On analogues of black brane solutions in the model with multicomponent anisotropic fluid

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Abstract

A family of spherically symmetric solutions with horizon in the model with m-component anisotropic fluid is presented. The metrics are defined on a manifold that contains a product of $n-1$ Ricci-flat "internal" spaces. The equation of state for any s-th component is defined by a vector U^s belonging to $\mathbb{R}^{n+1}$. The solutions are governed by moduli functions H_s obeying non-linear differential equations with certain boundary conditions imposed. A simulation of black brane solutions in the model with antisymmetric forms is considered. An example of solution imitating $M_2 - M_5$ configuration (in $D = 11$ supergravity) corresponding to Lie algebra A_2 is presented.

1 Introduction

In this paper we continue our investigations of spherically-symmetric solutions with horizon (e.g., black brain ones) defined on product manifolds containing several Ricci-flat factor-spaces (with diverse signatures and dimensions). These solutions appear either in models with antisymmetric forms and scalar fields [1]-[11] or in models with (multi-component) anisotropic fluid [12]-[15]. For black brane solutions with 1-dimensional factor-spaces (of Euclidean signatures) see [16, 17, 18] and references therein.

These and more general brane cosmological and spherically symmetric solutions were obtained by reduction of the field equations to the Lagrange equations corresponding to Toda-like systems [2, 19]. An analogous reduction for models with multicomponent anisotropic fluids was performed earlier in [20, 21]. For cosmological-type models with antisymmetric forms without scalar fields any brane is equivalent to an anisotropic fluid with the equations of state:

$$\hat{p}_i = -\hat{\rho} \qquad \text{or} \qquad \hat{p}_i = \hat{\rho}, \tag{1.1}$$

when the manifold M_i belongs or does not belong to the brane world volume, respectively (here $\hat{p}_i$ is the effective pressure in M_i and $\hat{\rho}$ is the effective density).

In this paper we present spherically-symmetric solutions with horizon (e.g the analogues of intersecting black brane solutions) in a model with multi-component anisotropic fluid (MCAF), when certain relations on fluid parameters are imposed. The solutions are governed by a set of moduli functions H_s obeying non-linear differential master equations with certain boundary conditions imposed. These master equations are equivalent to Toda-like equations and depend upon the non-degenerate $(m \times m)$ matrix A. It was conjectured earlier that the functions H_s should be polynomials when A is a Cartan matrix for some semi-simple finite-dimensional Lie algebra (of rank m) [6]. This conjecture was verified for Lie algebras: A_m, C_{m+1}, $m \geq 1$ [7, 8]. A special case of black hole solutions with MCAF corresponding to semisimple Lie algebra $A_1 \oplus ... \oplus A_1$ was considered earlier in [13] (for $m = 1$ see [12]).

The paper is organized as follows. In Section 2 the model is formulated. In Section 3 spherically-symmetric MCAF solutions with horizon corresponding to black-brane-type solutions, are presented. In Section 4 a polynomial structure of moduli functions H_s for semi-simple finite-dimensional Lie algebras is discussed. In Section 5 a simulation of intersecting black brane solutions is considered and an analogue of $M2 - M5$ dyonic solution is presented.

2 The model

In this paper we deal with a family of spherically symmetric solutions to Einstein equations with an anisotropic matter source

$$R^M_N - \frac{1}{2}\delta^M_N R = k^2 T^M_N, \tag{2.1}$$

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defined on the manifold

$$M = \underset{\substack{\text{radial} \\ \text{variable}}}{\mathbb{R}_*} \times \underset{\substack{\text{spherical} \\ \text{variables}}}{(M_0 = S^{d_0})} \times \underset{\text{time}}{(M_1 = \mathbb{R})} \times \ldots \times M_n, \tag{2.2}$$

with the block-diagonal metrics

$$ds^2 = \mathrm{e}^{2\gamma(u)} du^2 + \sum_{i=0}^{n} \mathrm{e}^{2\beta^i(u)} h^{[i]}_{m_i n_i} dy^{m_i} dy^{n_i}. \tag{2.3}$$

Here $\mathbb{R}_* \subseteq \mathbb{R}$ is an open interval. The manifold M_i with the metric $h^{[i]}$, $i = 1, 2, \ldots, n$, is a Ricci-flat space of dimension d_i:

$$R_{m_i n_i}[h^{[i]}] = 0, \tag{2.4}$$

and $h^{[0]}$ is the standard metric on the unit sphere S^{d_0}, so that

$$R_{m_0 n_0}[h^{[0]}] = (d_0 - 1) h^{[0]}_{m_0 n_0}; \tag{2.5}$$

u is a radial variable, κ^2 is the gravitational constant, $d_1 = 1$ and $h^{[1]} = -dt \otimes dt$.

