On Classical Analogs of Quantum Schwarzschild and Reissner-Nordstrom Black Holes. Solving the "Mystery of $\log 3$"

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Abstract

The model is built in which the main global properties of classical and quasi-classical black holes become local. These are the event horizon, "no-hair", temperature and entropy. Our construction is based on the features of a quantum collapse, discovered while studying some quantum black hole models. But it is purely classical, and this allows to use the Einstein equations and classical (local) thermodynamics and explain in this way the "$\log 3$" - puzzle.

Classical definition of the black hole is based on the existence of the event horizon [1]. The notion of the latter is global and requires the knowledge of the whole history, both past and future. Classical "black hole has no hair" [2] and is described by only few parameters. The process of becoming bald is also global, its duration, formally, is infinite. At late times the frequencies of the decaying modes are complex with equidistantly growing imaginary parts, they are called quasi-normal frequencies. Their real parts approaches the finite limit. The appearance of damping oscillations of this type points to the existence of some resonance frequency inherent in a black hole. Investigation of the processes near the event horizon showed that they can be reversible and irreversible [3, 4], like in thermodynamics, and the black hole area cannot decrease.

These features were generalized by suggestion [5] that the Schwarzschild black hole, indeed, can be considered as some thermal equilibrium state having both the temperature and entropy, the latter being a reflection of the "no hair" property, i.e., the black hole with given parameters could be formed by enormously many different ways. The suggested proportionality of the black hole entropy to the horizon area was then confirmed by extending the four laws of thermodynamics to the general type of black holes, possessing mass electric charge and angular momentum [6]. Finally, calculations made by S.Hawking [7] showed that the temperature is real, black holes should evaporate, and the entropy is one fourth of the horizon area.

Numerous attempts to quantize black holes taught us that their mass spectrum is discrete, and the temperature and entropy are the properties of the quasi-classical stage. The detailed description and understanding of such a quasi-classical regime is very important, but at the same time is very difficult because of the global character of the main features of the classical black holes. In this paper we construct the so-called classical analogs of quantum black holes for which global properties of classical and semiclassical black holes become local, what makes their description and understanding much more easy.

Our starting point is a quantum mechanical model for a spherically symmetric self-gravitating thin dust shell. Such a shell is a simplest generalization of a point particle with the advantage that it has the dynamical degree of freedom - the shell radius, and the corresponding classical problem has the exact solution with the full account for back reaction of the matter source on the space-time metric. Due to the spherical symmetry the shell's radius is the only dynamical variable of the whole system: space-time plus a gravitating source, and the quantum functional Wheeler-DeWitt equation is reduced to the following stationary Schroedinger equation in finite differences [8, 9]:

$$\Psi(m, m_{in}, S+i\zeta) + \Psi(m, m_{in}, S-i\zeta) = \frac{F_{in} + F_{out} - \frac{M^2}{4m^2 S}}{\sqrt{F_{in}}\sqrt{F_{out}}} \Psi(m, m_{in}, S), \quad (1)$$

where $m = m_{out} = m_{tot}$ - the total mass of the system, m_{in} - the Schwarzschild mass inside, M is the bare mass of the shell, $S = \frac{R^2}{4G^2 m^2}$ (R - radius, G - gravitational constant), $F = 1 - \frac{2Gm}{R}$, $\zeta = \frac{m_{Pl}^2}{2m^2}$ ($m_{Pl} = \sqrt{\frac{\hbar c}{G}}$ is the Planckian mass and we use units with $\hbar = c = k = 1$, $\hbar$ - Planck constant, c -

speed of light, k - Bolzmann constant). By investigation of wave functions in the vicinity of singular points (infinities and singularities) and around the branching points (apparent horizons) the following discrete mass spectrum for bound states was found ($\Delta m = m_{out} - m_{in}$) [9]:

$$\begin{aligned} \frac{2(\Delta m)^2 - M^2}{\sqrt{M^2 - (\Delta m)^2}} &= \frac{2m_{Pl}^2}{\Delta m + m_{in}}\, n\,, \\ M^2 - (\Delta m)^2 &= 2(1+2p)\, m_{Pl}^2\,, \end{aligned} \tag{2}$$

