

RELATIONS BETWEEN CLASSES OF DRAGILEV SPACES $(f_1)_\sigma$ AND $(f_2)_\sigma$ FOR ARBITRARY FUNCTION $f_1^{-1} \circ f_2$

V. Kashirin, A. Vakulenko

Abstract

Necessary and sufficient condition under which Dragilev classes $(f_1)_\sigma$ and $(f_2)_\sigma$ coincide have been found for an arbitrary function $f_1^{-1} \circ f_2$.

Introduction

A matrix (a_{pn}) of non-negative scalars satisfying:

1. $\forall n, p \quad a_{pn} \leq a_{p+1, n} \quad ; \quad 2. \forall n \sup a_{pn} > 0.$

is called a Kothe matrix and the sequence space

$$\ell_1(a_{pn}) = \left\{ (\xi_n) : \|\xi_n\|_p = \sum_n |\xi_n| \cdot a_{pn} < \infty \forall p \right\}$$

topologized by seminorms $(\|\cdot\|_p)$ is called a Kothe space.

For each fixed function $f(u), u \in \mathbf{R}$, which is odd, increasing and logarithmically convex for $u \geq 0$, and for $\lambda \in \{-1, 0, 1, \infty\}$ we define the class $(f)_\lambda$ of Kothe spaces $L_f(a_n, \lambda) = \ell_1(a_{pn})$, where $a_{pn} = \exp f(\lambda_p, a_n)$, $(p, n = 1, 2, \dots)$, $a_n \uparrow \infty, \lambda_p \uparrow \lambda$ (introduced by Dragilev [1] and called Dragilev spaces or generalized power series spaces). In [1] M. Dragilev proved that if logarithmically convex functions f_1, f_2, φ are connected by relation $f_1(u) = f_2(\varphi(u))$ then the classes $(f_1)_{\delta_1}$ and $(f_2)_{\delta_2}$ coincide or do not intersect. In [2] ([3]) M. Kocatepe obtained the necessary and sufficient condition when the classes $(f_1)_\infty$ and $(f_2)_1$ (correspondingly, $(f_1)_0$ and $(f_2)_{-1}$) intersect, but $f_1^{-1} \circ f_2$ is not logarithmically convex.

Function $g(u)$ is called rapidly increasing, if for $\forall \alpha > 1 \lim_{u \rightarrow \infty} g(\alpha u)/g(u) = \infty$:

We shall consider rapidly increasing functions f, f_1, f_2 only but function $f_1^{-1} \circ f_2$ is arbitrary.

Following V. Zahariuta [4] we shall say that for locally convex spaces X and Y , $(X, Y) \in R$ if all continuous linear maps from X into Y are also compact.

Results.

Lemma 1. *Suppose $\ell_1(a_{pn})$ is a Schwartz Kothe space,*

$$a_{pn} \leq 1 \text{ and}$$

$$\forall p \exists q : \log(a_{qn}) / \log(a_{pn}) \rightarrow 0 \text{ for } n \rightarrow \infty \tag{1}$$

$$\text{If } \exists q_0 \forall p \liminf_n \frac{f^{-1}(\log a_{pn})}{f^{-1}(\log a_{q_0 n})} \geq 1 \tag{2}$$

then the relation $(L_f(b_n, -1), \ell_1(a_{pn})) \in R$ takes place for each $(b_n) \uparrow \infty$.

Proof. We shall use methods [4], [5]. Let $b_{pn} = \exp f \left[\left(-1 - \frac{1}{p}\right) \cdot b_n \right]$ and $T : \ell_1(b_{pn}) \rightarrow \ell_1(a_{pn})$ be linear and continuous, then T is represented by a matrix (t_{ij}) and continuity of T gives a strictly increasing function $q = q(p)$ of $\mathbf{N} \rightarrow \mathbf{N}$ and positive constant $C(p)$ such that:

$$\sup_n \sum_i |t_{in}| \frac{a_{pi}}{b_{qn}} \leq C(p) < \infty. \tag{3}$$

We shall now use the fact that $\ell_1(a_{pn})$ is a Montel space, and for the conclusion of the lemma it is enough to show:

$$\exists q_0 \forall p \exists M(p) : \sup_n \sum_i |t_{in}| \frac{a_{pi}}{b_{q_0 n}} \leq M(p) < \infty. \tag{4}$$

