphi > &= \lim_\alpha \overline{P(f^* * e_\alpha^* * \varphi)(e_G)} \\ &= \lim_\alpha \overline{P(f^* * P(e_\alpha^* * \varphi))(e_G)} \\ &= \lim_\alpha < f, P(e_\alpha^* * \varphi) >= 0 \end{aligned}$$

which implies that $P'\varphi \in (X_\perp)^\perp = X$. Thus the range of P' is a subset of X.

If $\varphi \in X, g \in X_\perp$, then $< g, f^* * \varphi >=< f*g, \varphi >= 0$ and so $f^* * \varphi \in (X_\perp)^\perp = X$. Hence $f^* * \varphi \in X \cap RUC(1/\omega)$ and

$$< f, P'\varphi >= \overline{P(f^* * \varphi)(e_G)} = \overline{f^* * \varphi(e_G)} =< f, \varphi >;$$

that is, $P'\varphi = \varphi$ for all $\varphi \in X$ and so, the range of $P' is X$. Obviously, P' is linear and since

$$\begin{aligned} | < f, P'\varphi > | &= |P(f^* * \varphi)(e_G)| \\ &= \omega(e_G)\, |P(f^* * \varphi)(e_G)| \,/\omega(e_G) \\ &\leq \omega(e_G) \operatorname*{ess\ sup}_{x \in G} |P(f^* * \varphi)(x)| \,/\omega(x) \\ &= \omega(e_G) \parallel P(f^* * \varphi) \parallel_\omega^\infty \\ &\leq \omega(e_G) \parallel P \parallel \parallel f^* * \varphi \parallel_\omega^\infty \\ &\leq \delta\omega(e_G) \parallel P \parallel \parallel f \parallel_\omega^1 \parallel \varphi \parallel_\omega^\infty \end{aligned}$$

P' is continuous with $\parallel P' \parallel \leq \delta\omega(e_G) \parallel P \parallel$. Consequently P' is an extension of P to $L^\infty(1/\omega)$ as a bounded projection.

Now let $\{e_\alpha\}$ be an a.i for $L^1(\omega)$ bounded by M. Set $I = X_\perp$ and $C = (1 + \delta\ \omega(e_G) \parallel P \parallel)M$. Let E' be an element of $(L^\infty(1/\omega))'$ which is $\sigma((L^\infty(1/\omega)', L^\infty(1/\omega))$ closure of $\{e_\alpha\}$ in $L^1(\omega)$. Define a linear functional E on $L^\infty(1/\omega)$ such that

$$< \varphi, E >=< \varphi - P'\varphi, E' >$$

for all $\varphi \in L^\infty(1/\omega)$. Then $\| E \| \leq (1+ \| P' \|) \| E' \| \leq (1 + \delta\omega(e_G) \| P \|)M = C$ and $< \varphi, E >= 0$ for all $\varphi \in I^\perp$. Thus $E \in B_c(I'') = \{F \in I'' : \| F \| \leq C\}$ where I'' denotes the continuous bidual of I. By the Alaoglu Theorem $B_c(I)$ is weak $*$-dense in $B_c(I'')$, and so, there exists a net $\{u_\beta\}$ in $B_c(I)$ such that $u_\beta \to E$ with respect to $\sigma(I'', I')$ topology. Since X is complemented in $L^\infty(1/\omega)$, it is easy to see that, also $u_\beta \to E$ with restpect to $\sigma(L^1(\omega)'', L^\infty(1/\omega))$. It has a right a. i. bounded by C if and only if it has a weak right a.i. bounded by C (see, for example ([3])) and so, we need only to show that $\{u_\beta\}$ is a week right a.i. for I. To this end, let $f \in I, \varphi \in L^\infty(1/\omega)$. Then we have

$$\begin{aligned}
\lim_\beta < f * u_\beta - f, \varphi > &= \lim_\beta < f * u_\beta, \varphi > - < f, \varphi > \\
&= \lim_\beta < u_\beta, f^* * \varphi > - < f, \varphi > \\
&= < f^* * \varphi - P'(f^* \varphi), E' > - < f, \varphi > \\
&= < f^* * \varphi, E' > - < f, \varphi > - < P'(f^* * \varphi), E' > \\
&= \lim_\alpha < e_\alpha, f^* * \varphi > - < f, \varphi > - \lim_\alpha < e_\alpha, P'(f^* * \varphi) > \\
&= \lim_\alpha < f * e_\alpha, \varphi > - \lim_\alpha \overline{P(e_\alpha^* * f^* * \varphi)(e_G)} - < f, \varphi > \\
&= - \lim_\alpha \overline{P((f * e_\alpha)^* * \varphi)(e_G)} \\
&= - < f, P'\varphi >= 0 \quad \text{since } P'\varphi \in X)
\end{aligned}$$

which implies that $\{u_\beta\}$ is a bounded weak right a.i.. This completes the proof of the implication $(iv) \Longrightarrow (v)$.