where n and $p \geq 0$ are integers. The appearance of two quantum numbers instead of one in conventional quantum mechanics is due to the nontrivial causal structure of the complete Schwarzschild manifold which contains the so-called Einstein-Rosen bridge and, correspondingly, two isometric regions with spatial infinities. Unlike the classical shell motion confined to only one of these regions, the wave function of the quantum shell "feels" both infinities. The principal quantum number n comes from the boundary condition at "our" infinity, while the new, second, quantum number p - from the other one. The above spectrum is not universal in the sense that the corresponding wave functions form a two-parameter family $\Psi_{n,p}$. But for the quantum Schwarzschild black hole we expect a one-parameter family of solutions, because quantum black holes should not have "no hairs", otherwise there will be no smooth classical limit. This means that our spectrum is not a quantum black hole spectrum, and corresponding quantum shells do not collapse (like an electron in hydrogen atom). Physically, it is quite understandable, because the radiation was not included into consideration. Therefore, quantum gravitational collapse (even spherically symmetric) is accompanied with radiation. This is true also for the unbound motion because, though a principal quantum number n disappears in this case, the second quantum number p still exists and the collapsing shell is eventually settled into some bound state. The appearance of two quantum numbers instead of one leads to yet another consequence: the quantum gravitational collapse proceeds via production new shells, increasing the inner mass m_{in} inside the primary shell. Such a process can go in many different ways, so, the quantum collapse is accompanied with the loss of information, thus converting an initially pure quantum state into some thermal mixed one. But how could quantum collapse be stopped? The natural limit is the transition from a black hole-like shell to a wormhole-like shell by crossing an Einstein-Rosen bridge, since such a transition requires (at least in a quasi-classical regime) insertion of infinitely large volume, which probability is, of course, zero. Computer simulations show that the process of quantum gravitational process stops when the principal quantum number becomes zero, $n = 0$. The point $n = 0$ in our spectrum is very special. In this state the shell does not "feel" not only the outer region (what is natural for the spherically symmetric configuration), but it does not know anything about what is going on inside. It "feels" only itself. Such a situation reminds the "no hair" property of a classical black hole. Finally, when all the shells (both the primary one and newly produced) are in the corresponding states $n_i = 0$, the whole system does not "remember" its own history. And it is this "no- memory" state that can be called "the quantum black hole". Note, that the total masses of all the shells obey the relation

$$\Delta m_i = \frac{1}{\sqrt{2}} M_i. \tag{3}$$

The subsequent quantum Hawking's evaporation can proceed via some collective excitations.

The final state of quantum gravitational collapse, the quantum black hole, can be viewed as some stationary matter distribution. Therefore, we may hope that for massive enough quantum black hole such a distribution is described approximately by a classical static spherically symmetric perfect fluid with energy density ε and pressure p obeying classical Einstein equations. This is what we call a classical analog of a quantum black hole. Of course, in such a case the corresponding classical distribution has to be very specific. To study its main features let us consider the situation in more details.

Any static spherically symmetric metric can be written in the form

$$ds^2 = e^{\nu} dt^2 - e^{\lambda} dr^2 - r^2 (d\vartheta^2 + \sin^2\vartheta d\varphi^2). \tag{4}$$

Here r is the radius of a sphere with the area $A = 4\pi r^2, \nu = \nu(r), \lambda = \lambda(r)$. The Einstein equations are (prime denotes differentiation in r):

$$\begin{aligned} - e^{-\lambda}\left(\frac{1}{r^2} - \frac{\lambda'}{r}\right) + \frac{1}{r^2} &= 8\pi G\varepsilon\,, \\ -e^{-\lambda}\left(\frac{1}{r^2} + \frac{\nu'}{r}\right) + \frac{1}{r^2} &= -8\pi G p\,, \\ -\frac{1}{2}\left(\nu'' + \frac{{\nu'}^2}{2} + \frac{\nu' - \lambda'}{r} - \frac{\nu'\lambda'}{2}\right) &= -8\pi G p\,. \end{aligned} \tag{5}$$

