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Stochastic stability of the discrete-time constrained extended Kalman filter

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Abstract

In this paper, stability of the projection-based constrained discrete-time extended Kalman filter (EKF) as applied to nonlinear systems in a stochastic framework has been studied. It has been shown that like the unconstrained EKF, the estimation error of the EKF with known constraints on the states remains bounded when the initial error and noise terms are small, and the solution of the Riccati difference equation remains positive definite and bounded. Stability results are verified and performance of the constrained EKF is demonstrated through simulations on a nonlinear engineering example.

Key Words: *Extended Kalman filter, constrained Kalman filter, stochastic stability.*

1. Introduction

The Kalman filter, under Gaussian assumption, is the optimal state estimator for linear dynamic systems as it uses all the available information about the system in order to obtain a state estimate. Although originally devised for linear systems, nonlinear systems can also be addressed by the Kalman filter through some modifications to it as approximations to the optimal state estimator. The extended Kalman filter (EKF) is one of the most popular estimation techniques that has been largely investigated for state estimation of nonlinear systems [1-4]. The EKF uses the standard Kalman filter equations to the first-order approximation of the nonlinear model about the last estimate. It is very sensitive to initialization, and filter divergence is inevitable if the arbitrary noise matrices have not been chosen appropriately. There are many research papers that address these issues as well as analyze the stability and robustness of the filter [4-11].

When Kalman filters are used in state estimation, it is often assumed that the system model is known and any additional information about the system is overlooked. However, in the application of state estimators, there may be specific additional information about the system that the standard Kalman filter does not incorporate.

For instance, there may be defined constraints on the states that would help produce better estimates if utilized. However, such information regarding state constraints are not incorporated in the standard Kalman filter. In such cases, the Kalman filter can be modified to exploit this additional information. There are many ways to incorporate the constraint information which require modifying the Kalman filter [12]. If either the system and/or measurements are nonlinear with nonlinear or linear constraints, then the resulting modified filter is classified as a constrained extended Kalman filter.

Recently, constrained Kalman filtering has become a focus of increased attention and the use of state constraints has increased in practical engineering problems, both in linear and nonlinear systems. Applications where constraints are utilized vary from chemical processes [13] to target tracking [14], from vision-based systems [15] to biomedical systems [16], from robotics [17] to navigation [18], as well as fault diagnosis [19]. Thus, analysis of the stability and/or the stability conditions of the constrained EKF is of great importance. The convergence and stability properties of the constrained EKF have been treated for the zero noise case in [20], where it was proved that the EKF is exponentially stable for deterministic nonlinear systems; i.e., the estimation error is bounded, even when the states are constrained. In this paper, the previous study has been extended by removing the restriction imposed on the noise term. Then, using the direct method of Lyapunov, it has been proved that under certain conditions, the EKF is still an exponential observer; i.e., the dynamics of the estimation error is exponentially stable even when the states are constrained. This is an important outcome as real life systems are generally not noise free.

This paper discusses the stability of stochastic discrete-time extended Kalman filters when applied to nonlinear systems with state constraints. First, the stochastic stability of the unconstrained extended Kalman filter is considered; then the analysis is extended to the projection-based constrained extended Kalman filter. Due to stochasticity, the exponential stability of the nonlinear system is analyzed in the mean square error sense. Estimation projection is an analytical method of incorporating state equality constraints in the Kalman filter, where the filter was generalized in such a way that known relations among the state variables are satisfied by the state estimate [18]. The main contribution of this work is proving that under certain conditions, the estimation error of the extended Kalman Filter, when the states are constrained, remains bounded.

In Section 2, we recall the state estimation problem for nonlinear stochastic discrete-time systems when the states are constrained, and present some auxiliary results from stochastic stability theory. In Section 3, the constrained extended Kalman Filter is introduced and similar to the unconstrained case in [6], the boundedness of the error is proved. Section 4 presents some numerical simulation results, which verify that the estimation error for the stochastic constrained EKF remains bounded if the conditions are met. Conclusions drawn from these results are also discussed.

2. State estimation and stochastic boundedness

Consider a nonlinear discrete-time system defined by

$$x_{n+1} = f(x_n, u_n) + G_n w_n \tag{1}$$

$$y_n = h(x_n) + H_n v_n \tag{2}$$

where $n \in N_0$ is the discrete time index, $x_n \in R^q$ is the state, $u_n \in R^p$ is the known input, and $y_n \in R^m$ is the output. Moreover, v_n and w_n are R^k and R^l valued uncorrelated zero-mean white noise processes with identity covariance, respectively, and H_n and G_n are time varying matrices of size $m \times k$ and $q \times l$, respectively. The functions f and h are assumed to be C^1-functions.

Where there are known relationships among the state components, there is an additional constraint:

$$Dx_n = d_n \tag{3}$$

where D is a known $s \times q$ constant matrix, d_n is a known $s \times 1$ vector, s is the number of constraints, q is the number of states, and $s \leq q$. Also, it is assumed that D is full rank[1], i.e.,$rank(D) = s$.

