# THE BINOMIAL IDEAL OF THE INTERSECTION AXIOM FOR CONDITIONAL PROBABILITIES 

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#### Abstract

The binomial ideal associated with the intersection axiom of conditional probability is shown to be radical and is expressed as an intersection of toric prime ideals. This solves a problem in algebraic statistics posed by Cartwright and Engström.


Conditional independence contraints are a family of natural constraints on probability distributions, describing situations in which two random variables are independently distributed given knowledge of a third. Statistical models built around considerations of conditional independence, in particular graphical models in which the constraints are encoded in a graph on the random variables, enjoy wide applicability in determining relationships among random variables in statistics and in dealing with uncertainty in artificial intelligence.

One can take a purely combinatorial perspective on the study of conditional independence, as does Studený 10, conceiving of it as a relation on triples of subsets of a set of observables which must satisfy certain axioms. A number of elementary implications among conditional independence statements are recognised as axioms. Among these are the semi-graphoid axioms, which are implications of conditional independence statements lacking further hypotheses, and hence are purely combinatorial statements. The intersection axiom is also often added to the collection, but unlike the semi-graphoid axioms it is not uniformly true; it is our subject here.

Formally, a conditional independence model $\mathcal{M}$ is a set of probability distributions characterised by satisfying several conditional independence constraints. We will work in the discrete setting, where a probability distribution $p$ is a multi-way table of probabilities, and we follow the notational conventions in [1].

Consider the discrete conditional independence model $\mathcal{M}$ given by

$$
\left\{X_{1} \Perp X_{2}\left|X_{3}, X_{1} \Perp X_{3}\right| X_{2}\right\}
$$

where $X_{i}$ is a random variable taking values in the set $\left[r_{i}\right]=\left\{1, \ldots, r_{i}\right\}$. Throughout we assume $r_{1} \geq 2$. Let $p_{i j k}$ be the unknown probability $P\left(X_{1}=i, X_{2}=j, X_{3}=\right.$ $k$ ) in a distribution from the model $\mathcal{M}$. The set of distributions in the model $\mathcal{M}$ is the variety whose defining ideal $I_{\mathcal{M}} \subseteq S=\mathbb{C}\left[p_{i j k}\right]$ is

$$
\begin{aligned}
I_{\mathcal{M}}= & \left(p_{i j k} p_{i^{\prime} j^{\prime} k}-p_{i j^{\prime} k} p_{i^{\prime} j k}: i, i^{\prime} \in\left[r_{1}\right], j, j^{\prime} \in\left[r_{2}\right], k \in\left[r_{3}\right]\right) \\
& +\left(p_{i j k} p_{i^{\prime} j k^{\prime}}-p_{i j k^{\prime}} p_{i^{\prime} j k}: i, i^{\prime} \in\left[r_{1}\right], j \in\left[r_{2}\right], k, k^{\prime} \in\left[r_{3}\right]\right) .
\end{aligned}
$$

[^0]The intersection axiom is the axiom whose premises are the statements of $\mathcal{M}$ and whose conclusion is $X_{1} \Perp\left(X_{2}, X_{3}\right)$. This implication requires the further hypothesis that the distribution $p$ is in the interior of the probability simplex, i.e. that no individual probability $p_{i j k}$ is zero. It is thus a natural question to ask what can be inferred about distributions $p$ which may lie on the boundary of the probability simplex. In algebraic terms, we are asking for a primary decomposition of $I_{\mathcal{M}}$.

Our Proposition 1 resolves a problem posed by Dustin Cartwright and Alexander Engström in [1 p. 152]. The problem concerned the primary decomposition of $I_{\mathcal{M}}$; they conjectured a description in terms of subgraphs of a complete bipartite graph, which we show here to be correct.

In the course of this project the author carried out computations of primary decompositions for the ideal $\mathcal{M}_{I}$ for various values of $r_{1}, r_{2}$, and $r_{3}$ with the computer algebra system Singular [4, 5]. Thomas Kahle has recently written dedicated Macaulay2 code [3] for binomial primary decompositions [7], in which the same computations may be carried out.

A broad generalisation of this paper's results to the class of binomial edge ideals of graphs has been obtained by Herzog, Hibi, Hreinsdóttir, Kahle, and Rauh [6.