We see that there are three equations for four unknown functions of one variable, namely, $\nu(r), \lambda(r)$, $\varepsilon(r)$ and $p(r)$. But, even we would know an equation of state for our perfect fluid, $p = p(\varepsilon)$, the closed (formally) system of equations would have too many solutions. We need, therefore, some selection rules in order to single out the classical analog of quantum black hole. Surely, the "no hair" feature should be the main criterium. Thus, we have to adjust our previous definition of the "no-memory" state to the case of a continuum matter distribution. For this, let us integrate the first of Eqns.(5):

$$e^{-\lambda} = 1 - \frac{2Gm(r)}{r}, \tag{6}$$

where

$$m(r) = 4\pi \int_0^r \varepsilon \tilde{r}^2 d\tilde{r} \tag{7}$$

is the mass function that must be identified with m_{in}. Now, the "no memory" principle is readily formulated as the requirement, that $m(r) = ar$, i.e.,

$$\begin{aligned} e^{-\lambda} &= 1 - 2Ga = const\,, \\ \varepsilon &= \frac{a}{4\pi\, G\, r^2}. \end{aligned} \tag{8}$$

Note, that in static case, the inverse metric coefficient $e^{-\lambda}$ is an invariant which in the general spherically symmetric space-time reads as $\Delta = -e^{-\lambda} = g^{ik} R_{,i} R_{,k}$ and is nothing more but a squared normal vector to the surface of constant radius $R(x^i) = R(t,q) = const$. We can also introduce a bare mass function $M(r)$ (the mass of the system inside a sphere of radius r without the gravitational mass defect).

$$M(r) = \int \varepsilon dV = 4\pi \int_0^r \varepsilon(\tilde{r}) e^{\frac{\lambda}{2}}(\tilde{r}) \tilde{r}^2 d\tilde{r} = \frac{ar}{\sqrt{1-2Ga}}\,. \tag{9}$$

The remaining two equations (5) can now be solved for $p(r)$ and $e^{\nu}(r)$. The general solution is rather complex, but the correct non-relativistic limit for the pressure $p(r)$ (we are to reproduce the famous equation for hydrostatic equilibrium) has only the following one-parameter family:

$$p(r) = \frac{b}{4\pi r^2}\,, \tag{10}$$

where

$$b = \frac{1}{G}\left(1 - 3Ga - \sqrt{1-2Ga}\sqrt{1-4Ga}\right) . \tag{11}$$

We see that the solution exists only for $a \le \frac{1}{4G}$, then $b \le a$. The physical meaning of these inequalities is that the speed of sound cannot exceed the speed of light, $v_{sound}^2 = \frac{b}{a} \le 1 = c^2$, the equality being reached just for $a = b = \frac{1}{4G}$. Finally, for the temporal metric coefficient $g_{00} = e^\nu$ we get:

$$e^\nu = C_0 r^{\frac{4b}{a+b}} = C_0 r^{2G\frac{a+b}{1-2Ga}} . \tag{12}$$

Thus, demanding the "no-memory" feature and existence of the correct non-relativistic limit, we obtained the two-parameter family of static solutions. But we need a one-parameter family, so we have to continue our search.

Calculation of the Riemann curvature tensor $R^\mu_{\nu\lambda\sigma}$ shows that it is divergent at $r = 0$ for $b < a$. But, if $a = b = \frac{1}{4G}$ we are witnessing a miracle, the (before) divergent components become zero, and the remaining nonzero ones equal

$$\begin{aligned} R^0_{202} &= -(1-2Ga) = -\frac{1}{2}\,, \quad \left(R^2_{020} = \frac{1}{2}C_0^2\right)\,; \\ R^0_{303} &= -(1-2Ga) = -\frac{1}{2}\,, \quad \left(R^0_{030} = \frac{1}{2}C_0^2\right)\,; \\ R^2_{323} &= 2Ga\sin^2\theta = \frac{1}{2}\sin^2\theta\,, \quad \left(R^3_{232} = \frac{1}{2}\right)\,, \end{aligned} \tag{13}$$

and the only nonzero component of the Ricci tensor $R_{\mu\nu} (= R^\alpha_{\mu\alpha\nu})$ equals to

$$R_{00} = C_0^2. \tag{14}$$

Thus, demanding, in addition to the previous two very natural requirements, the third one (also natural), namely, the absence of the real singularity at $r = 0$, we arrive at the following one-parameter family solutions to the Einstein equations (5):

$$\begin{aligned} g_{00} &= e^\nu = C_0^2 r^2, \\ g_{11} &= -e^\lambda = -\sqrt{2}, \\ \varepsilon &= p = \frac{1}{16\pi G r^2}. \end{aligned} \tag{15}$$