First pick p_1 using (2) so that:

$$\forall p \liminf_n \frac{f^{-1}(\log a_{pn})}{f^{-1}(\log a_{p_1 n})} \geq 1 \tag{5}$$

For the given p_1 we pick q_1 so that (3) takes place. Let $q_0 > q_1$, we can pick $C > 0$ such that:

$$\left(1 + \frac{1}{q_0}\right) \cdot (1 + C) < \left(1 + \frac{1}{q_1}\right). \tag{6}$$

Let p be fixed and set

$$N_n = N_n(C, p) = \left\{ i : - \left(1 + \frac{1}{q_0}\right) (1 + C) \cdot b_n \leq f^{-1}(\log a_{pi}) \right\} \tag{7}$$

Now consider

$$\sum_i |t_{in}| \frac{a_{pi}}{b_{q_0 n}} = \sum_1 + \sum_2,$$

where \sum_1 is for $i \notin N_n$ and \sum_2 is for $i \in N_n$.

First we estimate \sum_1 . Using (1) pick p_2 such that

$$\log(a_{p_2 n}) / \log(a_{pn}) \rightarrow 0 \quad (8)$$

For the given p_2 , pick q_2 for which the relation (3) takes place.

Consider

$$\sum_i = \sum_{i \notin N_n} |t_{in}| \frac{a_{pi}}{a_{p_2 i}} \frac{a_{p_2 i}}{b_{q_2 n}} \frac{b_{q_2 n}}{b_{q_0 n}} \leq C(p_2) \sup_{i \notin N_n} \exp 6_{in}^{(1)}(p),$$

where

$$\begin{aligned} 6_{in}^{(1)}(p) &= \log a_{pi} - \log a_{p_2 i} + f \left[\left(-1 - \frac{1}{q_2} \right) \cdot b_n \right] - \\ &- f \left[\left(-1 - \frac{1}{q_0} \right) \cdot b_n \right] \leq \log a_{pi} - \log a_{p_2 i} + \\ &+ f \left[\left(1 + \frac{1}{q_0} \right) \cdot b_n \right] \leq \log a_{pi} - \log a_{p_2 i} - \\ &- f \left[\left(\frac{1}{C+1} \right) \cdot f^{-1}(\log a_{pi}) \right]. \end{aligned}$$

Write $\phi = f^{-1} \circ \log$, we have now

$$6_{in}^{(1)} \leq \log a_{pi} \left[1 - \frac{\log a_{p_2 i}}{\log a_{pi}} - \frac{f \left(\frac{1}{1+C} \phi(a_{pi}) \right)}{f \left(\phi(a_{pi}) \right)} \right]$$

$f(u)$ is rapidly increasing, that is $6_{in}^{(1)} \leq 0$ for $i \geq i_0, i \notin N_n$; $6_{in}^{(1)}(p)$ is bounded, and thus $\sup_n \sum_1 \leq M_1(p) < \infty$.

Now consider \sum_2 :

$$\sum_2 = \sum_{i \in N_n} |t_{in}| \frac{a_{pi}}{a_{p_1 i}} \frac{a_{p_1 i}}{b_{q_1 n}} \frac{b_{q_1 n}}{b_{q_0 n}} \leq C(p_1) \sup_{i \in N_n} \exp 6_{in}^{(2)},$$

where

$$6_{in}^{(2)} = \log a_{pi} - \log a_{p_1i} + f \left[\left(-1 - \frac{1}{q_1} \right) \cdot b_n \right] - f \left[\left(-1 - \frac{1}{q_0} \right) \cdot b_n \right].$$

If $p < p_1$, then $6_{in}^{(2)} \leq 0$. Otherwise, we can find $\epsilon > 0$ from (6) such that

$$\left(1 + \frac{1}{q_0} \right) \cdot \frac{1+C}{1-\epsilon} < 1 + \frac{1}{q_1}.$$