For this system, the unconstrained state estimate is given by

$$\widehat{x}_{n+1} = f(\widehat{x}_n, u_n) + K_n(y_n - h(\widehat{x}_n)) \tag{4}$$

where the observer gain K_n is a matrix-valued stochastic process of size $q \times m$. Using the unconstrained state estimate $\widehat{x}_n$, the constrained state estimate $\widetilde{x}_n$ can be given as [18]

$$\widetilde{x}_n = \widehat{x}_n - W^{-1}D^T(DW^{-1}D^T)^{-1}(D\widehat{x}_n - d_n) \tag{5}$$

where W is any symmetric positive definite matrix.

Since, f and h are C^1-functions, they can be expanded via

$$f(x_n, u_n) - f(\widehat{x}_n, u_n) = A_n(x_n - \widehat{x}_n) + \varphi(x_n, \widehat{x}_n, u_n) \tag{6}$$

and

$$h(x_n) - h(\widehat{x}_n) = C_n(x_n - \widehat{x}_n) + \chi(x_n, \widehat{x}_n) \tag{7}$$

with

$$A_n = \frac{\partial f}{\partial x}(\widehat{x}_n, u_n) \tag{8}$$

$$C_n = \frac{\partial h}{\partial x}(\widehat{x}_n) \tag{9}$$

We define the constrained estimation error as

$$\vartheta_n = x_n - \widetilde{x}_n \tag{10}$$

$$\vartheta_n = x_n - (\widehat{x}_n - W^{-1}D^T(DW^{-1}D^T)^{-1}(D\widehat{x}_n - d_n)).$$

If the unconstrained estimation error is given by [6]

$$\varsigma_n = x_n - \widehat{x}_n \tag{11}$$

then employing (3) yields

$$\vartheta_n = (I - W^{-1}D^T(DW^{-1}D^T)^{-1}D)\varsigma_n. \tag{12}$$

Taking the recursive expression for the unconstrained estimation error directly from [6],

$$\varsigma_{n+1} = (A_n - K_nC_n)\varsigma_n + r_n + s_n \tag{13}$$

[1] If D is not full rank, then there are redundant state constraints, which must be removed.

where

$$r_n = \varphi(x_n, \widehat{x}_n, u_n) - K_n \chi(x_n, \widehat{x}_n) \tag{14}$$

$$s_n = G_n w_n - K_n H_n v_n. \tag{15}$$

Thus, the constrained estimation error is given by

$$\vartheta_{n+1} = (I - W^{-1}D^T(DW^{-1}D^T)^{-1}D)\varsigma_{n+1}. \tag{16}$$

For the analysis of the constrained error dynamics in (16), let us recall two definitions [21, 22]:

Definition 1. The stochastic process ϑ_n is said to be exponentially bounded in the mean square sense if there are real numbers $\eta, \nu > 0$ and $0 < \varphi < 1$ such that

$$E\{||\vartheta_n||^2\} \leq \eta||\vartheta_0||^2\varphi^n + \nu \tag{17}$$

is satisfied for every $n \geq 0$.

Definition 2. The stochastic process ϑ_n is said to be bounded with probability one, if

$$\sup_{n\geq 0} \|\vartheta_n\| < \infty \tag{18}$$

holds with probability one.

Next, some standard results concerning the boundedness of stochastic processes are given.

Lemma 1. Assume there is a stochastic process $V_n(\vartheta_n)$ and real numbers $\underline{v}, \overline{v}, \mu$ and $0 < \alpha \leq 1$ such that

$$\underline{v}\,\|\vartheta_n\|^2 \leq V_n(\vartheta_n) \leq \overline{v}\,\|\vartheta_n\|^2 \tag{19}$$

and

$$E(V_{n+1}(\vartheta_{n+1})\,|\vartheta_n) - V_n(\vartheta_n) \leq \mu - \alpha V_n(\vartheta_n) \tag{20}$$

are satisfied for every solution of (16). Then, the stochastic process is exponentially bounded in the mean square sense: i.e., we have

$$E(\|\vartheta_n\|^2) \leq -\frac{\overline{v}}{\underline{v}} E(\|\vartheta_0\|^2)(1-\alpha)^n + \frac{\mu}{\underline{v}}\sum_{i=1}^{n-1}(1-\alpha)^i$$

for every $n \geq 0$. Moreover, the stochastic process is bounded with probability one.

Proof. See [6], [21], [22], [23], [24] for proof.

