

CERTAIN CLASSES OF ANALYTIC AND MULTIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

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Abstract

We introduce a subclass $K_{n+p-1}^*(A, B)$ of analytic and p -valent functions with negative coefficients. Coefficient estimates, some properties, distortion theorems and closure theorems of functions belonging to the class $K_{n+p-1}^*(A, B)$ are determined. Also we obtain radii of close-to-convexity, starlikeness and convexity for the class $K_{n+p-1}^*(A, B)$. We also obtain class preserving integral operator of the form

$$F(z) = \frac{c+p}{z^c} \int_0^z t^{c-1} f(t) dt, c > -p$$

for the class $K_{n+p-1}^*(A, B)$. Conversely when $F(z) \in K_{n+p-1}^*(A, B)$ radius of p -valence of $f(z)$ defined by the above equation is obtained.

1. Introduction

Let $S(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k} (p \in \mathbb{N} = \{1, 2, \dots\}), \quad (1.1)$$

which are analytic and p -valent in the unit disc $U = \{z : |z| < 1\}$. Let $f(z)$ be in $S(p)$ and $g(z)$ be in $S(p)$. Then we denote by $f \star g(z)$ the Hadamard product of $f(z)$ and $g(z)$, that is, if $f(z)$ is given by (1.1) and $g(z)$ is given by

$$g(z) = z^p + \sum_{k=1}^{\infty} b_{p+k} z^{p+k} (p \in \mathbb{N}), \quad (1.2)$$

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then

$$f * g(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} b_{p+k} z^{p+k}. \tag{1.3}$$

The $(n + p - 1)$ -th order Ruscheweyh derivative $D^{n+p-1} f(z)$ of a function $f(z)$ of $S(p)$ is defined by

$$D^{n+p-1} f(z) = \frac{z^p (z^{n-1} f(z))^{n+p-1}}{(n + p - 1)!} \tag{1.4}$$

where n is any integer such that $n > -p$. It is easy to see that

$$D^{n+p-1} f(z) = \frac{z^p}{(1 - z)^{n+p}} * f(z) \tag{1.5}$$

$$= z^p + \sum_{k=1}^{\infty} \delta(n, k) a_{p+k} z^{p+k}, \tag{1.6}$$

where

$$\delta(n, k) = \binom{n + p + k - 1}{n + p - 1}. \tag{1.7}$$

Particularly, the symbol $D^n f(z)$ was named the n -th order Ruscheweyh derivative of $f(z) \in S(1)$ by Al-Amiri [1].

Let $T(p)$ denote the subclass of $S(p)$ consisting of functions of the form

$$f(z) = z^p - \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \quad (a_{p+k} \geq 0; p \in \mathbb{N}). \tag{1.8}$$

Also let $K_{n+p-1}^*(A, B)$ denote the class of functions $f(z) \in T(p)$ such that

$$\left| \frac{2 \left(\frac{D^{n+p} f(z)}{D^{n+p-1} f(z)} - 1 \right)}{2B \frac{D^{n+p} f(z)}{D^{n+p-1} f(z)} - (A + B)} \right| < 1, (z \in U), \tag{1.9}$$

where $-1 \leq A < B \leq 1, 0 < B \leq 1$, and $n > -p$.

We note that:

- (i) $K_{n+p-1}^*(-1, 1) = K_{n+p-1}^*$ (Owa [3]);
- (ii) $K_0^*((4\gamma - 3)\beta, \beta) = T^*(\gamma, \beta) (0 \leq \gamma < 1, 0 < \beta \leq 1)$ (Gupta and Jain [2]);
- (iii) $K_0^*(4\gamma - 3, 1) = T^*(\gamma) (0 \leq \gamma < 1)$ (Silverman [4]).

2. Coefficient Estimates

Theorem 1. *Let the function $f(z)$ be defined by (1.8). Then $f(z)$ is in the class $K_{n+p-1}^*(A, B)$ if and only if*

$$\sum_{k=1}^{\infty} D_k \delta(n, K) a_{p+k} \leq (B - A)(n + p), \tag{2.1}$$

where

$$D_k = [2k(B + 1) + (B - A)(n + p)]. \tag{2.2}$$

The result is sharp.

