Uniform density static fluid sphere in higher dimensions and its universality

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In Newtonian theory, gravity inside a constant density static sphere is independent of spacetime dimension. Interestingly this general result is also carried over to Einsteinian as well as higher order Lovelock gravity notwithstanding their nonlinear character. We establish the universality of Schwarzschild interior solution describing a uniform density sphere for all $n \geq 4$.

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I. INTRODUCTION

In Newtonian gravity, the gravitational potential at any point inside a fluid sphere is given by $-M(r)/r^{n-3}$ for $n \geq 4$ dimensional spacetime. Now $M(r) = \int \rho r^{n-2} dr$ which for constant density will go as ρr^{n-1} and then the potential will go as $\rho r^{n-1}/r^{n-3} = \rho r^2$ and is therefore independent of the dimension. This is an interesting general result: for uniform density sphere, gravity has the universal character that it is independent of the dimension of spacetime. It is then a natural question to ask does this result carry over to Einsteinian gravity? In general relativistic language it is equivalent to ask: does Schwarzschild interior solution which describes uniform density sphere in 4 dimension remain good for all $n \geq 4$? The main purpose of this paper is to show that it is indeed the case not only for Einstein gravity but also for higher order Lovelock gravity. It is remarkable that this general feature holds true notwithstanding highly nonlinear character of the theory.

In static spherically symmetric spacetime, we have two equations to handle, one for density which easily integrates to give g_{rr} and the other is pressure isotropy equation determining g_{tt}. So long as density remains constant the former equation will always integrate to give g_{rr} in all dimensions with constant density redefined. Then we just need to make the latter equation free of dimension n so that constant density Schwarzschild interior solution becomes universally true for all n. In particular it turns out for Einstein theory that the condition required to make the latter equation independent of n does in fact determine g_{rr}. That is, it is indeed equivalent to integration of the former equation for constant density. Thus we have Schwarzschild interior solution valid for all $n \geq 4$.

Higher dimension is a natural playground for string theory and string inspired investigations (see a comprehensive review [1]). The most popular studies have been of higher dimensional black holes [2] with a view to gain greater and deeper insight into quantum phenomena, black hole entropy and the well known AdS/CFT correspondence [3]. There have also been studies of fluid spheres in higher dimensions [4]. We shall however focus on universal character of constant density solution in Einstein and Lovelock theory and its matching with the corresponding exterior solution. The paper is organized as follows. In the next section, we establish the universality of uniform density solution for Einstein and Einstein-Gauss-Bonnet theories and demonstrate the matching with exterior solution for the 5-dimensional Gauss-Bonnet black hole. We conclude with a discussion.

II. UNIFORM DENSITY SPHERE

A. Einstein case

We begin with the general static spherically symmetric metric given by

$$ds^2 = e^{\nu} dt^2 - e^{\lambda} dr^2 - r^2 d\Omega_{n-2}^2 \qquad (1)$$

where $d\Omega_{n-2}^2$ is the metric on a unit $(n-2)$-sphere. For the Einstein equation in the natural units $(8\pi G = c = 1)$,

$$G_{AB} = R_{AB} - \frac{1}{2} R g_{AB} = -T_{AB} \qquad (2)$$

and for perfect fluid, $T_A^B = diag(\rho, -p, -p, ..., -p)$, we write

$$e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{n-3}{r^2}\right) + \frac{n-3}{r^2} = \frac{2}{n-2}\rho \qquad (3)$$

$$e^{-\lambda}\left(\frac{\nu'}{r} + \frac{n-3}{r^2}\right) - \frac{n-3}{r^2} = \frac{2}{n-2}p \qquad (4)$$

$$e^{-\lambda}\left(2\nu'' + \nu'^2 - \lambda'\nu' - 2\frac{\nu'}{r}\right) \\ -2(n-3)\left(\frac{e^{-\lambda}\lambda'}{r} + 2\frac{e^{-\lambda}}{r^2} - \frac{2}{r^2}\right) = 0. \qquad (5)$$

Let us rewrite the last equation as

$$e^{-\lambda}(2\nu'' + {\nu'}^2 - \lambda'\nu' - 2\frac{\nu' + \lambda'}{r} - \frac{4}{r^2}) + \frac{4}{r^2}$$
$$-2(n-4)((n-1)(\frac{e^{-\lambda}}{r^2} - \frac{1}{r^2}) + \frac{2\rho}{n-2}) = 0. \qquad (6)$$