Let $K_{p, q}$ be the complete bipartite graph with bipartitioned vertex set $[p] \amalg[q]$. We say that a subgraph $G$ of $K_{r_{2}, r_{3}}$ is admissible if $G$ has vertex set $\left[r_{2}\right] \amalg\left[r_{3}\right]$ and all connected components of $G$ are isomorphic to some complete bipartite graph $K_{p, q}$ with $p, q \geq 1$.

Given a subgraph $G$ with edge set $\operatorname{Edges}(G)$, the prime $P_{G}$ to which it corresponds is defined to be

$$
\begin{equation*}
P_{G}=P_{G}^{(0)}+P_{G}^{(1)} \tag{1}
\end{equation*}
$$

where

$$
\begin{aligned}
P_{G}^{(0)}= & \left(p_{i j k}: i \in\left[r_{1}\right],(j, k) \notin \operatorname{Edges}(G)\right) \\
P_{G}^{(1)}= & \left(p_{i j k} p_{i^{\prime} j^{\prime} k^{\prime}}-p_{i j^{\prime} k^{\prime}} p_{i^{\prime} j k}: i, i^{\prime} \in\left[r_{1}\right]\right. \\
& \left.j, j^{\prime} \in\left[r_{2}\right] \text { and } k, k^{\prime} \in\left[r_{3}\right] \text { in the same connected component of } G\right) .
\end{aligned}
$$

Note that $j$ and $j^{\prime}$, and $k$ and $k^{\prime}$, need not be distinct. That is, for $\left(p_{i j k}\right)$ on the variety $V\left(P_{G}\right), p_{i j k}=0$ for $(j, k) \notin \operatorname{Edges}(G)$, and any pair of vectors $p \cdot j k$ and $p \cdot j^{\prime} k^{\prime}$ are proportional for $(j, k)$ and $\left(j^{\prime}, k^{\prime}\right)$ two edges in Edges $(G)$ in the same connected component of $G$. Later we will also want to refer to the individual summands $P_{C}^{(1)}$ of $P_{G}^{(1)}$, where $P_{C}^{(1)}$ includes only the generators $\left\{p_{i j k}:(j, k) \in C\right\}$ arising from edges in the connected component $C$.

Proposition 1. The set of minimal primes of the ideal $I_{\mathcal{M}}$ is

$$
\left\{P_{G}: G \text { an admissible graph on }\left[r_{2}\right] \amalg\left[r_{3}\right]\right\} .
$$

In particular, the value of $r_{1}$ is irrelevant to the combinatorial nature of the primary decomposition.

Proposition 1 was the original conjecture of Cartwright and Engström. It is a purely set-theoretic assertion, and is equivalent to the fact that

$$
\begin{equation*}
V\left(I_{\mathcal{M}}\right)=\bigcup_{G} V\left(P_{G}\right) \tag{2}
\end{equation*}
$$

as sets, where the union is over admissible graphs $G$. The ideas of a proof of Proposition 1 were anticipated in part 4 of the problem stated in [1, §6.6] which was framed for the prime corresponding to the subgraph $G$, the case where the conclusion of the intersection axiom is valid; they extend without great difficulty to the general case.

We will prove a stronger ideal-theoretic result. Let $\prec_{d p}$ be the revlex term order on $S$ over the lexicographic variable order on subscripts, with earlier subscripts more significant: thus under $\prec_{\mathrm{dp}}$, we have $p_{111} \prec_{\mathrm{dp}} p_{112} \prec_{\mathrm{dp}} p_{211}$.
Theorem 2. The primary decomposition

$$
\begin{equation*}
I_{\mathcal{M}}=\bigcap_{G} P_{G} \tag{3}
\end{equation*}
$$

holds and is an irredundant decomposition, where the union is over admissible graphs $G$ on $\left[r_{2}\right] \amalg\left[r_{3}\right]$. We moreover have

$$
\operatorname{in}_{\prec_{\mathrm{dp}}} I_{\mathcal{M}}=\operatorname{in}_{\prec_{\mathrm{dp}}} \bigcap_{G} P_{G}=\bigcap_{G} \operatorname{in}_{\prec_{\mathrm{dp}}} P_{G} .
$$

Furthermore, each primary component $\mathrm{in}_{\prec_{\mathrm{dp}}} P_{G}$ is squarefree, so $\mathrm{in}_{\prec_{\mathrm{dp}}} I_{\mathcal{M}}$ and hence $I_{\mathcal{M}}$ are radical ideals.