The energy-momentum tensor is adopted in the following form for each component of the fluid:

$$(T^{s\,M}_{\ \ N}) = \mathrm{diag}(-\hat{\rho}^s, \hat{p}^s_0 \delta^{m_0}_{k_0}, \hat{p}^s_1 \delta^{m_1}_{k_1}, \ldots, \hat{p}^s_n \delta^{m_n}_{k_n}), \tag{2.6}$$

where $\hat{\rho}^s$ and $\hat{p}^s_i$ are the effective density and pressures respectively, depending on the radial variable u.

We assume that the following "conservation laws"

$$\nabla_M T^{(s)M}_{\ \ N} = 0 \tag{2.7}$$

are valid for all components.

We also impose the following equations of state

$$\hat{p}^s_i = \left(1 - \frac{2U^s_i}{d_i}\right) \hat{\rho}^s, \tag{2.8}$$

where U^s_i are constants, $i = 0, 1, \ldots, n$.

The physical density and pressures are related to the effective ones (with "hats") by the formulae

$$\rho^s = -\hat{p}^s_1, \quad p^s_u = -\hat{\rho}^s, \quad p^s_i = \hat{p}^s_i \quad (i \neq 1). \tag{2.9}$$

In what follows we put $\kappa = 1$ for simplicity.

3 Spherically symmetric solutions with horizon

We will make the following assumptions:

$$\begin{aligned} &1^o. \quad U^s_0 = 0 \ \Leftrightarrow\ \hat{p}^s_0 = \hat{\rho}^s, \\ &2^o. \quad U^s_1 = 1 \ \Leftrightarrow\ \hat{p}^s_1 = -\hat{\rho}^s, \\ &3^o. \quad (U^s, U^s) = U^s_i G^{ij} U^s_j > 0, \\ &4^o. \quad 2(U^s, U^l)/(U^l, U^l) = A_{sl}, \end{aligned} \tag{3.1}$$

where $A = (A_{sl})$ is non-degenerate matrix,

$$G^{ij} = \frac{\delta^{ij}}{d_i} + \frac{1}{2 - D}, \tag{3.2}$$

are components of the matrix inverse to the matrix of the minisuperspace metric [22]

$$(G_{ij}) = (d_i \delta_{ij} - d_i d_j), \tag{3.3}$$

$i, j = 0, 1, ..., n$ and $D = 1 + \sum\limits_{i=0}^{n} d_i$ is the total dimension.

The conditions 1^o and 2^o in brane terms mean that brane "lives" in the time manifold M_1 and does not "live" in M_0. Due to assumptions 1^o and 2^o and the equations of state (2.8), the energy-momentum tensor (2.6) reads as follows:

$$(T^{(s)}{}^M_N) = \mathrm{diag}(-\rho^s, \ \rho^s \delta^{m_0}_{k_0}, \ -\rho^s, \ p^s_2 \delta^{m_2}_{k_2}, \ \dots, p^s_n \delta^{m_n}_{k_n}). \tag{3.4}$$

Under the conditions (2.8) and (3.1) we have obtained the following black-hole solutions to the Einstein equations (2.1):

$$ds^2 = J_0 \left(\frac{dr^2}{1 - 2\mu/R^d} + R^2 d\Omega^2_{d_0} \right) - J_1 \left(1 - \frac{2\mu}{R^d}\right) dt^2 + \sum_{i=2}^{n} J_i h^{[i]}_{m_i n_i} dy^{m_i} dy^{n_i}, \tag{3.5}$$

$$\rho^{(s)} = - \frac{A_s}{J_0 R^{2d_0}} \prod_{l=1}^{m} H_l^{-A_{sl}}, \tag{3.6}$$

which may be derived by analogy with the black brane solutions [7, 8]. Here $d = d_0 - 1$,