3. Error bounds for the constrained extended Kalman filter

Definition 3. A discrete-time constrained extended Kalman Filter is given by the following coupled difference equations:

$$\widehat{x}_{n+1} = f(\widehat{x}_n, u_n) + K_n(y_n - h(\widehat{x}_n)),$$

and after each step,

$$\widetilde{x}_{n+1} = \widehat{x}_{n+1} - W^{-1}D^T(DW^{-1}D^T)^{-1}(D\widehat{x}_{n+1} - d_{n+1}),$$

and the Riccati difference equation:

$$P_{n+1} = A_n P_n A_n^T + Q_n - K_n(C_n P_n C_n^T + R_n)K_n^T \tag{21}$$

where

$$A_n = \frac{\partial f}{\partial x}(\widehat{x}_n, u_n) \tag{22}$$

$$C_n = \frac{\partial h}{\partial x}(\widehat{x}_n) \tag{23}$$

where K_n is the Kalman gain given by

$$K_n = A_n P_n C_n^T (C_n P_n C_n^T + R)^{-1}. \tag{24}$$

In the above equations, Q_n and R_n are the symmetric, positive definite covariance matrices of the process and measurement noise, respectively.

Typically, the matrices Q_n and R_n are expressed in terms of the covariance of the corrupting noise terms in (1) and (2) as

$$Q_n = G_n G_n^T$$

$$R_n = H_n H_n^T.$$

Theorem 1. Consider a nonlinear stochastic system described by (1), (2) and a constrained Kalman Filter as stated in Definition 3. Let the following assumptions hold:

1. There are positive real numbers $\overline{a}, \overline{c}, \underline{p}, \overline{p}, \overline{w}, \overline{d} > 0$ such that the following bounds on various matrices are fulfilled for every $n \geq 0$:

2.

$$||A_n|| \leq \overline{a} \tag{25a}$$

$$||C_n|| \leq \overline{c} \tag{25b}$$

$$\underline{p}I \leq P_n \leq \overline{p}I \tag{25c}$$

$$\underline{q}I \leq Q_n \tag{25d}$$

$$\underline{r}I \leq R_n \tag{25e}$$

$$||W|| \leq \overline{w} \tag{25f}$$

$$||D|| \leq \overline{d} \tag{25g}$$

   Note that (25f) and (25g) are imposed due to the state equality constraints.

3. A_n is nonsingular for every $n \geq 0$.

4. There are positive real numbers $\in_{\varphi}, \in_{\chi}, \kappa_{\varphi}, \kappa_{\chi} > 0$ such that the nonlinear functions φ, χ in (14) are bounded via

5.

$$||\varphi(x, \widehat{x}, u)|| \leq \kappa_{\varphi}||x - \widehat{x}||^2 \tag{26a}$$

$$||\chi(x, \widehat{x})|| \leq \kappa_{\chi}||x - \widehat{x}||^2. \tag{26b}$$

Then, the estimation error ϑ_n given by (12) is exponentially bounded in the mean square sense with probability one, provided that the initial estimation error satisfies

$$||\vartheta_0|| <\in \tag{27}$$

and the covariance matrices of the noise terms are bounded via

$$G_n G_n^T \leq \delta I \tag{28}$$

$$H_n H_n^T \leq \delta I \tag{29}$$

for some $\delta, \in > 0$.

The proof of this theorem for unconstrained states is divided into several lemmas in [6]. The same lemmas will be restated here as they are used to prove Theorem 1.

Lemma 2. Under the conditions of Theorem 1, there is a real number $0 < \alpha < 1$ such that $\Pi_n = P_n^{-1}$ satisfies the inequality

$$(A_n - K_n C_n)^T \Pi_{n+1}(A_n - K_n C_n) \leq (1 - \alpha)\Pi_n \tag{30}$$

for $n \geq 0$ with K_n given by (24).

Proof. See [6] for proof.

Lemma 3. Let the conditions of Theorem 1 be satisfied, and $\Pi_n = P_n^{-1}$ and K_n, r_n be given by (24) and (14), respectively. Then, there are positive real numbers $\in', K_{nonl} > 0$ such that

$$r_n^T \Pi_n[2(A_n - K_n C_n)(x_n - \widehat{x}_n) + r_n] \leq K_{nonl}||x_n - \widehat{x}_n||^3 \tag{31}$$

holds for $||x_n - \widehat{x}_n|| \leq \in'$ and $\kappa_{nonl} = \kappa' \frac{1}{\underline{p}}(2(\overline{a} + \overline{a}\ \overline{p}\ \overline{c}\frac{1}{\underline{r}}\overline{c}) + \kappa' \in')$.

Proof. See [6] for proof.

Lemma 4. Let the conditions of Theorem 1 hold, and $\Pi_n = P_n^{-1}$ and K_n, s_n be given by (24) and (15), respectively. Then, there is a positive real number $K_{noise} > 0$ independent of δ, such that

$$E\{s_n^T \Pi_{n+1} s_n\} \leq K_{noise}\delta \tag{32}$$

holds with $K_{noise} = \frac{q}{\underline{p}} + \frac{\overline{a}^2\overline{c}^2\overline{p}^2 m}{\underline{p}\underline{r}^2}$ and

$$trace(G_n G_n^T) \leq \delta trace(I) = q\delta$$
$$trace(H_n H_n^T) \leq \delta trace(I) = m\delta.$$

Proof. See [6] for proof.