For $i \geq i_0$ from (5) we have

$$f^{-1}(\log a_{pi}) \leq (1 - \epsilon) \cdot f^{-1}(\log a_{p_1i}).$$

In other words

$$\begin{aligned} 6_{in}^{(2)} &\leq -\log a_{p_1i} - f \left[\left(1 + \frac{1}{q_1} \right) \cdot b_n \right] + f \left[\left(1 + \frac{1}{q_0} \right) \cdot b_n \right] \leq \\ &\leq -f \left[\left(\frac{1}{1-\epsilon} \right) \cdot f^{-1}(\log a_{pi}) \right] + f \left[\left(1 + \frac{1}{q_0} \right) \cdot b_n \right] - \\ &- f \left[\left(1 + \frac{1}{q_1} \right) \cdot b_n \right] \leq f \left[\left(1 + \frac{1}{q_0} \right) \frac{1+C}{1-\epsilon} \cdot b_n \right] + \\ &+ f \left[\left(1 + \frac{1}{q_0} \right) \cdot b_n \right] - f \left[\left(1 + \frac{1}{q_1} \right) \cdot b_n \right]. \end{aligned}$$

$f(u)$ is rapidly increasing, also $6_{in}^{(2)}$ is bounded, and thus $\sup_n \sum_2 \leq M_2(p) < \infty$. \square

Lemma 2. *Let $\ell_1(b_{pn})$ be a Schwarts K 'othe space, $b_n \equiv 1 (n = 1, 2, \dots)$, $f(u)$ be rapidly increasing function and*

$$\forall p \exists q : \log(b_{pn}) / \log(b_{qn}) \rightarrow 0 \text{ for } n \rightarrow \infty \quad (9)$$

$$\text{If } \exists_{q_0} \forall_p \limsup_n \frac{f^{-1}(\log b_{pn})}{f^{-1}(\log b_{q_0n})} \leq 1 \quad (10)$$

then the relation $(\ell, (b_{pn}), L_f(a_n, 1)) \in R$ takes place for each $(a_n), a_n \uparrow \infty$.

Proof. We pick out arbitrary sequence $(a_n), a_n \uparrow \infty$ and let

$$a_{pn} = \exp f\left(\left(1 - \frac{1}{p}\right)a_n\right) \quad (n, p = 1, 2, \dots).$$

Let further $T : \ell_1(b_{pn}) \rightarrow \ell_1(a_{pn})$ be a linear continuous operator. Then

$$\forall_p \exists_q : \sup_n \sum_i |t_{in}| \frac{a_{pi}}{b_{qn}} \leq C(p) < \infty. \quad (11)$$

For Montel space $\ell_1(a_{pn})$ we need to prove that

$$\exists q_0 \forall_p \exists M(p) : \sup_n \sum_i |t_{in}| \frac{a_{pi}}{b_{q_0 n}} \leq M(p) < \infty. \quad (12)$$

For fixed index p_1 (for example, $p_1 = 1$) we can find q_1 such that condition (3) takes place. From (1), (2) and using the Schwarts property we can pick up $q_0 > q_1$ such that when $n \rightarrow \infty$ we have

$$\log(b_{q_1, n}) / \log(b_{q_0, n}) \rightarrow 0, \quad (13)$$

$$f^{-1}(\log b_{qn}) / f^{-1}(\log b_{q_0, n}) \rightarrow 1 \quad (14)$$

for every $q > q_0$, and $b_{q_0, n} \rightarrow \infty$

Further $\forall_p \exists C$ ($0 < C < 1$) such that

$$(1 - \frac{1}{p}) / (1 - C) < 1 - \frac{1}{p+1} \quad (15)$$

Let

$$N_h = N_h(C, p) = \{i : (1 - \frac{1}{p})a_i < (1 - C f^{-1}(\log b_{q_0 n}))\} \quad (16)$$

Let further

$$\sum_i |t_{in}| \frac{a_{pi}}{b_{q_0 n}} = \sum_1 + \sum_2$$

where

$$\sum_1 = \sum_{i \notin N_h} |t_{in}| \frac{a_{pi}}{b_{q_0 n}} \text{ and } \sum_2 = \sum_{i \in N_u} |t_{in}| \frac{a_{pi}}{b_{q_0 n}}$$

We begining from the sum \sum_1 .