$$d\Omega^2_{d_0} = h^{[0]}_{m_0 n_0} dy^{m_0} dy^{n_0} \tag{3.7}$$

is the d_0-dimensional spherical element (corresponding to the metric on S^{d_0}),

$$J_i = \prod_{s=1}^{m} H_s^{-2h_s U^{si}}, \tag{3.8}$$

$i = 0, 1, ..., n$, $\mu > 0$ is integration constant and

$$U^{si} = G^{ij} U^s_j = \frac{U^s_i}{d_i} + \frac{1}{2-D} \sum_{j=0}^{n} U^s_j, \tag{3.9}$$

$$h_s = K_s^{-1}, \quad K_s = (U^s, U^s). \tag{3.10}$$

It follows from 1^o and (3.9) that

$$U^{s0} = \frac{1}{2-D} \sum_{j=1}^{n} U^s_j. \tag{3.11}$$

Functions $H_s > 0$ obey the equations

$$R^{d_0} \frac{d}{dR} \left[\left(1 - \frac{2\mu}{R^d}\right) \frac{R^{d_0}}{H_s} \frac{dH_s}{dR} \right] = B_s \prod_{l=1}^{m} H_l^{-A_{sl}}, \tag{3.12}$$

with $B_s = 2 K_s A_s$ and the boundary conditions imposed:

$$H_s \to H_{s0} \neq 0, \quad \text{for} \quad R^d \to 2\mu, \tag{3.13}$$

and

$$H_s(R = +\infty) = 1, \tag{3.14}$$

$s = 1, ..., m$, i.e. the metric (3.5) has a regular horizon at $R^d = 2\mu$ and has an asymptotically flat $(2 + d_0)$-dimensional section.

Due to to (3.1) and (3.9) the metric reads

$$ds^2 = J_0 \left[\frac{dr^2}{1 - 2\mu/R^d} + R^2 d\Omega^2_{d_0} - \left(\prod_{s=1}^{m} H_s^{-2h_s} \right) \left(1 - \frac{2\mu}{R^d}\right) dt^2 + \sum_{i=2}^{n} Y_i h^{[i]}_{m_i n_i} dy^{m_i} dy^{n_i} \right], \tag{3.15}$$

where

$$Y_i = \prod_{s=1}^{m} H_s^{-2h_s U^s_i / d_i}. \tag{3.16}$$

The solution (3.5), (3.6) may be verified just by a straightforward substitution into equations of motion. A detailed derivation of this solution will be given in a separate paper [26]. A special orthogonal case when $(U^s, U^l) = 0$, for $s \neq l$, was considered earlier in [13] (for $m = 1$ see [12]) More general solutions in orthogonal case (with more general condition instead of 2^o) were obtained in [15] (for $m = 1$ see [14].)

4 Polynomial structure of H_s for Lie algebras

Now we deal with solutions to second order non-linear differential equations (3.12) that may be rewritten as follows

$$\frac{d}{dz}\left(\frac{(1-2\mu z)}{H_s}\frac{d}{dz}H_s\right) = \bar{B}_s \prod_{l=1}^{m} H_l^{-A_{sl}}, \tag{4.1}$$

where $H_s(z) > 0$, $z = R^{-d} \in (0, (2\mu)^{-1})$ $(\mu > 0)$ and $\bar{B}_s = B_s/d^2 \neq 0$. Eqs. (3.13) and (3.14) read

$$H_s((2\mu)^{-1} - 0) = H_{s0} \in (0, +\infty), \tag{4.2}$$

$$H_s(+0) = 1, \tag{4.3}$$

$s = 1, ..., m$.

It was conjectured in [6] that equations (4.1)-(4.3) have polynomial solutions when $(A_{ss'})$ is a Cartan matrix for some semisimple finite-dimensional Lie algebra $\mathcal{G}$ of rank m. In this case we get

$$H_s(z) = 1 + \sum_{k=1}^{n_s} P_s^{(k)} z^k, \tag{4.4}$$

where $P_s^{(k)}$ are constants, $k = 1, \ldots, n_s$; $P_s^{(n_s)} \neq 0$, and

$$n_s = b_s \equiv 2 \sum_{l=1}^{m} A^{sl} \tag{4.5}$$

$s = 1, ..., m$, are the components of twice the dual Weyl vector in the basis of simple co-roots [24]. Here $(A^{sl}) = (A_{sl})^{-1}$.