Proof of Theorem 1. In the unconstrained case, the function

$$V_n(\varsigma_n) = \varsigma_n^T \Pi_n \varsigma_n$$

has been chosen for the proof. Here, for the constrained case we choose

$$V_n(\vartheta_n) = \vartheta_n^T (Cov(\vartheta_n))^{-1} \vartheta_n \tag{33}$$

with $\vartheta_n = (I - W^{-1}D^T(DW^{-1}D^T)^{-1}D)\varsigma_n$ and $\prod_n = P_n^{-1}$, which exists since P_n is positive definite. From (25c) we have

$$\frac{1}{4\,\bar{p}}||\vartheta_n||^2 \leq V_n(\vartheta_n) \leq \frac{1}{4\underline{p}}||\vartheta_n||^2 \tag{34}$$

i.e., (19) with $\underline{v} = {}^1\!/_{\overline{p}}$ and $\overline{v} = {}^1\!/_{\underline{p}}$.

$$\vartheta_{n+1} = (I - W^{-1}D^T(DW^{-1}D^T)^{-1}D)((A_n - K_nC_n)\varsigma_n + r_n + s_n)$$

and

$$Cov(\vartheta_{n+1}) = (I - W^{-1}D^T(DW^{-1}D^T)^{-1}D)Cov(\varsigma_{n+1})(I - W^{-1}D^T(DW^{-1}D^T)^{-1}D)^T$$
$$V_{n+1}(\vartheta_{n+1}) = \vartheta_{n+1}^T(Cov(\vartheta_{n+1}))^{-1}\vartheta_{n+1}$$

so we have

$$\begin{aligned} V_{n+1}(\vartheta_{n+1}) = \; & \varsigma_{n+1}^T(I - W^{-1}D^T(DW^{-1}D^T)^{-1}D)^T(I - W^{-1}D^T(DW^{-1}D^T)^{-1}D)^{-T}\Pi_{n+1} \\ & \times(I - W^{-1}D^T(DW^{-1}D^T)^{-1}D)^{-1}(I - W^{-1}D^T(DW^{-1}D^T)^{-1}D)\varsigma_{n+1} \end{aligned}$$

$$V_{n+1}(\vartheta_{n+1}) = V_{n+1}(\varsigma_{n+1})$$

To satisfy the requirements for application of Lemma 1, we need an upper bound on $E(V_{n+1}(\vartheta_{n+1})\,|\vartheta_n\,)$ as in (20). From (16), we have

$$\begin{aligned} V_{n+1}(\vartheta_{n+1}) = \; & (s_n^T + r_n^T + \varsigma_n^T(A_n - K_nC_n)^T)(I - W^{-1}D^T(DW^{-1}D^T)^{-1}D)^T \\ & (I - W^{-1}D^T(DW^{-1}D^T)^{-1}D)^{-T}\prod\nolimits_{n+1}(I - W^{-1}D^T(DW^{-1}D^T)^{-1}D) \\ & (I - W^{-1}D^T(DW^{-1}D^T)^{-1}D)^{-1}((A_n - K_nC_n)\varsigma_n + r_n + s_n) \end{aligned}$$

and applying Lemma 1, we obtain in conjunction with (33)

$$\begin{aligned} V_{n+1}(\vartheta_{n+1}) \leq \quad & (1-\alpha)V_n(\varsigma_n) + r_n^T \prod_{n+1}(2(A_n - K_nC_n)\varsigma_n + r_n) \\ & +2s_n^T \prod_{n+1}((A_n - K_nC_n)\varsigma_n + r_n) + s_n^T \Pi_{n+1} s_n \end{aligned} \tag{35}$$

Taking the conditional expectation $E\left[V_{n+1}(\vartheta_{n+1})\,|\vartheta_n\right]$ and considering the white noise property, it can be seen that the term $E\left[s_n^T \Pi_{n+1}((A_n - K_nC_n)\varsigma_n + r_n)\,|\vartheta_n\right]$ vanishes, since neither $\prod_{n+1} = P_{n+1}^{-1}$ nor $A_n, C_n, K_n, r_n, \varsigma_n$, depend on v_n or w_n. The remaining terms are estimated via Lemmas 2 and 3, yielding

$$E(V_{n+1}(\vartheta_{n+1})\,|\vartheta_n) - V_n(\vartheta_n) \leq -\alpha V_n(\vartheta_n) + K_{nonl}||\varsigma_n||^3 + K_{noise}\delta \tag{36}$$

for $||\vartheta_n|| \leq 2||\varsigma_n|| \leq 2 \in'$. Defining

$$\in = \min\left(\in', \frac{\alpha}{2\overline{p}K_{nonl}}\right) \tag{37}$$

with (33) and (34) for $\|\vartheta_n\| \leq 2 \in$, we obtain

$$K_{nonl}||\varsigma_n||||\varsigma_n||^2 \leq \frac{\alpha}{2\overline{p}}||\varsigma_n||^2 \leq \frac{\alpha}{2}V_n(\varsigma_n) = \frac{\alpha}{2}V_n(\vartheta_n).$$