So, the equation of state of our perfect fluid is the stiffest possible one. The constant of integration C_0 can be determined by matching the interior and exterior metrics at some boundary radius $r = r_0$. Let us suppose that for $r > r_0$ the space-time is empty, so, the interior should be matched to the Schwarzschild metric, labeled by the mass parameter m. Of course, to compensate the jump in the pressure $\Delta p \, (= p(r_0) = p_0)$ we must include in our model some surface tension Σ, so, actually, we are dealing with a some sort of liquid. It is easy to check, that

$$\begin{aligned} C_0^2 &= \frac{1}{2r_0^2}\,; \;\; \Delta p = \frac{2\Sigma}{\sqrt{2} r_0}\,; \\ e^{\nu} &= \frac{1}{2}\left(\frac{r}{r_0}\right)^2; \;\; p_0 = \varepsilon_0 = \frac{1}{16\pi G r_0^2}\,; \\ m &= m_0 = \frac{r_0}{4G}\,. \end{aligned} \tag{16}$$

Note, that the bare mass $M = \sqrt{2} m$, the relation is exactly the same as for the shell "no memory" state (3), and $r_0 = 4Gm_0$, so, the size of our analog of quantum black hole is twice as that of classical black hole. But how about the special point in our solution, $r = 0$? It is not a trivial coordinate singularity, like in a three-dimensional spherically symmetric case, because

$$ds^2(r = 0) = 0. \tag{17}$$

This looks rather like an event horizon. Indeed, it can be easily shown that the two-dimensional $(t - r)$-part of our metric describes a locally flat manifold. Since the static observers at $r = const$ are, in fact, accelerated, this is a Rindler space-time with the event horizon at $r = 0$. By definition, the surface $r = 0$ can not be crossed and it is in this sense that the generally global event horizon becomes local. The corresponding Rindler parameter which in more general case is called the "surface gravity $\varkappa$", equals

$$\varkappa = \frac{1}{2}\left|\frac{d\nu}{dr}\right| e^{\frac{\nu-\lambda}{2}} = \frac{C_0}{\sqrt{2}} = \frac{1}{2r_0}. \tag{18}$$

It is well known that uniformly accelerated Rindler observers register particles with Planckian spectrum. The corresponding temperature was calculated by W.G.Unruh by investigating a quantum field theory on two-dimensional manifolds with an event horizon [10]. The Unruh temperature equals

$$T_U = \frac{a}{2\pi}. \tag{19}$$

The same value can be obtained by considering an Euclidean version of the Rindler space-time, demanding the absence of conical singularity that replaces there the event horizon, and then equating the inverse period of the imaginary time to the temperature. Calculated in this way, it can be called the "topological temperature", T_{top}. The absence of the conical singularity means that the unfolded cone has no angle deficit, and the period of the corresponding azimuth angle equals 2π. One must remember that the temperature is not an invariant (scalar) but the temporal component of the heat flow four-vector, so, its value depends on the choice of clocks (= time coordinate). The Unruh temperature T_U is the temperature measured by the observer who is using the proper time, i.e., with $g_{00} = 1$. When the same observer is using some local time, then he must deal with the local temperature

$$T_{loc} = \frac{T_U}{\sqrt{g_{00}}}. \tag{20}$$

Up to now the model is the same as was elaborated by the author in 2003 [11]. But at this point we encounter a dilemma. The problem is that writing the Unruh temperature in terms of the total mass m we get

$$T_U = \frac{1}{4\pi r_0} = \frac{1}{16\pi\, G\, m}\,, \tag{21}$$

what is two times less than the Hawking temperature [7]

$$T_H = \frac{1}{8\pi\, G\, m}\,. \tag{22}$$

This is the Unruh temperature measured by the observer sitting at rest just at the event horizon and, at the same time, by the distant inertial observer at spatial infinity where $g_{00} = 1$. Adopting the "natural" boundary condition that the temperature inside is equal to that of outside we get in the Euclidean section a discontinuity (the period of the azimuthal angle corresponding to the imaginary time equals π inside and 2π outside). In addition, as one can easily check, we have a jump in the local temperatures measured by the Rindler observers just inside and outside the boundary $r = r_0$. Such a jump is only partly compensated by the surface tension Σ and can be considered as caused the start of the irreversible process of converting the energy (mass) of the inner region into the radiation. Besides, it is difficult to explain why local observers just inside and outside the boundary $r =$

r_0 who know nothing about what is going on elsewhere (especially in the Euclidean section) should give different interpretations to the intensities of particle creation in heir detectors. On the other hand, if we make the second choice, i.e., that the temperature inside is just the Unruh temperature for the corresponding Rindler space-time, $T_U = \frac{1}{4\pi\, r_0}$, then everything is smooth in the Euclidean region and there is no problem with local observations, but now we should somehow explain the origin of the Hawking temperature $T_H = \frac{1}{8\pi\, G\, m}$ because the latter is obviously measured at infinity by detecting the heat flux.