It is noted in [1, §6.6] that the number $\eta(p, q)$ of admissible graphs $G$ on $[p] \amalg[q]$ is given by the generating function

$$
\begin{equation*}
\exp \left(\left(e^{x}-1\right)\left(e^{y}-1\right)\right)=\sum_{p, q \geq 0} \eta(p, q) \frac{x^{p} y^{q}}{p!q!} \tag{4}
\end{equation*}
$$

which in that reference is said to follow from manipulations of Stirling numbers. This equation (4) can also be obtained as a direct consequence of a bivariate form of the exponential formula for exponential generating functions [9, §5.1], using the observation that

$$
\left(e^{x}-1\right)\left(e^{y}-1\right)=\sum_{p, q \geq 1} \frac{x^{p} y^{q}}{p!q!}
$$

is the exponential generating function for complete bipartite graphs with $p, q \geq 1$, and these are the possible connected components of admissible graphs.

We now review some standard facts on binomial and toric ideals [2]. Let $I$ be a binomial ideal in $\mathbb{C}\left[x_{1}, \ldots, x_{n}\right]$, generated by binomials of the form $x^{v}-x^{w}$ with $v, w \in \mathbb{N}^{n}$. There is a lattice $L_{I} \subseteq Z^{n}$ such that the localisation $I_{x_{1} \cdots x_{n}} \subseteq$ $\mathbb{C}\left[x_{1}^{ \pm 1}, \ldots, x_{n}^{ \pm 1}\right]$ has the form $\left(x^{v}-1: v \in L_{I}\right)$, provided that this localisation is a proper ideal, i.e. $I$ contains no monomial. If $\phi_{I}: \mathbb{Z}^{n} \rightarrow \mathbb{Z}^{m}$ is a $\mathbb{Z}$-linear map whose kernel is $L_{I}$, then $\phi_{I}$ provides a multigrading with respect to which $I$ is homogeneous. In statistical terms $\phi_{I}$ computes the minimal sufficient statistics for the statistical model associated to $I$.

Given a multivariate Laurent polynomial $f \in \mathbb{C}\left[x_{1}^{ \pm 1}, \ldots, x_{n}^{ \pm 1}\right], f$ lies in $I_{x_{1} \cdots x_{n}}$ if and only if, for each fiber $F$ of $\phi_{I}$, the sum of the coefficients on all monomials $x^{v}$ with $v \in F$ is zero. With respect to $\mathbb{C}\left[x_{1}, \ldots, x_{n}\right]$ a modified statement holds, as follows. For each fiber $F$, consider the graph $\Gamma_{F}(I)$ whose vertices are the set of vectors in $F$ with all entries nonnegative, and whose edge set is $\left\{(v, w): x^{v}-x^{w}\right.$ is a monomial multiple of a generator of $I\}$. In the statistical context these edges are known as moves. Then $f$ lies in $I \subseteq \mathbb{C}\left[x_{1}, \ldots, x_{n}\right]$ if and only if, for each connected
component $C$ of each $\Gamma_{F}(I)$, the sum of the coefficients on all monomials $x^{v}$ with $v \in C$ is zero. In particular $I$ is determined by this set of connected components.

Viewing $I \subseteq \mathbb{C}\left[x_{1}^{ \pm 1}, \ldots, x_{n}^{ \pm 1}\right]$ as the ideal of the toric subvariety of $\left(\mathbb{C}^{*}\right)^{n}$ associated to the lattice polytope $A$, Sturmfels in [8] shows that the radicals of the monomial initial ideals of $I$ are exactly the Stanley-Reisner ideals of regular triangulations of $A$. The Stanley-Reisner ideal $I_{\Delta}$ of a simplicial complex $\Delta$ on a set $T$ is the monomial ideal of $\mathbb{C}\left[x_{t}: t \in T\right]$ generated as a vector space by the products of variables $x_{t_{1}} \cdots x_{t_{k}}$ for which $\left\{t_{1}, \ldots, t_{k}\right\}$ does not contain a face of $\Delta$. Every squarefree monomial ideal is the Stanley-Reisner ideal of some simplicial complex, and primary decompositions of Stanley-Reisner ideals are easily described: $I_{\Delta}$ is the intersection of the ideals $\left(x_{t}: t \notin F\right)$ over all facets $F$ of $\Delta$.