Inserting this into (36) yields

$$E\left[V_{n+1}(\vartheta_{n+1})\,|\vartheta_n\right] - V_n(\vartheta_n) \leq -\frac{\alpha}{2}V_n(\vartheta_n) + K_{noise}\delta \tag{38}$$

for $|\vartheta_n|| \leq 2 \in$. Therefore, we are able to apply Lemma 1 with $||\vartheta_0|| \leq 2 \in$, $\underline{v} = {}^1\!/_{\overline{p}}$, $\overline{v} = {}^1\!/_{\underline{p}}$ and $\mu = \kappa_{noise}\delta$. However, we have to ensure that for $2\widetilde{\in} \leq ||\vartheta_n|| \leq 2 \in$ with some $\widetilde{\in} < \in$ the supermartingale inequality

$$E\left[V_{n+1}(\vartheta_{n+1})\,|\vartheta_n\right] - V_n(\vartheta_n) \leq -\frac{\alpha}{2}V_n(\vartheta_n) + K_{noise}\delta \leq 0 \tag{39}$$

is fullfilled to guarantee that the estimation error is bounded [25]. Choosing

$$\delta = \frac{\alpha\widetilde{\in}^2}{2\overline{p}K_{noise}} \tag{40}$$

with some $\widetilde{\in} < \in$, we have for $||\vartheta_n|| \geq 2\widetilde{\in}$

$$K_{noise}\delta \leq \frac{\alpha}{2\overline{p}}\frac{\|\vartheta_n\|^2}{4} \leq \frac{\alpha}{2}V_n(\vartheta_n);$$

i.e., (39) holds, which completes the proof.

4. Simulations

In the preceding section it was proved that the estimation error of the projection-based, discrete-time, constrained extended Kalman filter remains bounded under certain assumptions. These assumptions include the requirement of a sufficiently small initial estimation error and sufficiently small noise. Some numerical simulations are presented in this section in order to illustrate the significance of these assumptions. For this purpose, a nonlinear stochastic system is considered with constraints imposed on the parameters to be estimated. The proposed system models the ingestion and subsequent metabolism of a drug in a given individual. A two compartment model is used to characterize the ingestion, distribution and metabolism of the drug in the individual [4].

Let x_1 and x_2 denote the drug mass in the first compartment, the gastrointestinal tract, and in the second compartment, the bloodstream of the individual, respectively. The rate of change of the drug mass in the gastrointestinal tract is equal to the rate at which the drug is ingested minus the rate at which the drug is distributed from the gastrointestinal tract to the bloodstream. The latter rate is assumed to be proportional to the drug mass in the gastrointestinal tract. k_1 is a positive constant characterizing the gastrointestinal tract of the given individual. The rate of change of the drug mass in the bloodstream is equal to the rate at which the drug is distributed from the gastrointestinal tract minus the rate at which the drug is metabolized and eliminated from bloodstream. The positive constant, k_2, characterizes the metabolic and excretory processes of the individual. The output variable, y, is the drug mass, x_2, in the bloodstream, as this is the variable that is indicative of the effect of the drug on the individual. The state variables are given by the drug mass in the gastrointestinal tract, x_1, and the mass of drug in the bloodstream. Thus, the state space model of the two-compartment model is given by

$$x_{n+1} = \begin{bmatrix} x_{1,n+1} \\ x_{2,n+1} \end{bmatrix} = \begin{bmatrix} 1-k_1\Delta_t & 0 \\ k_1\Delta_t & 1-k_2\Delta_t \end{bmatrix} \begin{bmatrix} x_{1,n} \\ x_{2,n} \end{bmatrix} + G_n w_n \tag{41}$$

$$y_n = [0\ 1]\, x_n + H_n v_n \tag{42}$$

where Δ_t is the integration-time interval sub-divider, which is chosen as 0.01, and it is assumed that $k_1 \neq k_2$. Typically, the physiological constants satisfy the inequality $k_1 > k_2$, due to the fact that x_1 decays more rapidly than does x_2 [26]. Thus, $x_{1,n}$ and $x_{2,n}$ are the states, and k_1 and k_2 are unknown parameters, which can be constant or time-varying, to be estimated along with the states.