Before coming to the thermodynamics we should describe two interesting and very important features of quasi-classical black holes. The first of them is the quantization of entropy. In 1973 J.Bekenstein made the remarkable observation [5] that the horizon area of a non-extremal black hole behaves as a classical adiabatic invariant. In the spirit of the Ehrenfest principle he conjectured that the horizon area and, therefore, the quantum black hole entropy, should have a discrete spectrum of the form

$$S = \gamma\, n, \quad n = 1, 2, 3, ... \tag{23}$$

Applying statistical physics arguments, J.Bekenstein and V.Mukhanov showed [12, 13] that the spacing coefficient must be equal to $\gamma = \log k\,, \ \ k = 2, 3, ...$ Such a value does not contradict the $\log 2$-prediction coming from the information theory which connects the entropy production to the information loss, and the very famous claim by J.A.Wheeler "It from Bit". The second feature is the existence of the proper frequency, inherent in the black hole with given parameters. This frequency was discovered when studying the behavior of various types of perturbations (scalar, vector, tensor) around a black hole (see, e.g., [14]) in the attempts to understand how the process of gravitational collapse proceeds resulting eventually in the black hole baldness. The evolution of a small perturbation is governed by a one-dimensional Schroedinger-like wave equation, first derived by T.Regge and J.A.Wheeler [15] in the case of Schwarzschild black hole. For scalar massless (long range) perturbations it reads as follows, assuming the time dependence of the form $e^{-i\omega t}$:

$$\frac{d^2\Psi}{dr^{\star 2}} + \left[w^2 - V(r)\right]\Psi = 0\,, \tag{24}$$

where the tortoise radial coordinate $r^\star$ is related to the radius r by $dr^\star = \frac{dr}{1-\frac{2Gm}{r}}$, m is the Schwarzschild black hole mass, and the effective potential is given by (l is the multipole moment)

$$V(r) = \left(1 - \frac{2\,G\,m}{r}\right)\left(\frac{l(l+1)}{r^2} + \frac{2}{r^3}\right)\,. \tag{25}$$

It appeared that, at late times, all perturbations are radiated away in a manner reminiscent of the last pure dying tones of a ringing bell. To describe these free oscillations of the black hole the notion of quasi-normal modes was introduced [16]. The quasi-normal frequencies (ringing frequencies) are characteristic of the black hole itself, they correspond to solutions of the above wave equation with the physical boundary conditions of purely outgoing waves at spatial infinity ($r^\star \to \infty$) and purely ingoing wave crossing the event horizon ($r^\star \to -\infty$). There are infinite number (for a given harmonic index) of complex frequencies with decreasing relaxation times, i.e., increasing imaginary parts. Their real parts, on the other hand, approaches an asymptotic constant value. For the Schwarzschild black hole of mass m the quasi-normal frequencies equal ($n \gg 1$) [17]

$$G\,m\,w_n = 0.0437123 - \frac{i}{4}\left(n + \frac{1}{2}\right) + O[(n+1)^{-1/2}]. \tag{26}$$

In 1998 S.Hod recognized [18] that the real part is actually equal to $\frac{\log 3}{8\pi}$ and, using the famous Bohr's corresponding principle: "transitions frequencies at large quantum numbers should equal classical oscillation frequencies" and the relations $dm = Re\,w_n$, $A = 4\pi r_g^2 = 16\pi G^2 m^2 = 4G\,S$, deduced that $\gamma = \log 3$. This value is also in agreement with the general result obtained by J.Bekenstein and V.Mukhanov, but contradicts the value $\log 2$ advocated by the "It from Bit" claim.

Note, that both the entropy quantization and the quantum nature of radiation suggest the discrete nature of the quantum (and, correspondingly, quasi-classical) black hole constituents. We can imagine some number of quasi-particles, black hole phonons, interference between which results in equidistant spectrum of excitations, and transition from different energy states to their neighbors produces quanta of quasi-normal frequencies.