Proof. Assume that the inequality (2.1) holds and let $|z| = 1$. Then we get

$$\begin{aligned} & |2(D^{n+p} f(z) - D^{n+p-1} f(z))| - |2BD^{n+p} f(z) - (A + B)D^{n+p-1} f(z)| \\ &= \left| -2 \sum_{k=1}^{\infty} \left(\frac{k}{n+p}\right) \delta(n, k) a_{p+k} z^{p+k} \right| - |(B - A)z^p - \\ & \sum_{k=1}^{\infty} \left[2B \left(\frac{k}{n+p}\right) + (B - A) \right] \delta(n, k) a_{p+k} z^{p+k} | \\ &\leq \sum_{k=1}^{\infty} 2 \left(\frac{k}{n+p}\right) \delta(n, k) a_{p+k} - (B - A) + \sum_{k=1}^{\infty} \left[2B \left(\frac{k}{n+p}\right) + (B - A) \right] \delta(n, k) a_{p+k} \\ &= \sum_{k=1}^{\infty} \left[\frac{2k}{(n+p)} (B + 1) + (B - A) \right] \delta(n, k) a_{p+k} - (B - A) \\ &\leq 0, \text{ by hypotheses.} \end{aligned}$$

Hence by the maximum modulus theorem $f(z) \in K_{n+p-1}^*(A, B)$.

Conversely, suppose that

$$\begin{aligned} & \left| \frac{2 \left(\frac{D^{n+p} f(z)}{D^{n+p-1} f(z)} - 1 \right)}{2B \frac{D^{n+p} f(z)}{D^{n+p-1} f(z)} - (A + B)} \right| \\ &= \left| \frac{-2 \sum_{k=1}^{\infty} \left(\frac{k}{n+p}\right) \delta(n, k) a_{p+k} z^k}{(B - A) - \sum_{k=1}^{\infty} \left[2B \left(\frac{k}{n+p}\right) + (B - A) \right] \delta(n, k) a_{p+k} z^k} \right| \leq 1, z \in U. \tag{2.3} \end{aligned}$$

Since $|\operatorname{Re}(z)| \leq |z|$ for all z , we have

$$\operatorname{Re} \left\{ \frac{-2 \sum_{k=1}^{\infty} \left(\frac{k}{n+p}\right) \delta(n, k) a_{p+k} z^k}{(B-A) - \sum_{k=1}^{\infty} \left[2B \left(\frac{k}{n+p}\right) + (B-A) \right] \delta(n, k) a_{p+k} z^k} \right\} < 1. \quad (2.4)$$

Choose values of z on the real axis so that $\frac{D^{n+p} f(z)}{D^{n+p-1} f(z)}$ is real. Upon clearing the denominator in (2.4) and letting $z \rightarrow 1^-$ through real values, we obtain

$$2 \sum_{k=1}^{\infty} \left(\frac{k}{n+p}\right) \delta(n, k) a_{p+k} \leq (B-A) - \sum_{k=1}^{\infty} \left[2B \left(\frac{k}{n+p}\right) + (B-A) \right] \delta(n, k) a_{p+k}.$$

This gives the required condition.

Finally, the function

$$f(z) = z^p - \frac{(B-A)(n+p)}{D_k \delta(n, k)} z^{p+k} \quad (k \geq 1) \quad (2.5)$$

is an extremal function for the theorem. □

Corollary 1. *Let the function $f(z)$ defined by (1.8) be in the class $K_{n+p-1}^*(A, B)$. Then*

$$a_{p+k} \leq \frac{(B-A)(n+p)}{D_k \delta(n, k)} \quad (k \geq 1). \quad (2.6)$$

The result is sharp for the function $f(z)$ given by (2.5).