Let us assume $\Phi_n(\psi)$ is a known vector that is a function of some unknown vector, given as $\psi = \begin{bmatrix} k_1 & k_2 \end{bmatrix}^T$. Now ψ can be thought as a random walk process, that is

$$\psi_{n+1} = \psi_n + \delta_n, \tag{43}$$

where δ_n is any zero mean white noise sequence uncorrelated with the measurement noise variance ν_n and with the pre-assigned positive definite variances $\mathrm{Var}(\delta_n) = S_n$. In applications, S_n may be chosen as $S_n = S > 0$ for all n. The nonlinear model is as follows:

$$\begin{aligned} \begin{bmatrix} x_{n+1} \\ \psi_{n+1} \end{bmatrix} &= \begin{bmatrix} \Phi_n(\psi_n)x_n \\ \psi_n \end{bmatrix} + G_n \begin{bmatrix} w_n \\ \delta_n \end{bmatrix} \\ y_n &= [0\ 1\ 0\ 0] \begin{bmatrix} x_n \\ \psi_n \end{bmatrix} + H_n v_n, \end{aligned} \tag{44}$$

where w_n is the process noise variance that is also uncorrelated with ν_n. Considering its nonlinear nature, the EKF can be applied to the problem in hand in order to estimate the state vector, which contains ψ_n as one of its components. Also, let us impose a constraint on the parameters k_1 and k_2 and force them to satisfy $k_1 + k_2 = 1$. This constraint could be incorporated in the EKF through the use of the equality $Dx_n = d_n$ given by (3), where $D = [0\ 0\ 1\ 1]$ and $d_n = 1$.

Simulations for the initial values and noise terms given in Table 1 have been conducted, where the results are displayed in Figures 1 and 2, illustrating the estimation error on k_1 and k_2 for small and large initial estimation error and process noise terms, respectively. As can be seen from Figure 1, if the conditions described by (27) to (29) are satisfied, then the estimation error remains bounded; however, if, as shown in Figure 2, the conditions defined by (27) to (29) are violated, then the estimation error grows without bound.

Table 1. Initial state values and noise terms used in simulations.

	Stability conditions are met	Stability conditions are violated
Initial State - $\widehat{x}_0$	$[10\ 10\ 0.7\ 0.3]^T$	$[15\ 15\ 0.9\ 0.1]^T$
Process noise - G_n	$\sqrt{10^{-9}}I_4$	$\sqrt{5\times10^{-2}}I_4$
Measurement Noise -H_n	$\sqrt{10^{-5}}$	$\sqrt{5\times10^{-2}}$

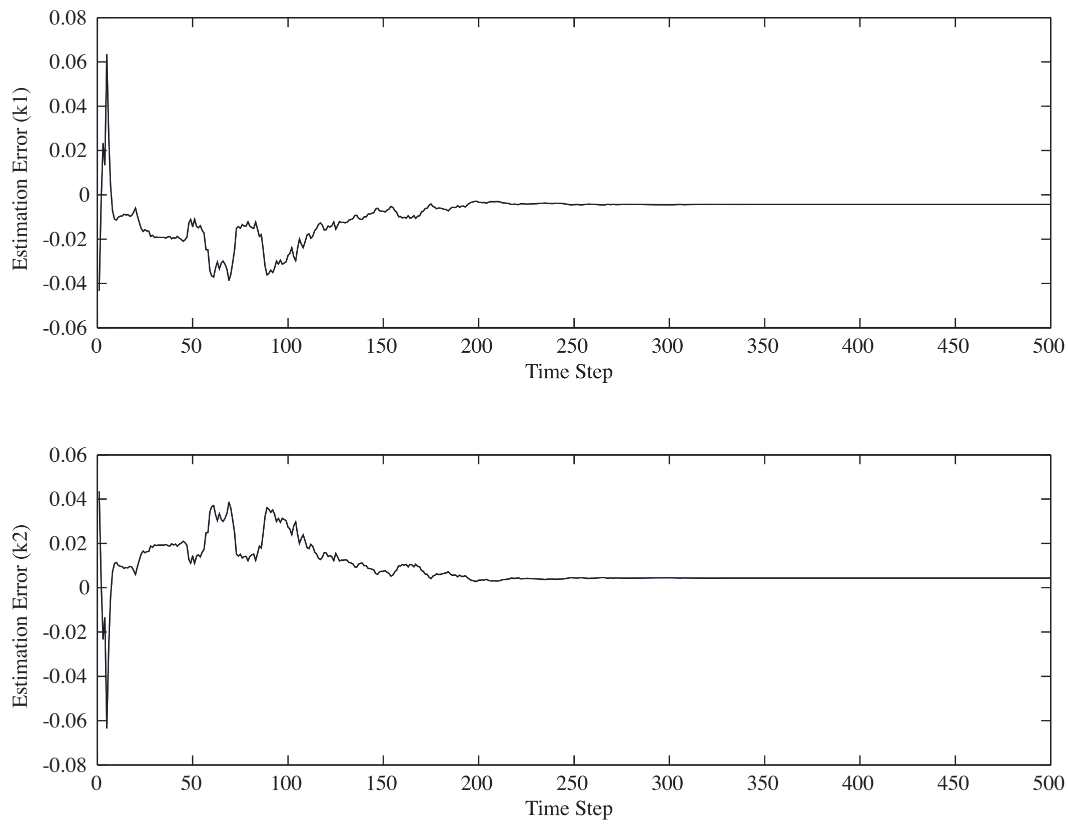

Figure 1. Estimation error for the parameters k_1 and k_2 (small initial error and process noise).