This conjecture was verified for $\mathbf{A_m}$ and $\mathbf{C_{m+1}}$ series of Lie algebras in [7, 8]. In extremal case $(\mu = +0)$ an a analogue of this conjecture was suggested (implicitly) in [25].

$\mathbf{A_1} \oplus \ldots \oplus \mathbf{A_1}$ -case.

The simplest example occurs in orthogonal case : $(U^s, U^l) = 0$, for $s \neq l$ [1, 2] (see also [16, 17, 18] and refs. therein). In this case $(A_{sl}) = \mathrm{diag}(2, \ldots, 2)$ is a Cartan matrix for semisimple Lie algebra $\mathbf{A_1} \oplus \ldots \oplus \mathbf{A_1}$ and

$$H_s(z) = 1 + P_s z, \tag{4.6}$$

with $P_s \neq 0$, satisfying

$$P_s(P_s + 2\mu) = -\bar{B}_s = -2K_s A_s/d^2, \tag{4.7}$$

$s = 1, ..., m$. When all $A_s < 0$ (or, equivalently, $\rho^s > 0$) there exists a unique set of numbers $P_s > 0$ obeying (4.7).

A_2-case.

For the Lie algebra $\mathcal{G}$ coinciding with $\mathbf{A_2} = sl(3)$ we get $n_1 = n_2 = 2$ and

$$H_s = 1 + P_s z + P_s^{(2)} z^2, \tag{4.8}$$

where $P_s = P_s^{(1)}$ and $P_s^{(2)} \neq 0$ are constants, $s = 1, 2$.

It was found in [6] that for $P_1 + P_2 + 4\mu \neq 0$ (e.g. when all $P_s > 0$) the following relations take place

$$P_s^{(2)} = \frac{P_s P_{s+1}(P_s + 2\mu)}{2(P_1 + P_2 + 4\mu)}, \qquad \bar{B}_s = -\frac{P_s(P_s + 2\mu)(P_s + 4\mu)}{P_1 + P_2 + 4\mu}, \tag{4.9}$$

$s = 1, 2$.

Here we denote $s + 1 = 2, 1$ for $s = 1, 2$, respectively.

Other solutions.

At the moment the "master" equations were integrated (using the Maple) in [9, 10] for Lie algebras $\mathbf{C_2}$ and $\mathbf{A_3}$, respectively.

Special solutions $H_s(z) = (1 + P_s z)^{b_s}$ with b_s from (4.5) appeared earlier in [3, 4, 5] in a context of so-called block-orthogonal configurations.

5 Examples

5.1 Simulation of intersecting black branes

The solution from the previous section for MCAF allows to simulate the intersecting black brane solutions [11] in the model with antisymmetric forms without scalar fields. In this case the parameters U_i^s and pressures have the following form:

$$\begin{array}{rclrcrl} U_i^s & = & d_i, & p_i^{(s)} & = & -\rho^s, & i \in I_s; \\ & & 0, & & & \rho^s, & i \notin I_s. \end{array} \tag{5.1}$$

Here $I_s = \{i_1^s, \ldots, i_{k_s}^s\} \in \{1, \ldots n\}$ is the index set [11] corresponding to brane submanifold $M_{i_1^s} \times \ldots \times M_{i_{k_s}^s}$.

The relation 4^o (3.1) leads us to the following dimensions of intersections of brane submanifolds ("worldvolumes") [2, 11]:

$$d(I_s \cap I_l) = \frac{d(I_s)d(I_l)}{D-2} + \frac{1}{2}K_l A_{sl}, \tag{5.2}$$

$s \neq l;\ s, l = 1, \ldots, m$. Here $d(I_s)$ and $d(I_l)$ are dimensions of brane world-volumes.

5.2 $M_2 - M_5$-analogue for Lie algebra A_2

In [13] examples of simulation by MCAF of $M2 \cap M5$, $M2 \cap M2$, $M5 \cap M5$ black brane solutions in $D = 11$ supergravity, with the standard (orthogonal) intersection rules were considered.

Now we consider a solution with 2-component anisotropic fluid that simulates $M_2 - M_5$ dyonic configuration in $D = 11$ supergravity [6], corresponding to Lie algebra $\mathbf{A_2}$.