For every p exists P_2 such that $p_2 > p + 1$. Using (3) for p_2 we pick q_2 . Further we have

$$\sum_1 = \sum_{i \notin N_n} |t_{in}| \frac{a_{pi}}{a_{p_2 i}} \cdot \frac{a_{p_2 i}}{b_{q_2 n}} \cdot \frac{b_{q_2 n}}{b_{q_0 n}} \leq C(p_2) \sup_{i \notin N_n} \exp 6_{in}^{(1)}(p),$$

where

$$\begin{aligned} 6_{in}^{(1)}(p) &= f\left(\left(1 - \frac{1}{p}\right)a_i\right) - f\left(\left(1 - \frac{1}{p_2}\right)a_i + \log b_{q_2n} - \log b_{q_0n}\right) \leq \\ &\leq f\left(\left(1 - \frac{1}{p}\right)a_i\right) - f\left(\left(1 - \frac{1}{p_2}\right)a_i\right) + \log b_{q_2n} \end{aligned}$$

Using (6), (7) we can find ϵ such that

$$f^{-1}(\log b_{q_2n}) \leq (1 + \epsilon)f^{-1}(\log b_{q_0n}) \quad (17)$$

for sufficiently large n and

$$\frac{\left(1 - \frac{1}{p}\right) \cdot (1 + \epsilon)}{1 - c} < 1 - \frac{1}{p+1} < 1 - \frac{1}{p_2} \quad (18)$$

The inequality (9) we rewrite in more convenient form

$$\log b_{q_2n} \leq f((1 + \epsilon)f^{-1}(\log b_{q_0n})) \quad (19)$$

Using (11) and the inequalities (8), (10) we have

$$\begin{aligned} 6_{in}^{(1)} &\leq f\left(\left(1 - \frac{1}{p}\right)a_i\right) - f\left(\left(1 - \frac{1}{p_2}\right)a_i\right) + \\ &+ f((1 + \epsilon)f^{-1}(\log b_{q_0n})) \leq \\ &\leq f\left(\left(1 - \frac{1}{p}\right)a_i\right) - f\left(\left(1 - \frac{1}{p_2}\right)a_i\right) + f\left(\left(1 - \frac{1}{p+1}\right)a_i\right) \leq \\ &\leq 0 \end{aligned}$$

for sufficiently large n , because f is a rapidly increasing function.

Now we consider \sum_2 .

$$\begin{aligned} \sum_2 &= \sum_{i \in N_u} |t_i n| \frac{a_{pi}}{a_{p_1 i}} \cdot \frac{a_{p_1 i}}{b_{q_1 n}} \cdot \frac{b_{q_1 n}}{b_{q_0 n}} \leq \\ &\leq C(P_1) \sup_{i \in N_n} \exp 6_{in}^{(2)}, \end{aligned}$$

where

$$\begin{aligned} 6_{in}^{(2)} &= f\left(\left(1 - \frac{1}{p}\right)a_i\right) - f\left(\left(1 - \frac{1}{p_1}\right)a_i\right) + \log b_{q_1n} - \log b_{q_0n} \leq \\ &\leq f\left(\left(1 - \frac{1}{p}\right)a_i\right) + \log b_{q_1n} - \log b_{q_0n} \end{aligned}$$

Using inequality (18) we have

$$6_{in}^{(2)} \leq f((1-C)f^{-1}(\log b_{q_0n})) + \log b_{q_1n} - \log b_{q_0n}$$

Introducing new function $\phi(u) = f^{-1}(\log u)$, we obtain

$$\begin{aligned} 6_{in}^{(2)} &= f((1-C)\phi(b_{q_0n})) + f(\phi(b_{q_1n})) - f(\phi(b_{q_0n})) = \\ &= f(\phi(b_{q_0n})) \cdot \left(\frac{f((1-C)\phi(b_{q_0n}))}{f(\phi(b_{q_0n}))} + \frac{\log b_{q_1n}}{\log b_{q_0n}} - 1 \right) \end{aligned}$$

Using the relation (5) we obtain that $6_{in}^{(2)} \leq 0$ for sufficiently large n , because f is a rapidly increasing function. \square

Theorem 1. *The classes $(f_1)_{-1}, (f_2)_{-1}$ coincide iff $\forall \alpha > 0$ the two inequalities take place:*

$$\liminf_u \frac{f_2^{-1} \left(f_1((1+\alpha) \cdot u) \right)}{f_2^{-1}(f_1(u))} > 1 \quad (20)$$

and

$$\liminf_u \frac{f_1^{-1} \left(f_2((1+\alpha) \cdot u) \right)}{f_1^{-1}(f_2(u))} > 1 \quad (21)$$