Moreover, in order to demonstrate the performance improvement with the use of constraints, both the constrained and unconstrained estimates of the parameter k_1 are displayed in Figure 3, along with the true value of the parameter (0.7). As can be seen from Figure 3, when the additional information provided by the known constraints is utilized, the estimation error performance improves, i.e., the estimation error is reduced.

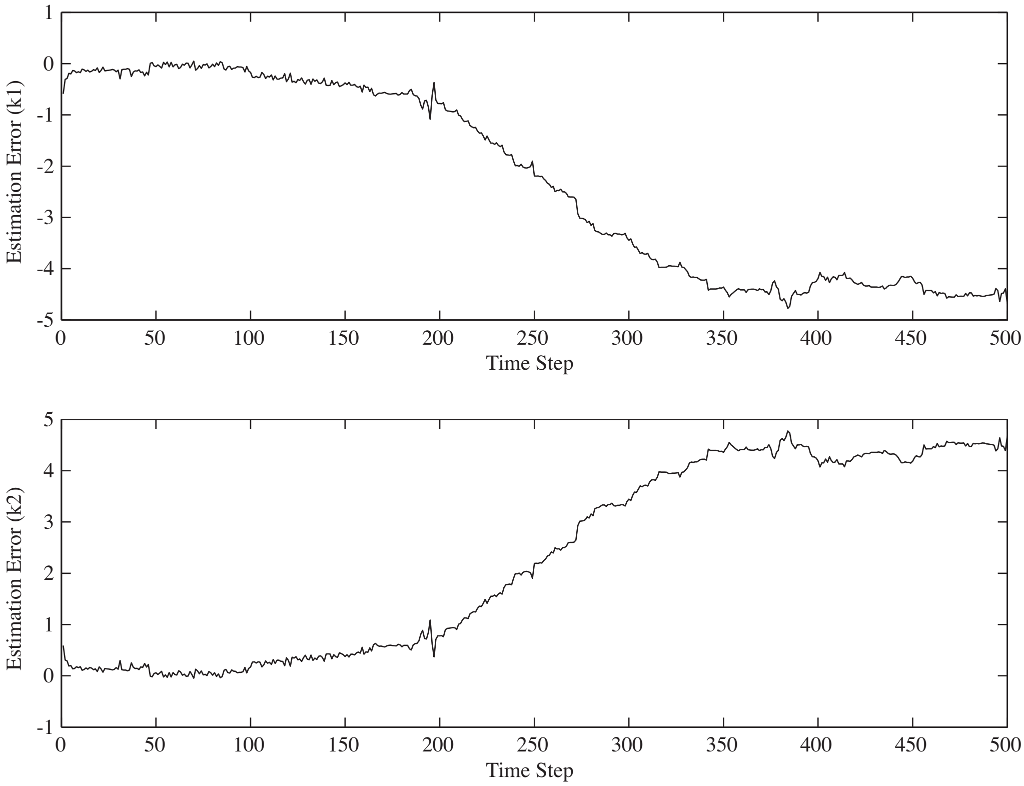

Figure 2. Estimation error for the parameters k_1 and k_2 (large initial error and process noise).

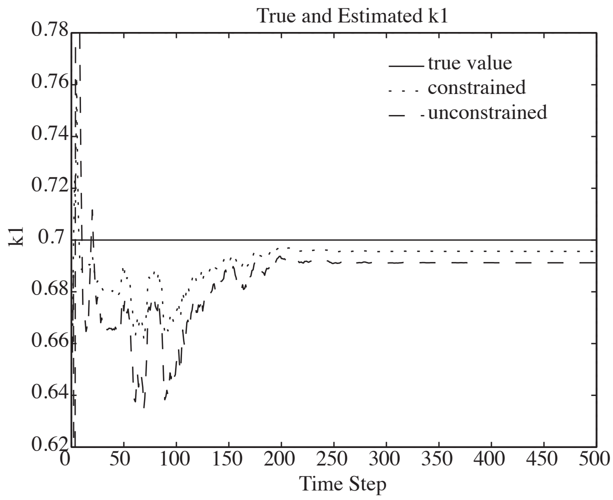

Figure 3. Estimated k_1, with and without the use of constraint information.

5. Conclusion

In this paper, the error behavior of the constrained extended Kalman Filter, when applied to general estimation problems for nonlinear stochastic discrete-time systems, has been analyzed. The analysis considered the stability of the unconstrained EKF, which was then extended to the case where there were assumed constraints imposed on the states. The analysis has shown that under certain conditions, similar to the unconstrained error, the constrained estimation error is bounded in the mean square sense with probability one, under the conditions that the initial estimation error and the disturbing noise terms are small enough, and that the solution of the Riccati difference equation remains positive definite and bounded. Finally, some numerical simulation results are presented in order to illustrate the stability, significance of the stability conditions and demonstrate the estimation performance of the constrained EKF.