The solution is defined on the manifold

$$M = (2\mu, +\infty) \times (M_0 = S^2) \times (M_1 = \mathbb{R}) \times M_2 \times M_3, \tag{5.3}$$

where $\dim M_2 = 2$ and $\dim M_3 = 5$. The U^s-vectors corresponding to fluid components obey (5.1) with $I_1 = \{1, 2\}$ and $I_2 = \{1, 3\}$.

The solution reads as following

$$g = H_1^{1/3} H_2^{2/3} \Bigg\{ \frac{dR \otimes dR}{1 - 2\mu/R} + R^2 d\Omega_2^2 \tag{5.4}$$

$$- H_1^{-1} H_2^{-1} \left(1 - \frac{2\mu}{R}\right) dt \otimes dt + H_1^{-1} h^{[2]} + H_2^{-1} h^{[3]} \Bigg\}, \tag{5.5}$$

$$\rho^{(1)} = -\frac{A_1}{J_0 R^4} H_1^{-2} H_2, \qquad \rho^{(2)} = -\frac{A_2}{J_0 R^4} H_1 H_2^{-2}, \tag{5.6}$$

where $J_0 = H_1^{1/3} H_2^{2/3}$; $h^{[2]}$ and $h^{[3]}$ are Ricci-flat metrics of Euclidean signatures, $\mu > 0$ and H_s are defined by (4.8), where $z = R^{-1}$ and parameters P_s, $P_s^{(2)}$, $\bar{B}_s = B_s = 4A_s$ $(s = 1, 2)$ obey (4.9).

This solution simulates $\mathbf{A_2}$-dyon from [6] consisting of electric $M2$-brane with worldvolume isomorphic to $(M_1 = \mathbb{R}) \times M_2$ and magnetic $M5$-brane with worldvolume isomorphic to $(M_1 = \mathbb{R}) \times M_3$. The branes are intersecting on the time manifold $M_1 = \mathbb{R}$. Here $K_s = (U^s, U^s) = 2$, for all $s = 1, 2$.

For $\mathbf{A_2}$-dyon from [6] we had $\bar{B}_s = B_s = -2Q_s^2$, where Q_s is the charge density parameter of s-th brane. Thus, for fixed Q_s the fluid parameters should obey the relations $A_s = -\frac{1}{2}Q_s^2$ and hence A_s are negative.

5.3 The Hawking temperature

The Hawking temperature of the black hole (3.5) (see also (3.15)) may be calculated using the relation from [23]. It has the following form:

$$T_H = \frac{d}{4\pi (2\mu)^{1/d}} \prod_{s=1}^{m} H_{s0}^{-h_s}, \tag{5.7}$$

where H_{s0}, $s = 1, 2$, are defined in (3.13).

For the dyonic solution from the previous subsection we get

$$T_H = \frac{1}{8\pi\mu} (H_{10} H_{20})^{-1/2}, \tag{5.8}$$

where T_H is a function of fluid parameters $A_s < 0$, $s = 1, 2$.

6 Conclusions

Here we have presented a family of spherically symmetric solutions with horizon in the model with multi-component anisotropic fluid with the equations of state (2.8) and the conditions (3.1) imposed. The metric of any solution contains $(n-1)$ Ricci-flat "internal" space metrics.

As in [6, 7, 8] the solutions are defined up to solutions of non-linear differential equations (equivalent to Toda-like ones) with certain boundary conditions imposed. These solutions may have a polynomial structure when the matrix A from (3.1) is coinciding with the Cartan matrix of some semi-simple finite-dimensional Lie algebra.

For certain equations of state (with $p_i = \pm\rho$) the metric of the solution may coincide with the metric of intersecting black branes (in a model with antisymmetric forms without dilatons). Here we have considered an example of simulating of $M2 - M5$ black brane (dyonic) solution in $D = 11$ supergravity with intersection rules corresponding to Lie algebra A_2.

An open problem is to generalize this formalism to the case when scalar fields are added into consideration. In a separate paper we also plan to calculate the post-Newtonian parameters β and γ corresponding to the 4-dimensional section of the metric (for $d_0 = 2$) and analyze the thermodynamic properties of the black-brane-like solutions in the model with MCAF.

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