Sturmfels also treats explicitly the ideal $I$ of $2 \times 2$ minors of an $r \times s$ matrix $Y=\left(y_{i j}\right)$, of which $P_{K}:=P_{K_{r_{2} r_{3}}}$ is a particular case. In this case the polytope $A$ is the product of two simplices, $\Delta_{r-1} \times \Delta_{s-1}$.

Theorem 3 ( 8 ). Let $I$ be the ideal of $2 \times 2$ minors of an $r \times s$ matrix of indeterminates. For any term order $\prec$, in ${ }_{\prec} I$ is a squarefree monomial ideal.

This immediately yields the radicality claim of Theorem 2 the in ${ }_{\prec} P_{G}$ are squarefree monomial ideals, so their associated primes are generated by subsets of the variables $\left\{p_{i j k}\right\}$.

We repeat from [8] one especially describable example of an initial ideal of this ideal $I$, namely $\operatorname{in}_{\prec_{\mathrm{dp}}} I$, corresponding to the case that $\Delta$ is the so-called staircase triangulation. Then the vertices of the simplices of $\Delta$ correspond to those sets $\pi$ of entries of the matrix $Y$ which form ("staircase") paths through $Y$ starting at the upper-left corner, taking only steps right and down, and terminating at the lower left corner. Hence to each such $\pi$ corresponds one primary component $Q_{G, \pi}$, generated by all $(r-1)(s-1)$ indeterminates not lying on $\pi$. Note that staircase paths are maximal subsets of indeterminates not including both $x_{i j^{\prime}}$ and $x_{i^{\prime} j}$ for any $i<i^{\prime}$ and $j<j^{\prime}$.

This framework suffices to understand the primary decomposition of in ${ }_{\prec} P_{G}$ for an arbitrary admissible graph $G$. Let the connected components of $G$ be $C_{1}, \ldots, C_{l}$, so that, from (11), $\operatorname{in}_{\prec} P_{G}$ is the sum of the ideal $\operatorname{in}_{\prec} P_{G}^{(0)}=P_{G}^{(0)}$ and the various ideals $\operatorname{in}_{\prec} P_{C_{i}}^{(1)}$, and moreover these summands use disjoint sets of variables. Suppose that $\mathrm{in}_{\prec} P_{C_{i}}^{(1)}=\bigcap_{j} Q_{C_{i}, j}$ are primary decompositions of the in ${ }_{\prec} P_{C_{i}}^{(1)}$. Then it follows that we have the primary decomposition

$$
\operatorname{in}_{\prec} P_{G}=\bigcap_{\mathbf{j}}\left(P_{G}^{(0)}+\sum_{i=1}^{l} \operatorname{in}_{\prec} Q_{C_{i}, j_{i}}\right)
$$

where $\mathbf{j}=\left(j_{1}, \ldots, j_{l}\right)$ ranges over the Cartesian product of the index sets in $\bigcap_{j} Q_{C, j}$.
Proof of Theorem 2, We begin by proving that the right side of (3) is an irredundant primary decomposition. Let $G$ be an admissible graph. For each connected component $C \subseteq G$ and fixed $i, P_{C}^{(1)}$ are the determinantal ideal of $2 \times 2$ minors of the matrix with $r_{1}$ rows and columns indexed by Edges $(C)$, whose $i,(j, k)$ entry is $p_{i j k}$. Being a determinantal ideal, $P_{C}^{(1)}$ is prime. The ideal $P_{G}^{(0)}$ is also prime, as it is generated by a collection of variables. Now $P_{G}$ is the sum of the prime
ideals $P_{G}^{(0)}$ and $P_{C}^{(1)}$ for each $C$, and the generators of these primes involve pairwise disjoint subsets of the unknowns $p_{i j k}$. It follows that $P_{G}$ itself is prime.