References

[1] B. D. O. Anderson, J. B. Moore, Optimal Filtering, Application in Nonlinear Filtering, Prentice Hall, 1979.

[2] A. E. Bryson, Y. C. Ho, Optimal Filtering and Prediction, Ginn and Company, 1969.

[3] A. H. Jazwinski, Stochastic Processes and Filtering Theory, Academic Pres, 1970.

[4] L. Ozbek, M. Efe, "An Adaptive Extended Kalman Filter with Application to Compartment Models, Communication in Statistics-Simulation and Computation, Vol 3, pp. 145-158, 2004.

[5] F. Aliev, L. Ozbek, "Evaluation of Convergence Rate in Central Limit Theorem for the Kalman Filter", IEEE Trans. Automatic Control, 44(10), pp. 1905-1909, 1999.

[6] K. Reif, S. Günther, E. Yaz and R. Unbehauen, "Stochastic Stability of the Discrete-Time Extended Kalman Filter", IEEE Transactions on Automatic Control, 44 (4), pp. 714-728, 1999.

[7] H. K. Chow, "Robust Estimation in Time Series: An Approximation to the Gaussian Sum Filter", Commun. Stat. Theory and Methods, 23(12), pp. 3491-3505, 1999.

[8] C. P. Lee, "Forecasting with incomplete Data", Commun. Stat. Theory and Methods, 24(5), pp. 1255-1269, 1995.

[9] L. Ozbek, F. Aliev, "Comments on Adaptive Fading Kalman Filter with an Application", Automatica, 34 (12), pp. 1663-1664, 1998.

[10] R. Romera, T. Cipra, "On practical implementation of Robust Kalman Filtering", Commun. Stat. Simul. Comput., 24(2), pp. 461-488, 1995.

[11] J. C. Spall, K. D. Wall, "Asymptotic Distribution Theory for The Kalman Filter State Estimator", Commun. Stat. Theory and Methods, 13(16), pp. 1981-2003, 1984.

[12] D. Simon, "Kalman filtering with state constraints: How an optimal filter can get even better". http://academic.csuohio.edu/simond/ConstrKF (retrieved on Nov. 13, 2008)

[13] P. Vachhani, R. Rengaswamy, V. Gangwal, and S. Narasimhan, "Recursive estimation in constrained nonlinear dynamical systems", AIChE Journal, vol. 51, no. 3, pp. 946-959, 2005.

[14] L. Wang, Y. Chiang, F. Chang, "Filtering method for nonlinear systems with constraints", IEEE Proceedings – Control Theory and Applications, vol. 149, no. 6 , pp. 525-531, 2002.

[15] S. Julier, J. LaViola, "On Kalman filtering with nonlinear equality constraints" IEEE Transactions on Signal Processing, vol. 55, no. 6, pp. 2774-2784, 2007.

[16] T. Chia, P. Chow, H. Chizek, "Recursive parameter identification of constrained systems: An application to electrically stimulated muscle," IEEE Transactions on Biomedical Engineering, vol. 38, no. 5, pp. 429-441, 1999.

[17] M. Spong, S. Hutchinson, M. Vidyasagar, Robot Modeling and Control, New York: John Wiley & Sons, 2005.

[18] D. Simon, T. Chia, "Kalman Filtering With State Equality Constraints", IEEE Transactions on Aerospace and Electronic Systems, vol. 39, pp. 128-136, 2002.

[19] D. Simon, D. L. Simon, "Kalman Filtering with Inequality Constraints for Turbofan 33 Engine Health Estimation", IEEE Proceedings – Control Theory and Applications, vol. 153, no. 3, pp. 371-378, 2006.

[20] E. K. Babacan, L. Ozbek, M. Efe, "Stability of the Extended Kalman Filter When the States Are Constrained", IEEE Transactions on Automatic Control, vol. 53, no. 11, pp. 2707-2711, 2008.

[21] R. G. Agniel, E. I. Jury, "Almost Sure Boundedness of Randomly Sampled Systems", IAM J. Contr., vol. 9, pp. 372-384, 1971.

[22] T. J. Tarn, Y. Rasis, "Observers for Nonlinear Stochastic Systems", IEEE Trans. Automat. Contr., vol. AC-21, pp. 441-448, 1976.

[23] T. Morozan, "Stability of Stochastic Discrete Systems", J. Math. Anal. Appl., vol. 23, pp. 1-9, 1968.

[24] R. F. Curtain, "Boundedness Properties for Stochastic Systems, Stability of Stochastic Systems", Lecture Notes in Mathematics 294, Ed. Berlin, Germany: Springer-Verlag, 1972.

[25] T. C. Gard, Introduction to Stochastic Differential Equations, New York: Marcel Dekker, 1988.

[26] N. H. McClamroch, State Models of Dynamic Systems. New York: Springer-Verlag, (1980).