Let us proceed with the thermodynamics. We begin with the first choice for the temperature and, following the line of reasoning presented in [11], have a look at the result. In what follows we distinguish between two types of thermodynamic relations, the local ones as seen and measured by the local static observer, and the global for the distant inertial observer at infinity. The local observer deals with the bare mass M defined as the following integral over some volume V:

$$M = \int T^{0\lambda}\xi_\lambda dV = \int T^0_0 \xi^0 dV = \int \varepsilon dV \,, \tag{27}$$

where T^λ_ν is the energy-momentum tensor, ξ^μ - the Killing vector normalized as $\xi^0 = 1$. Thus, this observer is using the local time and measures the local temperature $T_{loc} = \frac{T_U}{\sqrt{g_{00}}} = \frac{1}{\sqrt{2}\pi r}$. The first law of thermodynamics now reads as follows

$$dM = \varepsilon dV = T_{loc} dS - p dV + \mu dN \,. \tag{28}$$

Here μ is the chemical potential related to the number of black hole phonons (this is how the integer number enters our model), it ought to be included because in our model all the distributions are universal and the only parameter that changes is the boundary value of radius r_0, and this means the automatical changing of all the integrated extensive variables, M, S, V and N. Dividing the above expression by the volume element dV we get the first law in its local form

$$\varepsilon(r) = T_{loc}(r) s(r) - p(r) + \mu(r) n(r) \,, \tag{29}$$

where s and n are the entropy and particle densities, respectively. In our model $\varepsilon = p$, but what about s? The local observer can ask his global counterpart who is educated enough and knows that the total entropy of the black hole of mass m is $S = 4\pi G m^2 = \frac{\pi r_0^2}{4G}$. Having this information, our local observer can deduce that

$$s(r) = \frac{1}{8\sqrt{2} G r} \tag{30}$$

and

$$s(r) T(r) = \frac{1}{16\pi G r^2} \,. \tag{31}$$

Remembering now that $\varepsilon = \frac{1}{16\pi G r^2}$ we obtain

$$\varepsilon(r) = p(r) = s(r) T(R) = \mu(r) n(r) \,. \tag{32}$$

Our system is in thermal equilibrium because the Unruh temperature is constant everywhere in the inner region, and for the local temperature we have the well known relation $T_{loc}\sqrt{g_{00}} = const$. But in the thermal equilibrium also $\mu\sqrt{g_{00}} = const$, hence $\frac{T_{loc}(r)}{\mu(r)} = const$ and, consequently, $\frac{s(r)}{n(r)} = const$. From this we obtain the equidistant quantization for the entropy ($\gamma = const$, $N - integer$)

$$S = \gamma N\,, \quad N = 1, 2, 3\ldots\,. \tag{33}$$

The spacing coefficient γ is universal (does not depend on the value of r_0) and can be calculated by noticing that the free energy density $f(r)$ is exactly zero

$$f(r) = \varepsilon(r) - T_{loc}(r)s(r) = 0\,. \tag{34}$$

To do this, let us construct the partition function Z_1 for a small part of our system corresponding to one black hole phonon:

$$Z_1 = \Sigma_n e^{-\frac{\varepsilon_n}{T}}\,. \tag{35}$$

Here ε_n are the excitation energy levels and, since $\frac{\varepsilon_n}{T}$ is invariant under the change of time variable (clocks), we will use the proper time of local observers, so the temperature is just the Unruh temperature, $T = T_U = const$. What concerns the energy spectrum ε_n, we already mentioned that the existence of the intrinsic frequency ω for Schwarzschild black holes and the equidistant imaginary parts of the quasi-normal frequencies suggests the following relation (the phonon spectrum)

$$\varepsilon_n = \omega n\,, \quad n = 1, 2, 3\ldots\,. \tag{36}$$

After substituting this into the exponent in Eqn.(35) the summation can be easily performed:

$$Z_1 = \frac{e^{-\frac{\omega}{T}}}{1 - e^{-\frac{\omega}{T}}}\,. \tag{37}$$

The transition from the $n'th$ energy state to the $(n+1)'th$ (or the other way around) gives dM, hence $\frac{\omega}{T} = dS_{min} = \gamma$. From zero value for the free energy we have $Z_1 = 1$, and

$$\frac{\omega}{T} = \gamma = \log 2\,. \tag{38}$$

Let us summarize what we have got with the first choice for the temperature, $T_U = T_H$.