Irredundance is the assertion that for $G$ and $G^{\prime}$ distinct admissible graphs, $P_{G}$ is not contained in $P_{G^{\prime}}$. As above, we will think of the 3 -tensor $\left(p_{i j k}\right)$ as a size $r_{2} \times r_{3}$ table whose entries are vectors $(p \cdot j k)$ of length $r_{1}$. Then if $\left(p_{i j k}\right) \in V\left(P_{G}\right)$, all nonzero vectors in each subtable determined by a connected component of $G$ are proportional, while vectors outside of any subtable must be the zero vector. There is an open dense subset $U_{G} \subseteq V\left(P_{G}\right)$ such that for $\left(p_{i j k}\right) \in U_{G}$, no vector $p \cdot j k$ associated to a connected component of $G$ is zero, and no two associated to distinct components are dependent.

Now, $G$ may differ from $G^{\prime}$ in two fashions. If $G$ contains an edge $(j, k)$ that $G^{\prime}$ doesn't, the vector $\left(p \cdot{ }_{j k}\right)$ is zero on $V\left(P_{G^{\prime}}\right)$ but is nonzero on $U_{G}$ : hence $V\left(P_{G}\right) \nsubseteq$ $V\left(P_{G^{\prime}}\right)$. If not, $G \subseteq G^{\prime}$, but two edges $(j, k),\left(j^{\prime}, k^{\prime}\right)$ in different components of $G$ must be in the same component of $G^{\prime}$, in which case the vectors $(p \cdot j k)$ and $\left(p \cdot j^{\prime} k^{\prime}\right)$ are linearly dependent for $\left(p_{i j k}\right) \in V\left(P_{G^{\prime}}\right)$ but linearly independent on $U_{G}$ : hence also $V\left(P_{G}\right) \nsubseteq V\left(P_{G^{\prime}}\right)$. This proves irredundance.

Now we turn to proving (3). Let $\prec$ be $\prec_{\mathrm{dp}}$. Write $I=I_{\mathcal{M}}$. It is apparent that $I \subseteq P_{G}$ for each $G$. Indeed, given a generator $f$ of $I$, without loss of generality $f=p_{i j k} p_{i^{\prime} j^{\prime} k}-p_{i j^{\prime} k} p_{i^{\prime} j k}$, either both edges $(j, k)$ and $\left(j^{\prime}, k\right)$ lie in $\operatorname{Edges}(G)$, in which case $f$ is a generator of $P_{G}^{(1)}$, or one of these edges is not in $\operatorname{Edges}(G)$, in which case $f \in P_{G}^{(0)}$. Therefore the containments

$$
\operatorname{in}_{\prec} I \subseteq \operatorname{in}_{\prec} \bigcap_{G} P_{G} \subseteq \bigcap_{G} \operatorname{in}_{\prec} P_{G}
$$

hold. It now suffices to show an equality of Hilbert functions

$$
\begin{equation*}
H\left(S / \operatorname{in}_{\prec} I\right)=H\left(S / \bigcap_{G} \operatorname{in}_{\prec} P_{G}\right) . \tag{5}
\end{equation*}
$$

In the present case, the lattice $L_{I}$ associated to $I$ is generated by all vectors of the forms $e_{i j k}+e_{i^{\prime} j^{\prime} k}-e_{i j^{\prime} k}-e_{i^{\prime} j k}$ and $e_{i j k}+e_{i^{\prime} j k^{\prime}}-e_{i j k^{\prime}}-e_{i^{\prime} j k}$. The map $\phi_{I}: \mathbb{Z}^{r_{1} r_{2} r_{3}} \rightarrow \mathbb{Z}^{r_{1}+r_{2} r_{3}}$ sending $\left(u_{i j k}\right)$ to

$$
\left(\sum_{(j, k)} u_{1 j k}, \ldots, \sum_{(j, k)} u_{r_{1} j k}, \sum_{i} u_{i 11}, \ldots, \sum_{i} u_{i r_{2} r_{3}}\right)
$$

has kernel $L_{I}$ and thus induces the multigrading on $S$ by minimal sufficient statistics, with respect to which $I$ is homogeneous. In fact the analogue of (5) using Hilbert functions in the multigrading $\phi$ is also true, and it is this we will prove.