We now set the coefficient of $(n-4)$ to zero so that the equation remains the same for all $n \geq 4$. This then straightway determines $e^{-\lambda}$ without integration and it is given by

$$e^{-\lambda} = 1 - \frac{2}{(n-1)(n-2)}\rho r^2. \qquad (7)$$

Note that integration of Eqn (3) for constant density will also give the same expression plus k/r where k is an integration constant. Then we have to choose $k = 0$ to avoid singularity at the center. On the other hand the requirement of dimension independence determines $e^{-\lambda}$ straightaway with regularity at the center and this also implies constancy of density. An alternative identification of constant density is that the gravitational field inside a fluid sphere is independent of spacetime dimension ≥ 4. This universal property is therefore true if and only if density is constant.

As is well known, Eqn (6) on substituting Eqn (7) admits the general solution as given by

$$e^{\nu/2} = A + Be^{-\lambda/2} \qquad (8)$$

where A and B are constants of integration to be determined by matching to the exterior solution. This is the Schwarzschild interior solution for constant density sphere which is independent of the dimension except for redefinition of constant density as $\rho/(n-1)(n-2)$. This proves universality of Schwarzschild interior solution for all $n \geq 4$.

The Newtonian result that gravity inside a uniform density sphere is independent of the spacetime dimension is thus carried over to general relativity as well despite nonlinearity of the equations. That is Schwarzschild interior solution is valid for all $n \geq 4$. Since there exist more general actions like Lovelock polynomial and $f(R)$ than the linear Einstein-Hilbert, it would be interesting to see whether this result would carry through there as well. That is what we take up next.

B. Gauss-Bonnet case

There is a natural generalization of Einstein action to Lovelock action which is a homogeneous polynomial in Riemann curvature with Einstein being the linear order. It has the remarkable property that on variation it still gives the second order quasi-linear equation which is its distinguishing feature. The higher order terms make nonzero contribution in the equation only for dimension ≥ 5. The quadratic term in the polynomial is known as Gauss-Bonnet and for that we write the action as

$$S = \int d^n x\sqrt{-g}\left[\frac{1}{2}(R - 2\Lambda + \alpha L_{GB})\right] + S_{\text{matter}}, \qquad (9)$$

where α is the GB coupling constant and all other symbols having their usual meaning. The GB Lagrangian is the specific combination of Ricci scalar, Ricci and Riemann curvatures and it is given by

$$L_{GB} = R^2 - 4R_{AB}R^{AB} + R_{ABCD}R^{ABCD}. \qquad (10)$$

This form of action is known also to follow from low-energy limit of heterotic superstring theory [5]. In that case, α is identified with the inverse string tension and is positive definite which is also required for stability of Minkowski spacetime.

The gravitational equation following from the action (9) is given by

$$G^A_B + \alpha H^A_B = -T^A_B, \qquad (11)$$

where

$$H_{AB} \equiv 2\Big[RR_{AB} - 2R_{AC}R^C_B - 2R^{CD}R_{ACBD}$$
$$+R_A^{\ CDE}R_{BCDE}\Big] - \frac{1}{2}g_{AB}L_{GB}. \qquad (12)$$

After some manipulations the density equation could be cast in the form

$$(\tilde{\alpha}r^{n-5}f^2 + r^{n-3}f)' = \frac{2}{n-2}\rho r^{n-2} \qquad (13)$$

where $f = 1 - e^{-\lambda}$, $\tilde{\alpha} = (n-3)(n-4)\alpha$ and a prime denotes derivative w.r.t r. Defining $2\rho = (n-1)(n-2)/r_0^2$, we write on integration

$$\tilde{\alpha}r^{n-5}f^2 + r^{n-3}f = \frac{r^{n-1}}{r_0^2} + k \qquad (14)$$

where k is a constant of integration which should be set to zero for regularity of the solution at the center. Solving for f, we get

$$e^{-\lambda} = 1 - f = 1 - r^2/R_0^2 \qquad (15)$$

where

$$\frac{1}{R_0^2} = \frac{\sqrt{1 + 4\tilde{\alpha}/r_0^2} - 1}{2\tilde{\alpha}}. \qquad (16)$$

The appropriate choice of sign is made so as to admit the limit $\alpha \to 0$ yielding the Einstein solution. The pressure isotropy condition, $(T^1_1 = T^2_2)$, gives after substitution for $e^{-\lambda}$,

$$(2\nu'' + {\nu'}^2)(1 - \frac{r^2}{R_0^2}) - \frac{2\nu'}{r} = 0. \qquad (17)$$

This readily integrates to give

$$e^{\nu/2} = A + Be^{-\lambda/2} \tag{18}$$

which is the same as the solution given in Eqn. (8).