Proof. Sufficiency. For arbitrary sequence $(a_n) \uparrow \infty$ we consider sequence $b_n = f_2^{-1}(f_1(a_n))$. It is obvious that $(b_n) \uparrow \infty$. From (9) we have

$$\forall p \exists q, q > 1 : \frac{f_2^{-1} \left(f_1\left(\left(1 + \frac{1}{p}\right) \cdot u\right) \right)}{f_2^{-1}(f_1(u))} > 1 + \frac{1}{q}$$

for sufficiently large u . That is

$$f_2^{-1} \left(f_1\left(\left(1 + \frac{1}{p}\right) \cdot u\right) \right) \geq \left(1 + \frac{1}{q}\right) f_2^{-1}(f_1(u)),$$

$$f_1\left(\left(1 + \frac{1}{p}\right) \cdot u\right) \geq f_2 \left(\left(1 + \frac{1}{q}\right) f_2^{-1}(f_1(u)) \right),$$

$$f_1\left(\left(-1 - \frac{1}{p}\right) \cdot a_n\right) \leq f_2 \left(\left(-1 - \frac{1}{q}\right) \cdot b_n \right).$$

Analogously, from (10) we have

$$\forall p \exists q, q > 1 : f_1^{-1} \left(f_2 \left(\left(1 + \frac{1}{p} \right) \cdot u \right) \right) \leq \left(1 + \frac{1}{q} \right) f_1^{-1} (f_2(u))$$

or $f_2 \left(\left(-1 - \frac{1}{p} \right) \cdot b_n \right) \leq f_1 \left(\left(-1 - \frac{1}{q} \right) \cdot a_n \right)$. for sufficiently large n .

Necessity.

Assume, for example, that condition (20) does not take place, that is $\exists \alpha > 0$ and $(a_n) \uparrow \infty$ such that

$$\lim_{n \rightarrow \infty} \frac{f_2^{-1} \left(f_1 \left((1 + \alpha) a_n \right) \right)}{f_2^{-1} (f_1(a_n))} = 1.$$

Let $\ell_1(a_{pn})$ be a Kothe space with $a_{pn} = \exp f_1 \left[\left(-1 - \frac{1}{p} \right) a_n \right]$.

The function $f_1(u)$ is rapidly increasing, therefore the condition (1) takes place. Let $q_0 = 1/\alpha$, then

$$\liminf_n \frac{f_2^{-1}(\log a_{pn})}{f_2^{-1}(\log a_{q_0 n})} = \liminf_n \frac{f_2^{-1} \left(f_1 \left(\left(-1 - \frac{1}{p} \right) a_n \right) \right)}{f_2^{-1} \left(f_1 \left(\left(-1 - \frac{1}{q_0} \right) a_n \right) \right)} \geq \liminf_n \frac{f_2^{-1} \left(f_1(a_n) \right)}{f_2^{-1} \left(f_1 \left(\left(1 + \frac{1}{q_0} \right) a_n \right) \right)} = 1.$$

It means that the condition (2) of lemma 1 takes place and relation $(L_{f_2}(b_n, -1), L_{f_1}(a_n, -1)) \in R$ for every sequence (b_n) . The last means that $L_{f_1}(a_n, -1)$ is not isomorphic to some $L_{f_2}(b_n, -1)$ or classes $(f_1)_{-1}$ and $(f_2)_{-1}$ is not coincide. \square

Theorem 2. *The classes $(f_1)_{-1}$ and $(f_2)_{-1}$ do not intersect if one of the two conditions takes place:*

$$\forall (a_n) \exists (n(j)) \exists q > 1 : \lim_j \frac{f_2^{-1} \left(f_1 \left(\left(1 + \frac{1}{q} \right) \cdot a_{n(j)} \right) \right)}{f_2^{-1} (f_1(a_{n(j)}))} = 1 \quad (22)$$

or

$$\forall (a_n) \exists (n(j)) \exists q > 1 : \lim_j \frac{f_1^{-1} \left(f_1 \left(\left(1 + \frac{1}{q} \right) \cdot a_{n(j)} \right) \right)}{f_1^{-1} (f_2(a_{n(j)}))} = 1.$$

Proof. Assume that there are two isomorphic spaces $L_{f_1}(a_n, -1)$ and $L_{f_2}(b_n, -1)$ but the condition (11) holds.