3. Some Properties of the Class $K_{n+p-1}^*(A, B)$

Theorem 2. $K_{n+p}^*(A, B) \subset K_{n+p-1}^*(A, B)$ for $p \in \mathbb{N}, n > -p, -1 \leq A < B \leq 1$, and $0 < B \leq 1$.

Proof. Let the function $f(z)$ defined by (1.8) be in the class $K_{n+p}^*(A, B)$. Then, by Theorem 1, we have

$$\sum_{k=1}^{\infty} \left(\frac{2k(B+1)}{(n+p+1)} + (B-A) \right) \delta(n+1, k) a_{p+k} \leq (B-A) \quad (3.1)$$

and since

$$\left(\frac{2k(B+1)}{(n+p)} + (B-A)\right)\delta(n, k) \leq \left(\frac{2k(B+1)}{(n+p+1)} + (B-A)\right)\delta(n+1, k) \quad \text{for } k \geq 1, \quad (3.2)$$

we have

$$\begin{aligned} & \sum_{k=1}^{\infty} \left(\frac{2k(B+1)}{(n+p)} + (B-A)\right)\delta(n, k)a_{p+k} \\ & \leq \sum_{k=1}^{\infty} \left(\frac{2k(B+1)}{(n+p+1)} + (B-A)\right)\delta(n+1, k)a_{p+k} \leq (B-A). \end{aligned} \quad (3.3)$$

The result follows from Theorem 1. □

Theorem 3. Let $-1 \leq A_1 \leq A_2 < B_1 \leq B_2 \leq 1$ and $0 < B_1 \leq B_2 \leq 1$. Then we have

$$K_{n+p-1}^*(A_1, B_2) \supseteq K_{n+p-1}^*(A_2, B_1).$$

Proof. Theorem 3 is an immediate consequence of the definitin of the class $K_{n+p-1}^*(A, B)$. □

4. Distortion Theorems

Theorem 4. Let the function $f(z)$ defined by (1.8) be in the class $K_{n+p-1}^*(A, B)$. Then we have

$$|z|^p - \frac{(B-A)}{D_1}|z|^{p+1} \leq |f(z)| \leq |z|^p + \frac{(B-A)}{D_1}|z|^{p+1} \quad (4.1)$$

for $z \in U$. The result is sharp.

Proof. Since $f(z) \in K_{n+p-1}^*(A, B)$, in view of Theorem 1, we obtain

$$\begin{aligned} D_1\delta(n, 1)\sum_{k=1}^{\infty} a_{p+k} & \leq \sum_{k=1}^{\infty} D_k\delta(n, K)a_{p+k} \\ & \leq (B-A)(n+p), \end{aligned} \quad (4.2)$$

which implies that

$$\sum_{k=1}^{\infty} a_{p+k} \leq \frac{(B-A)}{D_1}. \tag{4.3}$$

Therefore we can show that

$$\begin{aligned} |f(z)| &\geq |z|^p - |z|^{p+1} \sum_{k=1}^{\infty} a_{p+k} \\ &\geq |z|^p - \frac{(B-A)}{D_1} |z|^{p+1} \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} |f(z)| &\leq |z|^p + |z|^{p+1} \sum_{k=1}^{\infty} a_{p+k} \\ &\leq |z|^p + \frac{(B-A)}{D_1} |z|^{p+1} \end{aligned} \tag{4.5}$$

for $z \in U$. This completes the proof of Theorem 4. Finally, by taking the function

$$f(z) = z^p - \frac{(B-A)}{D_1} z^{p+1}, \tag{4.6}$$

we can show that the result of Theorem 4 is sharp. □

Corollary 2. *Let the function $f(z)$ defined by (1.8) be in the class $K_{n+p-1}^*(A, B)$. Then $f(z)$ is included in a disc with its center at the origin and radius r_1 given by*

$$r_1 = \frac{D_1 + (B-A)}{D_1}. \tag{4.7}$$

Theorem 5 . *Let the function $f(z)$ defined by (1.8) be in the class $K_{n+p-1}^*(A, B)$. Then we have*

$$p|z|^{p-1} - \frac{(B-A)(p+1)}{D_1} |z|^p \leq |f'(z)| \leq p|z|^{p-1} + \frac{(B-A)(p+1)}{D_1} |z|^p \tag{4.8}$$

for $z \in U$. The result is sharp.