Finally to show $(v) \Longrightarrow (i)$, let $\{u_\alpha\}$ be a bounded right a.i. for $X_\perp$ and E be the weak $*$-limit point of the net $\{u_\alpha\}$ in $L^1(\omega)''$. Then the mapping $P : L^\infty(1/\omega) \to L^\infty(1/\omega)$, defined by,

$$< f, P\varphi >=< f, \varphi > - < f^* * \varphi, E >$$

For all $f \in L^1(\omega)$ and $\varphi \in L^\infty(1/\omega)$. It is not difficult to verified that P is a bounded projection with range X. For all $f \in L^1(\omega), \varphi \in L^\infty(1/\omega)$ since

$$\begin{aligned}
< g, P(f^*\varphi) > &= < g, f^*\varphi > - < g^* * f * \varphi, E > \\
&= < f^* * g, \varphi > - < g^* * f * \varphi, E > \\
&= f, f * P\varphi > \quad (g \in L^1(\omega))
\end{aligned}$$

we have $P(f*\varphi) = f*P(\varphi)$ and it follows that X is topologically invariantly complemented in $L^\infty(1/\omega)$, as required. This completes the proof of the Theorem.

Remark 6. The involution $\tilde{\varphi}(x) = \overline{\varphi(x^{-1})}$ $\left(resp f^*(x) = \overline{f(x^{-1})}\Delta(x^{-1})\right)$ on $L^\infty(1/\omega)$ $\left(resp. L^1(\omega)\right)$ transforms every left invariant subspace of $L^\infty(1/\omega)$ (resp. left ideals of $L^1(\omega)$) into a right invariant subspace of $L^\infty(1/\omega)$ (resp. right ideals of $L^1(\omega)$). Furthermore, $(\tilde{X})_\perp = (X_\perp)^*$ for $X \subseteq L^\infty(1/\omega)$ and a left ideal I of $L^1(\omega)$ has a bounded right a.i. if and only if I^* has a bounded left a.i. since the involution "$*$" is continuous. Thus the Theorem 5 has an obvious version for right invariant subspaces of $L^\infty(1/\omega)$.

Remark 7. If we take $\omega \equiv 1_G$ $(1_G(x) = 1, x \in G)$, then $L^1(\omega) = L^1(G)$ and so, this study is clearly a generalization of the work [2].

We also would like to mention here that the equivalaence $(i) \implies (v)$ in the Theorem 5 is true for arbitrary Banach algebras (see, ([4], 4.1.4)

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$L^{\infty}(1/\omega)$ UZAYININ INVARYANT TAMLANMIŞ ALT UZAYLARI VE $L^1(\omega)$' NİN YAKLAŞIK BİRİME SAHİP İDEALLERİ

Özet

$L^{\infty}(\omega)$ lokal kompakt G grubu üzerinde ağırlıklı grup cebiri ve X de $L^{\infty}(1/\omega)$ nin zayıf $*$-topolojisine göre kapalı, sol ötelemeye göre invaryant olan bir alt uzay olsun. Bu çalışmada;

(i) $X, L^{\infty}(1/\omega)$ içinde topolojik olarak sol ötelemeye göre invaryant ve tamlanmış bir alt uzaydır;

(ii) $X, L^{\infty}(1/\omega)$ da sol ötelemeye göre invaryant ve tamlanmış uzaydır;

(iii) $L^1(\omega)$ içindeki her $X_{\perp}$ sol idealinin yaklaşık birimi vardır;

önermelerinin denk olduklarını gösteriyoruz. Dolayısıyla X in $L^{\infty}(1/\omega)$'da tamlanmış bir alt uzayı olması için gerek ve yeter şart $X_{\perp}$ nin en az bir yaklaşık biriminin olmasıdır.

Ziya ARGÜN
Karadeniz Technical University
Mathematics Department Gazi Education Fac.
Gazi University Teknik Okullar,
06500 Ankara-TURKEY
Cevriye TONYALI
Mathematics Department The Fac. of Arts and Sci.
Gazi University Teknik Okullar,
06500 Ankara-TURKEY

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