Let $d \in \mathbb{Z}^{r_{1}+r_{2} r_{3}}$ be the multidegree of some monomial, and write its components as $d_{i}$ for $i \in\left[r_{1}\right]$ and $d_{j k}$ for $j, k \in\left[r_{2}\right] \times\left[r_{3}\right]$. Let $G(d)$ be the bipartite graph with vertex set $\left[r_{2}\right] \amalg\left[r_{3}\right]$ and edge set $\left\{(j, k): d_{j k} \neq 0\right\}$. We now prove the following two claims:
Claim 1. $I_{d}=\left(P_{G(d)}\right)_{d}$.
Claim 2. $\left(\bigcap_{G} \mathrm{in}_{\prec} P_{G}\right)_{d}=\left(\operatorname{in}_{\prec} P_{G(d)}\right)_{d}$.
These claims, and the fact that an ideal and its initial ideal have the same Hilbert function, imply

$$
H\left(\operatorname{in}_{\prec} I\right)(d)=H(I)(d)=H\left(P_{G(d)}\right)(d)=H\left(\operatorname{in}_{\prec} P_{G(d)}\right)(d)=H\left(\bigcap_{G} \operatorname{in}_{\prec} P_{G}\right)(d),
$$

We conclude that (5) holds, proving Theorem 2,
Proof of Claim 1. Observe first that no polynomial homogeneous of multidegree $d$ can be divisible by any $p_{i j k}$ with $(j, k) \notin \operatorname{Edges}(G(d))$. Accordingly we have $\left(P_{G(d)}\right)_{d}=\left(P_{G(d)}^{(1)}\right)_{d}$, in the notation of (1), and we will work with $P_{G(d)}^{(1)}$ hereafter.

Since $I$ and $P_{G(d)}^{(1)}$ are binomial ideals generated by differences of monomials, it will suffice to show that the two graphs $\Gamma_{F}(I)$ and $\Gamma_{F}\left(P_{G(d)}^{(1)}\right)$ of moves on the fiber $F=\phi_{I}^{-1}(d)$ have the same partition into connected components. The refinement in one direction is clear: $\Gamma_{F}(I)$ is a subgraph of $\Gamma_{F}\left(P_{G(d)}^{(1)}\right)$, since $I_{d} \subseteq\left(P_{G(d)}\right)_{d}=$ $\left(P_{G(d)}^{(1)}\right)_{d}$, and indeed each generator of $I$ of multidegree at most $d$ is a monomial multiple of a generator of $P_{G(d)}^{(1)}$.

So given an edge of $\Gamma_{F}\left(P_{G(d)}^{(1)}\right)$, we must show that this edge is contained in a connected component of $\Gamma_{F}(I)$. Let $u, u^{\prime} \in F$ be the endpoints of an edge of $\Gamma_{F}\left(P_{G(d)}^{(1)}\right)$. Then $u=u^{\prime}+e_{i j k}+e_{i^{\prime} j^{\prime} k^{\prime}}-e_{i j^{\prime} k^{\prime}}-e_{i^{\prime} j k}$ for some $i, i^{\prime} \in\left[r_{1}\right]$ and $(j, k),\left(j^{\prime}, k^{\prime}\right)$ edges of $G(d)$ in the same component. By connectedness, there is a path of edges $e_{0}=\left(j^{\prime}, k^{\prime}\right), e_{1}, \ldots, e_{l}=(j, k)$ of $G(d)$ such that $e_{i}$ and $e_{i+1}$ share a vertex for each $i$. Corresponding to this path there exists a sequence of moves $\left(M_{m}\right)_{m=0, \ldots, l-1}$ in $I$, say $M_{m}=p^{u_{m}}-p^{u_{m+1}}$, where $u_{0}=u^{\prime}, u_{l}=u$, and where $M_{m}$ is a monomial multiple of