Proof. In view of Theorem 1, we have

$$\frac{D_1 \delta(n, 1)}{(p+1)} \sum_{k=1}^{\infty} (p+k) a_{p+k} \leq \sum_{k=1}^{\infty} D_k \delta(n, K) a_{p+k} \leq (B-A)(n+p) \tag{4.9}$$

which implies that

$$\sum_{k=1}^{\infty} (p+k) a_{p+k} \leq \frac{(B-A)(p+1)}{D_1}. \tag{4.10}$$

Hence, with the aid of (4.10), we have

$$\begin{aligned} |f'(z)| &\geq p|z|^{p-1} - |z|^p \sum_{k=1}^{\infty} (p+k) a_{p+k} \\ &\geq p|z|^{p-1} - \frac{(B-A)(p+1)}{D_1} |z|^p \end{aligned} \tag{4.11}$$

and

$$\begin{aligned} |f'(z)| &\leq p|z|^{p-1} + |z|^p \sum_{k=1}^{\infty} (p+k) a_{p+k} \\ &\leq p|z|^{p-1} + \frac{(B-A)(p+1)}{D_1} |z|^p \end{aligned} \tag{4.12}$$

for $z \in U$. Further the results of Theorem 5 are sharp for the function $f(z)$ given by (4.6). □

Corollary 3. *Let the function $f(z)$ defined by (1.8) be in the class $K_{n+p-1}^*(A, B)$. Then $f'(z)$ is included in a disc with its center at the origin and radius r_2 given by*

$$r_2 = \frac{PD_1 + (B-A)(p+1)}{D_1} \tag{4.13}$$

5. Closure Theorems

Let the functions $f_i(z)$ be defined, for $i = 1, 2, \dots, m$, by

$$f_i(z) = z^P - \sum_{k=1}^{\infty} a_{p+k,i} z^{p+k} \quad (a_{p+k,i} \geq 0) \tag{5.1}$$

for $z \in U$.

We shall prove the following results for the closure of functions in the class $K_{n+p-1}^*(A, B)$.

Theorem 6. *Let the functions $f_i(z)$ defined by (5.1) be in the class $K_{n+p-1}^*(A, B)$ for every $i = 1, 2, \dots, m$. Then the function $h(z)$ defined by*

$$h(z) = \sum_{i=1}^m c_i f_i(z) \quad (c_i \geq 0) \tag{5.2}$$

is also in the same class $K_{n+p-1}^*(A, B)$, where

$$\sum_{i=1}^m c_i = 1. \tag{5.3}$$

Proof. By means of the definition of $h(z)$, we obtain

$$h(z) = z^P - \sum_{k=1}^{\infty} \left(\sum_{i=1}^m c_i a_{p+k,i} \right) z^{p+k}. \tag{5.4}$$

Further, since $f_i(z)$ are in $K_{n+p-1}^*(A, B)$ for every $i = 1, 2, \dots, m$, we get

$$\sum_{k=1}^{\infty} D_k \delta(n, k) a_{p+k,i} \leq (B - A)(n + p) \tag{5.5}$$

for every $i = 1, 2, \dots, m$. Hence we can see that

$$\begin{aligned} & \sum_{k=1}^{\infty} D_k \delta(n, k) \left(\sum_{i=1}^m c_i a_{p+k,i} \right) \\ &= \sum_{i=1}^m c_i \left(\sum_{k=1}^{\infty} D_k \delta(n, k) a_{p+k,i} \right) \\ &\leq \left(\sum_{i=1}^m c_i \right) (B - A)(n + p) = (B - A)(n + p) \end{aligned} \tag{5.6}$$

with the aid of (5.5). This proves that the function $h(z)$ is in the class $K_{n+p-1}^*(A, B)$ by means of Theorem 1. Thus we have the theorem. \square