The good features are the equidistant quantization of the entropy and the value $\log 2$ for its spacing. And, of course, the very appearance of the temperature itself, but this is common for given distributions $\varepsilon(r)$ and $p(r)$ irrespective of the choice of the temperature T_U. From that fact that the inner distribution is in thermal equilibrium there comes out one more desirable property which can be called "indifference". If we remove (radiate away) some outer layer, the inner part would remain unperturbed. The "indifference" reflects the universality of our classical analog of quantum black holes which, in turn, is the "analytical continuation" of the classical black holes. Indeed, the energy density distribution is universal, the speed of sound equals the speed of light, the ratio of the resonance frequency to the Unruh temperature is the universal constant, thus explaining the rather unusual inverse proportionality of the black hole temperature to its mass. On the other hand, the inverse proportionality of the resonance frequency to the mass becomes quite understandable, because it is translated into inverse proportionality to the boundary radius, in direct analogy with the music instruments - the smaller the size, the higher the dominant tone, i.e., the resonance frequency.

The bad features (or, better, "not good") are the following. The total free energy of the Schwarzschild black hole equals $F - T_H S = \frac{m}{2}$, and it is impossible to explain why the inner part of our model has zero free energy. Also, it is impossible to imagine how to obtain the $\log 3$ in the real part of the quasi-normal frequencies.

The crucial test for the validity of our model, i.e., for the choice of the temperature, is the possibility of its generalization to a physically acceptable classical analog of the quantum Reissner-Nordstrom black hole. We constructed two of them [19]. The first model has continuous distributions both of mass and electric charge, while in the second the charge is concentrated in the thin massive shell at the boundary surface, the inner mass distribution being the same as in the Schwarzschild case. Let us describe briefly the first model. There are two parameters characterizing it completely, the boundary radius r_0 and the charge/mass ratio $\frac{e^2}{Gm^2}$. The radius r_0 is again a free parameter, and everything else: the energy density ε, radial and tangential pressures p_r and p_t, electric charge distribution $e(r)$ and the surface tension Σ, depends solely on the charge/mass ratio. The temperature matching condition $T_U = T_H$ results in some strange features of these parametric dependencies. The most awful is the change of sign in the radial pressure p_r (though energy dominance conditions are not violated) and, consequently, the change of sign in the surface tension Σ, the latter points to the instability - a potential wall (barrier) is substituted by a potential well. Moreover, the lack of the "indifference" means that the radiation of a single quantum with the charge/mass ration different from that of the given distribution would cause a complete "reloading" of the whole system, what is unacceptable and nonphysical for large semiclassical black holes. In the second model with shell-like distribution of the electric charge such an unpleasant thing is, of course, absent. But, unfortunately, this model has no smooth limit to the Schwarzschild uncharged case. Namely, the mare mass of the shell (and, thus, its total mass) does not vanishes when the charged becomes zero.

Thus, we are forced to make the second choice for the temperature,

$$T_U = \frac{1}{2} T_H \,, \tag{39}$$

i.e., the Unruh temperature of the inner region is only one half of the Hawking temperature measured at infinity. This jump is exactly compensated by the surface tension. Classically, the radiation is now absent because the heat in the inner region is thermodynamically locked by this surface tension which provides the equilibrium temperature gradient from the inner to outer Rindler observers. In this sense the model is self-consistent since no more back reaction corrections are needed. In quantum theory the radiation will be caused by the tunneling process. But let us proceed with the thermodynamical relations, and write down the first law of thermodynamics in the local form,

$$\varepsilon(r) = T_{loc}(r) s(r) - p(r) + \mu(r) n(r) \,. \tag{40}$$

Again, $\varepsilon(r) = p(r) = \frac{1}{16\pi G r^2}$, but now $T_{loc}(r) = \frac{1}{2\sqrt{2}\pi r}$, and we have

$$T_{loc}(r) s(r) = \frac{1}{2} \varepsilon \,, \quad \mu(r) n(r) = \frac{3}{2} \varepsilon \,. \tag{41}$$