This shows the universality of Schwarzschild interior solution for Einstein-Gauss-Bonnet gravity as it is valid for all $n \geq 4$. It is only the constant density gets redefined in terms of r_0 and R_0. As the result carries through the linear (Einstein) as well as quadratic (Gauss-Bonnet) order and hence it would hold good for any order in Lovelock polynomial, only density would be redefined by a new R_0. We have thus established the general result that gravity inside a uniform density sphere is universal for all dimensions as well as for the entire Lovelock polynomial action.

C. Matching with the exterior

Now we would like to demonstrate matching of the interior with the corresponding exterior 5-dimensional Gauss-Bonnet black hole solution [6]. In the interior pressure is given by

$$p = \frac{3(r_0 - \sqrt{r_0^2 + 8\alpha})}{4\alpha r_0} \left(1 - \frac{\sqrt{r_0^2+8\alpha}}{r_0} \left(1 + \frac{2A\sqrt{r_0\alpha}}{B\sqrt{r^2(r_0 - \sqrt{r_0^2+8\alpha}) + 4\alpha r_0}}\right)^{-1}\right). \tag{19}$$

At the boundary, $r = r_\Sigma$, pressure vanishes which is equivalent to the continuity of g'_{tt} and that is what we shall employ. Besides this, the metric should be continuous across r_Σ. The 5-dimensional Gauss-Bonnet black hole is given by the metric [6],

$$ds^2 = F(r)dt^2 - \frac{dr^2}{F(r)} - r^2(d\theta^2 + \sin^2(\theta)(d\varphi^2 + \sin^2(\psi)d\psi^2))$$

where

$$F(r) = 1 + \frac{r^2}{4\alpha}(1 - \sqrt{1 + 8M\alpha/r^4}).$$

Now matching g_{rr} means $[g_{rr}]_\Sigma = 0$ which after appropriate substitutions determines the mass enclosed inside the radius r_Σ,

$$M = \frac{1}{6}\rho r_\Sigma^4. \tag{20}$$

Further $[g_{tt}]_\Sigma = 0$ and $[g'_{tt}]_\Sigma = 0$ determine the constants,

$$A = (1 - B)\sqrt{1 - r_\Sigma^2/R_0^2} \tag{21}$$

and

$$B = -(1 + \frac{8\alpha M}{r_\Sigma^4})^{-1/2}. \tag{22}$$

This completes the matching of the interior and exterior solutions.

III. DISCUSSION

We have established that gravitational field inside a constant density fluid sphere has universal character for spacetime dimensions ≥ 4. This is true not only for Einstein-Hilbert action but also for the more general Lovelock action which is a homogeneous polynomial in Riemann curvature. We have explicitly shown this for the linear Einstein and the quadratic Gauss-Bonnet cases and similarly it could be shown to hold good for any order in Lovelock polynomial. That is, Schwarzschild interior solution describing gravitational field of constant density sphere is true for all spacetime dimensions ≥ 4 as well as for the higher order Lovelock polynomial gravity. It turns out that the necessary and sufficient condition for universality of fluid sphere is that its density must be constant.

This result is obvious but perhaps not so known in Newtonian gravity as argued in the opening of the paper. It is however not so for Einstein-Lovelock gravity because of its highly nonlinear character. Yet it is carried through because the equation of motion still remains second order quasi-linear. It is this feature which carries the general character of the solution into higher order gravity. Then it would not in general be carried along for $f(R)$ gravity which in general does not have the quasi-linear character. Apart from Lovelock's original derivation of the action [7], there are two other characterizations of Lovelock action [8, 9]. It is interesting to note that this could be yet another identifying property of Lovelock gravity.

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