Theorem 7. Let $f_p(z) = z^P$ and

$$f_{p+k}(z) = z^P - \frac{(B - A)(n + p)}{D_k \delta(n, k)} z^{p+k} \quad (k \geq 1) \tag{5.7}$$

for $p \in \mathbb{N}, n > -p, -1 \leq A < B \leq 1$ and $0 < B \leq 1$. Then $f(z)$ is in the class $K_{n+p-1}^*(A, B)$ if and only if it can be expressed in the form

$$f(z) = \sum_{k=0}^{\infty} \lambda_{p+k} f_{p+k}(z), \tag{5.8}$$

where $\lambda_{p+k} \geq 0 (k \geq 0)$ and $\sum_{k=0}^{\infty} \lambda_{p+k} = 1$.

Proof. Suppose that

$$\begin{aligned} f(z) &= \sum_{k=0}^{\infty} \lambda_{p+k} f_{p+k}(z) \\ &= z^P - \sum_{k=1}^{\infty} \frac{(B - A)(n + p)}{D_k \delta(n, k)} \lambda_{p+k} z^{p+k}. \end{aligned} \tag{5.9}$$

Then it follows that

$$\begin{aligned} &\sum_{k=1}^{\infty} D_k \delta(n, k) \frac{(B - A)(n + p)}{D_k \delta(n, k)} \lambda_{p+k} \\ &= (B - A)(n + p) \sum_{k=1}^{\infty} \lambda_{p+k} \\ &= (B - A)(n + p)(1 - \lambda_p) \leq (B - A)(n + p). \end{aligned} \tag{5.10}$$

So by Theorem 1, $f(z) \in K_{n+p-1}^*(A, B)$.

Conversely, assume that the function $f(z)$ defined by (1.8) belongs to the class $K_{n+p-1}^*(A, B)$. Then

$$a_{p+k} \leq \frac{(B - A)(n + p)}{D_k \delta(n, k)} \quad (k \geq 1). \tag{5.11}$$

Setting

$$\lambda_{p+k} = \frac{D_k \delta(n, k)}{(B - A)(n + p)} a_{p+k} \tag{5.12}$$

and

$$\lambda_p = 1 - \sum_{k=1}^{\infty} \lambda_{p+k}. \tag{5.13}$$

We can see that $f(z)$ can be expressed in the form (5.8). This completes the proof of Theorem 7. □

Corollary 4. *The extreme points of the class $K_{n+p-1}^*(A, B)$ are the functions $f_{p+k}(z) (k \geq 0)$ given by Theorem 7.*

6. Integral Operators

Theorem 8. *Let the function $f(z)$ defined by (1.8) in the class $K_{n+p-1}^*(A, B)$, and let c be a real number such that $c > -p$. Then the function $F(z)$ defined by*

$$F(z) = \frac{c + p}{z^c} \int_0^z t^{c-1} f(t) dt \tag{6.1}$$

also belongs to the class $K_{n+p-1}^(A, B)$.*

Proof. From the representation of $F(z)$, it follows that

$$F(z) = z^p - \sum_{k=1}^{\infty} b_{p+k},$$

where

$$b_{p+k} = \left(\frac{c + p}{c + p + k} \right) a_{p+k}.$$

Therefore,

$$\begin{aligned} \sum_{k=1}^{\infty} D_k \delta(n, k) b_{p+k} &= \sum_{k=1}^{\infty} D_k \delta(n, k) \left(\frac{c + p}{c + p + k} \right) a_{p+k} \\ &\leq \sum_{k=1}^{\infty} D_k \delta(n, k) a_{p+k} \leq (B - A)(n + p), \end{aligned}$$

since $f(z) \in K_{n+p-1}^*(A, B)$. Hence, by Theorem 1, $F(z) \in K_{n+p-1}^*(A, B)$.