$$
p_{i, e_{m}} p_{i_{m}, e_{m+1}}-p_{i_{m}, e_{m}} p_{i, e_{m+1}}
$$

for some $i_{m} \in\left[r_{1}\right]$. So $u$ and $u^{\prime}$ are in a single connected component of $\Gamma_{F}(I)$.
Proof of Claim 2. Again, one containment is straightforward, namely in ${ }_{\prec} P_{G(d)} \subseteq$ $\bigcap_{G} \mathrm{in}_{\prec} P_{G}$. There is an admissible graph $G$ such that $P_{G} \subseteq P_{G(d)}$. Such a $G$ can be constructed per the discussion of irredundance, if we take $p$ to be a generic point of $V\left(P_{G(d)}\right)$. Then $\mathrm{in}_{\prec} P_{G} \subseteq$ in $\prec P_{G(d)}$ and this latter initial ideal is one of the ideals being intersected in $\bigcap_{G} \in \prec P_{G}$.

For the other containment, let $C$ be any connected bipartite graph on vertex set $\left[r_{2}\right] \amalg\left[r_{3}\right]$, such that $d_{j k}=0$ for $(j, k) \notin E(C)$. By the Stanley-Reisner description of the initial ideal for $\prec_{\mathrm{dp}}$, a monomial $p^{u} \in S$ of degree $d$ lies in in in $^{2} P_{C}=\operatorname{in}_{\prec} P_{K_{r_{2} r_{3}}}$ if and only if $p^{u}$ is divisible by $p_{i j^{\prime} k^{\prime}} p_{i^{\prime} j k}$ for some $i<i^{\prime}$ and $(j, k)<\left(j^{\prime}, k^{\prime}\right)$ lexicographically.

So if $p^{u}$ is a monomial of multidegree $d$ lying in in $\prec_{\mathrm{dp}_{\mathrm{p}}} P_{G(d)}$, it's divisible by some $p_{i j^{\prime} k^{\prime}} p_{i^{\prime} j k}$ with $i<i^{\prime}$ in $\left[r_{1}\right]$ and $(j, k)<\left(j^{\prime}, k^{\prime}\right)$ two edges lying in the same connected component of $G(d)$; it cannot occur that instead $p^{u}$ is divisible by some indeterminate $p_{i j^{\prime \prime} k^{\prime \prime}}$ for $\left(j^{\prime \prime}, k^{\prime \prime}\right)$ not an edge of $G(d)$, since $p_{u}$ has multidegree $d$. Now let $G$ be any admissible graph. If $G(d)$ is not a subset of $G$, then $p^{u}$ is divisible by some indeterminate $p_{i j^{\prime \prime} k^{\prime \prime}}$ with $\left(j^{\prime \prime}, k^{\prime \prime}\right) \notin E(G)$, so $p^{u} \in \operatorname{in}_{\prec} P_{G}$. Otherwise $G(d) \subseteq G$. In this case the edges $(j, k)$ and $\left(j^{\prime}, k^{\prime}\right)$ lie in the same component of $G$, and so $p_{i j^{\prime} k^{\prime}} p_{i^{\prime} j k} \mid p^{u}$ implies $p^{u} \in \operatorname{in}_{\prec} P_{G}$ again. Therefore $\mathrm{in}_{\prec} P_{G(d)} \supseteq \bigcap_{G} \mathrm{in}_{\prec} P_{G}$.

We close with the remark that we can describe explicitly which components of (3) contain a given point of $V\left(I_{\mathcal{M}}\right)$. Let $p=\left(p_{i j k}\right) \in \mathbb{C}^{r_{1} r_{2} r_{3}}$, and define $G(p)$ to be the bipartite graph on $\left[r_{2}\right] \amalg\left[r_{3}\right]$ with edge set $\left\{(j, k): p_{i j k} \neq 0\right.$ for some $\left.i\right\}$. Then the components $V\left(P_{G}\right)$ containing $\left(p_{i j k}\right)$ are exactly those for which $G$ can be
obtained from $G(p)$ by adding edges which don't unite two connected components of the latter containing respective edges $(j, k)$ and $\left(j^{\prime}, k^{\prime}\right)$ such that $p \cdot j k$ and $p \cdot j^{\prime} k^{\prime}$ are not proportional. If $p \in U_{G(p)}$, then these components are exactly those for which $G$ adds only edges which don't unite two connected components of $G(p)$, neither of which is an isolated vertex.

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