Let $\exists q \exists (n(j))$:

$$\begin{aligned} & \liminf_j \frac{f_2^{-1} \left(f_1 \left(\left(-1 - \frac{1}{p} \right) \cdot a_{n(j)} \right) \right)}{f_2^{-1} \left(f_1 \left(\left(-1 - \frac{1}{q} \right) \cdot a_{n(j)} \right) \right)} \geq \\ & \geq \liminf_j \frac{f_2^{-1} (f_1(a_{n(j)}))}{f_2^{-1} \left(f_1 \left(\left(1 + \frac{1}{q} \right) \cdot a_{n(j)} \right) \right)} = 1. \end{aligned}$$

Using lemma 1 for a $a_{pj} = \exp f_1 \left(\left(-1 - \frac{1}{p} \right) a_{n(j)} \right)$ we have $(L_{f_2}(b_n, -1), L_{f_1}(a_{n(j)}, -1)) \in R$. Now it is easy to show that $L_{f_1}(a_n, -1)$ is not isomorphic to $L_{f_2}(b_n, -1)$ and we get a contradiction.

The next theorems are proved analogously. □

Theorem 3. *The classes $(f_1)_1, (f_2)_1$ coincide iff $\forall \alpha > 0$ the two inequalities take place:*

$$\liminf_u \frac{f_2^{-1} \left(f_1 \left((1 + \alpha) \cdot u \right) \right)}{f_2^{-1} (f_1(u))} > 1$$

and

$$\liminf_u \frac{f_1^{-1} \left(f_2 \left((1 + \alpha) \cdot u \right) \right)}{f_1^{-1} (f_2(u))} > 1$$

Theorem 4. *The classes $(f_1)_1$ and $(f_2)_1$ do not intersect if one of the two conditions takes place:*

$$\forall (a_n) \exists (n(j)) \exists q > 1 : \lim_j \frac{f_2^{-1} \left(f_1 \left(\left(1 - \frac{1}{q} \right) \cdot a_{n(j)} \right) \right)}{f_2^{-1} (f_1(a_{n(j)}))} = 1$$

or

$$\forall (a_n) \exists (n(j)) \exists q > 1 : \lim_j \frac{f_1^{-1} \left(f_2 \left(\left(1 - \frac{1}{q} \right) \cdot a_{n(j)} \right) \right)}{f_1^{-1} (f_2(a_{n(j)}))} = 1$$

The same theorems for classes $6 = 0, \infty$ were obtained in [6].

References

- [1] Dragilev, M.M., On Regular Bases in Nuclear Spaces, Amer. Math. Soc. Transl. 2, 93, 62-82, 1970.
- [2] Kocatepe, M. An Isomorphism Theorem for Dragilev Spaces. Arch. Math. 50, 281-286, 1988.
- [3] Kocatepe, M. Classification of Dragilev Spaces of Types -1 and 0. Math. Balkanica 2, 2-3, 265-275, 1988.
- [4] Zahariuta, V.P., On the Isomorphism of Cartesian Products of Locally Convex Spaces. Studia Math. 46, 201-221, 1973.
- [5] Alpseymen, M., Ramanujan, M.S., Terzioglu, T. Subspaces of Some Nuclear Sequence Spaces. Indag. Math. 82, 217-224, 1979.
- [6] Kashirin, V., Vakulenko, A. Classification of Relations Between Classes $(f_1)_\sigma$ and $(f_2)_\sigma$ for arbitrary function $f_1^{-1} \circ f_2$. (to appear).

DRAGİLEV UZAYLARI ARASINDAKİ BAZI BAĞLANTILAR

Özet

Bu çalışmada keyfi f_1 ve f_2 için $(f_1)_\sigma$ ve $(f_2)_\sigma$ Dragilev uzaylarının çalışmaları için yeter ve gerek koşullar verilmektedir.

V. KASHIRIN, A. VAKULENKO
Rostov State,
Building Academy

Received 21.11.1994