Putting $c = 1 - p$ in Theorem 8, we get the following. □

Corollary 5. *Let the function $f(z)$ defined by (1.8) be in the class $K_{n+p-1}^*(A, B)$ and let $F(z)$ be defined by*

$$F(z) = \frac{1}{z^{1-p}} \int_0^z \frac{f(t)}{t^p} dt. \tag{6.2}$$

Then $F(z) \in K_{n+p-1}^*(A, B)$.

Theorem 9. *Let c be a real number such that $c > -p$. If $F(z) \in K_{n+p-1}^*(A, B)$, then the function $f(z)$ defined by (6.1) is p -valent in $|z| < R_p^*$, where*

$$R_p^* = \inf_k \left\{ \frac{p(c+p)D_k\delta(n, k)}{(p+k)(c+p+k)(B-A)(n+p)} \right\}^{\frac{1}{k}} \quad (k \geq 1). \tag{6.3}$$

The result is sharp.

Proof. Let $F(z) = z^p - \sum_{k=1}^{\infty} a_{p+k} z^{p+k}$ ($a_{p+k} \geq 0$). It follows from (6.1) that

$$\begin{aligned} f(z) &= \frac{z^{1-c}(z^c F(z))'}{(c+p)} a_{p+k} z^{p+k}, \quad (c > -p) \\ &= z^p - \sum_{k=1}^{\infty} \left(\frac{c+p+k}{c+p} \right) a_{p+k} z^{p+k}. \end{aligned}$$

To prove the result, it suffices to show that

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| \leq p \text{ for } |z| < R_p^*.$$

Now

$$\begin{aligned} \left| \frac{f'(z)}{z^{p-1}} - p \right| &= \left| - \sum_{k=1}^{\infty} (p+k) \left(\frac{c+p+k}{c+p} \right) a_{p+k} z^k \right| \\ &\leq \sum_{k=1}^{\infty} (p+k) \left(\frac{c+p+k}{c+p} \right) a_{p+k} |z|^k. \end{aligned}$$

Thus $\left| \frac{f'(z)}{z^{p-1}} - p \right| \leq p$ if

$$\sum_{k=1}^{\infty} \left(\frac{p+k}{p}\right) \left(\frac{c+p+k}{c+p}\right) a_{p+k} |z|^k \leq 1. \tag{6.4}$$

But Theorem 1 confirms that

$$\sum_{k=1}^{\infty} \frac{D_k \delta(n, k)}{(B-A)(n+p)} a_{p+k} \leq 1. \tag{6.5}$$

Thus (6.4) will be satisfied if

$$\left(\frac{p+k}{p}\right) \left(\frac{c+p+k}{c+p}\right) |z|^k \leq \frac{D_k \delta(n, k)}{(B-A)(n+p)} \quad (k \geq 1),$$

or if

$$|z| \leq \left\{ \frac{p(c+p)D_k \delta(n, k)}{(p+k)(c+p+k)(B-A)(n+p)} \right\}^{\frac{1}{k}} \quad (k \geq 1). \tag{6.6}$$

The required result follows now from (6.6). The result is sharp for the function

$$f(z) = z^p - \frac{D_k \delta(n, k)(c+p+k)}{(B-A)(n+p)(c+p)} z^{p+k} \quad (k \geq 1). \tag{6.7}$$

□

7. Radii of Close-to-Convexity, Starlikeness and Convexity

Theorem 10. *Let the function $f(z)$ defined by (1.8) be in the class $K_{n+p-1}^*(A, B)$, then $f(z)$ is p -valently close-to-convex of order α ($0 \leq \alpha < p$) in $|z| < r_1(A, B, n, p, \alpha)$ where*

$$r_1(A, B, n, p, \alpha) = \inf_k \left[\frac{(p-\alpha)D_k \delta(n, k)}{(p+k)(B-A)(n+p)} \right]^{\frac{1}{k}} \quad (k \geq 1). \tag{7.1}$$

The result is sharp with the extremal function $f(z)$ given by (2.5).