From this it follows that the free energy is no more zero, but

$$\begin{aligned} f(r) &= T_{loc}(r) s(r) = \frac{1}{2} \varepsilon(r) \,, \\ F(r) &= \frac{1}{2} M = 2 T_{loc}(r_0) S \end{aligned} \tag{42}$$

as measured by local observers, $T_{loc}(r_0) = \frac{1}{2\sqrt{2}\pi r_0}$, S is the total entropy of the system, M is its bare mass. The distant observer at infinity measures the total mass $m = \frac{1}{\sqrt{2}} M$ and the Hawking temperature $T_H = 2 T_U = \frac{2}{\sqrt{2}} T_{loc}(r_0)$, hence, for him our free energy becomes

$$F_\infty = m - T_H S = \frac{1}{2} m \,, \tag{43}$$

and this guaranties the usual Schwarzschild black hole thermodynamical relation $dm = T_H dS$.

Surely, we again have the equidistant entropy quantization $S = \gamma N$, but with, perhaps, different spacing γ. To evaluate this spacing we need to know the partition function. Of course, it is the same as before, i.e., for the one-phonon patch we have

$$Z_1 = \frac{e^{-\frac{\omega}{T}}}{1 - e^{-\frac{\omega}{T}}}\,, \tag{44}$$

where, as was explained earlier, ω is an intrinsic black hole proper (resonance) frequency, and $T = T_U$. The total partition function equals $Z_{tot} = (Z_1)^N$. The partition function is an invariant. For any small part of our system we should have the usual relation between densities of its free energy and partition function, $f dV = -T_{loc} \log Z_{small}$. Integration over the volume gives us

$$\int \frac{f}{T_{loc}} dV = -\Sigma \log Z_{small} dV = -\log Z_{tot}\,. \tag{45}$$

The left hand side equals

$$\int \frac{f}{T_{loc}} dV = \frac{1}{2} \int \frac{\varepsilon}{T_{loc}} dV = 2\sqrt{2}\pi \int_0^{r_0} \frac{\varepsilon}{T_{loc}} r^2 dr = \frac{\pi r_0^2}{4G} = \frac{\pi r_g^2}{G} = S\,. \tag{46}$$

Here r_g is the Schwarzschild radius, and S is the total black hole entropy. Eventually, we obtain the the important relation

$$e^{-S} = Z_{tot} = (Z_1)^N\,, \tag{47}$$

from which it follows that

$$\begin{aligned} \frac{e^{-\frac{\omega}{T}}}{1 - e^{-\frac{\omega}{T}}} &= e^{-\frac{S}{N}} = e^{-\gamma}\,, \\ e^{\gamma} &= e^{\frac{\omega}{T}} - 1\,. \end{aligned} \tag{48}$$

To go further, let us consider the irreversible process of converting the mass (energy) of the system into radiation from a thermodynamical point of view. In our model such a process takes place just at the boundary $r = r_0$, and the thin shell with zero surface energy density and surface tension Σ serves as a convertor supplying the radiation with extra energy and extra entropy, and this resembles the "brick wall" model [20]. One can imagine that the near-boundary layer of thickness Δr_0 is converting into radiation, thus decreasing the boundary of the inner region to $(r_0 - \Delta r_0)$. Its energy equals $\Delta M = \epsilon \Delta V$ plus the energy released from the work done by the surface tension due to its shift, which is equal exactly to $\Sigma d(4\pi r_0^2) = p\Delta V = \varepsilon \Delta V = \Delta M$. Therefore, both the energy and its temperature becomes two times higher that that for any inner layer of the same thickness. And this double energy is gained by the radiating quanta. Clearly, they have the double frequency and exhibit double temperature, so

$$\frac{Re\, w}{T_H} = \frac{\omega}{T_U} = \log 3\,. \tag{49}$$

Substituting this into Eqn.(48) we obtain

$$\frac{e^{-\log 3}}{1 - e^{-\log 3}} = \frac{\frac{1}{3}}{1 - \frac{1}{3}} = \frac{1}{2} = e^{-\gamma} \tag{50}$$

and

$$\gamma = \log 2\,. \tag{51}$$

(Note, that substituting in Eqn.(48) the general value $\gamma = \log k, k = 2, 3...$, we obtain $\frac{\omega}{T} = \log (k+1)$). Since the radiated energy is thermalized, its energy density decreases during expansion due to the work done by the pressure and, thus, the interpretation of dm as equal to $Re\, w$ is an improper procedure. This resolves the "log 3-paradox".

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