Proof. We must show that $\left| \frac{f'(z)}{z^{p-1}} - p \right| \leq p - \alpha$ for $|z| < r_1(A, B, n, p, \alpha)$. We have

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| \leq \sum_{k=1}^{\infty} (p+k) a_{p+k} |z|^k.$$

Thus $\left| \frac{f'(z)}{z^{p-1}} - p \right| \leq p - \alpha$ if

$$\sum_{k=1}^{\infty} \frac{(p+k)}{(p-\alpha)} a_{p+k} |z|^k \leq 1. \tag{7.2}$$

Hence, by Theorem 1, (7.2) will be true if

$$\frac{(p+k)}{(p-\alpha)} |z|^k \leq \frac{D_k \delta(n, k)}{(B-A)(n+p)}$$

or if

$$|z| \leq \left[\frac{(p-\alpha) D_k \delta(n, k)}{(p+k)(B-A)(n+p)} \right]^{\frac{1}{k}}, (k \geq 1). \tag{7.3}$$

The theorem follows easily from (7.3). □

Theorem 11. *Let the function $f(z)$ defined by (1.8) be in the class $K_{n+p-1}^*(A, B)$, then $f(z)$ is p -valently starlike of order α ($0 \leq \alpha < p$) in $|z| < r_2(A, B, n, p, \alpha)$ where*

$$r_2(A, B, n, p, \alpha) = \inf_k \left[\frac{(p-\alpha) D_k \delta(n, k)}{(p+k-\alpha)(B-A)(n+p)} \right]^{\frac{1}{k}} (k \geq 1). \tag{7.4}$$

The result is sharp with the extremal function $f(z)$ given by (2.5).

Proof. It is sufficient to show that $\left| \frac{zf'(z)}{f(z)} - p \right| \leq p - \alpha$ for $|z| < r_2(A, B, n, p, \alpha)$. We have

$$\left| \frac{zf'(z)}{f(z)} - p \right| \leq \frac{\sum_{k=1}^{\infty} k a_{p+k} |z|^k}{1 - \sum_{k=1}^{\infty} a_{p+k} |z|^k}.$$

Thus $\left| \frac{zf'(z)}{f(z)} - p \right| \leq p - \alpha$ if

$$\sum_{k=1}^{\infty} \frac{(p+k-\alpha) a_{p+k} |z|^k}{(p-\alpha)} \leq 1. \tag{7.5}$$

Hence, by Theorem 1, (7.5) will be true if

$$\frac{(p+k-\alpha) |z|^k}{(p-\alpha)} \leq \frac{D_k \delta(n, k)}{(B-A)(n+p)}$$

or if

$$|z| \leq \left[\frac{(p - \alpha)D_k\delta(n, k)}{(p + k - \alpha)(B - A)(n + p)} \right]^{\frac{1}{k}} \quad (k \geq 1). \tag{7.6}$$

The theorem follows easily from (7.6). □

Corollary 6. *Let the function $f(z)$ defined by (1.8) be in the class $K_{n+p-1}^*(A, B)$, then $f(z)$ is p -valently convex of order α ($0 \leq \alpha < p$) in $|z| < r_3(A, B, n, p, \alpha)$ where*

$$r_3(A, B, n, p, \alpha) = \inf_k \left[\frac{p(p - \alpha)D_k\delta(n, k)}{(p + k)(p + k - \alpha)(B - A)(n + p)} \right]^{\frac{1}{k}} \quad (k \geq 1). \tag{7.7}$$

The result is sharp with the extremal function $f(z)$ given by (2.5).

8. Modified Hadamard Product

Let the functions $f_i(z)$ ($i = 1, 2$) be defined by (5.1). The modified Hadamard product of $f_1(z)$ and $f_2(z)$ is defined by

$$f_1 * f_2(z) = z^p - \sum_{k=1}^{\infty} a_{p+k,1} a_{p+k,2} z^{p+k}. \tag{8.1}$$

Theorem 12. *Let the function $f_1(z)$ defined by (5.1) be in the class $K_{n+p-1}^*(A, B)$ and the function $f_2(z)$ defined by (5.1) be in the class $K_{n+p-1}^*(C, E)$ ($-1 \leq C \leq E \leq 1, 0 < E \leq 1$). Then the modified Hadamard product $f_1 * f_2(z)$ belongs to the class*

$$K_{n+p-1}^* \left(1 - \frac{4(n + p)(B - A)(E - C)}{\delta(n, 1)D_1G_1 - (n + p)^2(B - A)(E - C)}, 1 \right) \tag{8.2}$$

where D_k ($k \geq 1$) is defined by (2.2) and G_k ($k \geq 1$) is given by

$$G_k = [2k(E + 1) + (E - C)(n + p)]. \tag{8.3}$$

The result is sharp.

Proof. From Theorem 1, we have

$$\sum_{k=1}^{\infty} \frac{D_k\delta(n, k)}{(B - A)(n + p)} a_{p+k} \leq 1 \tag{8.4}$$

and

$$\sum_{k=1}^{\infty} \frac{G_k \delta(n, k)}{(E - C)(n + p)} a_{p+k} \leq 1. \tag{8.5}$$

We want to find the largest $\beta = \beta(n, p, A, B, C, E)$ such that

$$\sum_{k=1}^{\infty} \frac{[4k + (1 - \beta)(n + p)] \delta(n, k)}{(1 - \beta)(n + p)} a_{p+k,1} a_{p+k,2} \leq 1. \tag{8.6}$$

From (8.4) and (8.5) by means of Cauchy-Schwarz inequality we obtain

$$\sum_{k=1}^{\infty} \sqrt{\frac{D_k G_k}{(B - A)(E - C)} \frac{\delta(n, k)}{(n + p)}} \sqrt{a_{p+k,1} a_{p+k,2}} \leq 1. \tag{8.7}$$

Hence (8.6) will be satisfied if

$$\sqrt{a_{p+k,1} a_{p+k,2}} \leq \frac{(1 - \beta)}{[4k + (1 - \beta)(n + p)]} \sqrt{\frac{D_k G_k}{(B - A)(E - C)}} (k \geq 1). \tag{8.8}$$

From (8.7) it follows that

$$\sqrt{a_{p+k,1} a_{p+k,2}} \leq \frac{(n + p)}{\delta(n, k)} \sqrt{\frac{(B - A)(E - C)}{D_k G_k}} (k \geq 1). \tag{8.9}$$

Therefore (8.6) will be satisfied if

$$\frac{(n + p)}{\delta(n, k)} \sqrt{\frac{(B - A)(E - C)}{D_k G_k}} \leq \frac{(1 - \beta)}{[4k + (1 - \beta)(n + p)]} \sqrt{\frac{D_k G_k}{(B - A)(E - C)}} (k \geq 1). \tag{8.10}$$

that is, that

$$\beta \leq 1 - \frac{4k(n + p)(B - A)(E - C)}{\delta(n, k) D_k G_k - (n + p)^2 (B - A)(E - C)}. \tag{8.11}$$

The right-hand side of (8.11) is an increasing function of k ($k \geq 1$). Therefore, setting $k = 1$ in (8.11) we get

$$\beta \leq 1 - \frac{4(n + p)(B - A)(E - C)}{\delta(n, 1) D_1 G_1 - (n + p)^2 (B - A)(E - C)}.$$

The result is sharp, with equality when

$$f_1(z) = z^p - \frac{(B - A)}{D_1} z^{p+1}$$

and

$$f_2(z) = z^P - \frac{(E - C)}{G_1} z^{p+1}.$$

□

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NEGATİV KATSAYILI BAZI ANALİTİK VE ÇOK KATLI FONKSİYON SINIFLARI

Özet

Analitik, p-katlı ve negatif katsayılı fonksiyonların bir $K_{n+p-1}^*(A, B)$ alt sınıfı tanımlanıp, bu sınıf için katsayı kestirmeleri bozulma teoremleri, kapanış teoremleri kanıtlanmıştır. Ayrıca bu sınıfın özellikleri incelenmiş ve elemanlarının yıldızlık, konvekslik, yaklaşık konvekslik çapları hesaplanmıştır.

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