Consistent Histories in Quantum Cosmology

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(Dated: January 25, 2010)

Abstract

We illustrate the crucial role played by decoherence (consistency of quantum histories) in extracting consistent quantum probabilities for alternative histories in quantum cosmology. Specifically, within a Wheeler-DeWitt quantization of a flat Friedmann-Robertson-Walker cosmological model sourced with a free massless scalar field, we calculate the probability that the univese is singular in the sense that it assumes zero volume. Classical solutions of this model are a disjoint set of expanding and contracting singular branches. A naive assessment of the behavior of quantum states which are superpositions of expanding and contracting universes may suggest that a "quantum bounce" is possible *i.e.* that the wave function of the universe may remain peaked on a non-singular classical solution throughout its history. However, a more careful consistent histories analysis shows that for arbitrary states in the physical Hilbert space the probability of this Wheeler-DeWitt quantum universe encountering the big bang/crunch singularity is equal to unity. A quantum Wheeler-DeWitt universe is inevitably singular, and a "quantum bounce" is thus not possible in these models.

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arXiv:1001.4311v1 [gr-qc] 25 Jan 2010

I. INTRODUCTION

Whatever form a theory of quantum gravity may ultimately take – string theory, loop quantum gravity, causal dynamical triangulations, causal sets, or something else[1] – extracting coherent physical predictions will require more than just the basic technical apparatus of the theory. Because one of the explicit aims of such theories is to provide a complete quantum description of the universe *as a whole*, a framework within which consistent predictions can be made in the absence of external observers or an external notion of "measurement" will also be required in an essential way.

In this essay we aim to outline an explicit realization of one such framework, the generalized "consistent histories" quantum theory of J.B. Hartle [2], in the context of the Wheeler-DeWitt quantization of a flat homogeneous and isotropic Friedman-Robertson-Walker (FRW) model universe populated with a free, massless, minimally coupled scalar field. (Previous discussions of decoherent histories formulations of quantum cosmological models include [3–8].) We will describe when and how, within the framework of decoherent histories, consistent quantum predictions may be extracted from the theory – namely, when the histories describing the possible outcomes decohere – emphasizing the fundamental quantum fact that it is *not* normally the case that a probability may sensibly be assigned to every history of a system. We illustrate with an example of a prediction that depends crucially on decoherence of the appropriate histories: whether or not Wheeler-DeWitt quantized FRW universes are singular. The classical solutions of this model are all singular and fall into disjoint sets of ever-expanding or ever-contracting universes. We shall show that decoherence is critical to a coherent *quantum* analysis of this question, concluding unambiguously within this framework that all quantum Wheeler-DeWitt FRW universes are singular as well. This stands in contrast to a naive analysis of the behavior of the quantum wave function alone, not taking care of the proper description and decoherence of the corresponding histories. This (erroneous) analysis of states which are superpositions of wavefunctions peaked on classically disjoint expanding and contracting solutions seems to suggest that quantum avoidance of the singularity *is* possible. However, a careful analysis of the quantum histories shows that the probability of avoidance of the singularity is zero and the probability that a Wheeler-DeWitt quantum universe encounters a singularity is unity.

The organization of this paper is as follows. Section II describes the overall framework of generalized quantum theory. In section III we give details of the cosmological model and its (canonical) quantization, including a discussion of the model's observables, and in section IV we discuss the analysis of the quantum singularity. (Complete technical details of the model and its decoherent histories formulation will be given elsewhere [9].)

II. GENERALIZED "CONSISTENT HISTORIES" QUANTUM THEORY

Quite generally, the job of any quantum theory is to do two things. For any given physical system, a quantum theory must, for all observable quantities, tell both *What are the possible values of those observables?* and *How likely is each of these values?* In Hilbert space formulations of quantum theory, where observables are represented by self-adjoint operators $\{\hat{A}\}$ on a Hilbert space $\mathcal{H}$, the answer to the first question is found in the eigenvalues $\{a\}$

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[1] See [1] for an overview.

of those operators, and to the second, in amplitudes constructed from the corresponding eigenstates $\{|a\rangle\}$ and the state $|\Psi(t)\rangle$ of the system at the moment in question:

$$p_a = |\langle a\,|\,\Psi(t)\rangle|^2. \tag{2.1}$$

In the textbook "Copenhagen" view, this framework is usually supplemented with the rule that once a particular outcome $\{a'\}$ has been "observed", the state of the system is replaced by that of the observed outcome, $|\Psi(t)\rangle \rightarrow \{|a'\rangle\}$. This is commonly referred to as the "collapse of the wave function".

But when does it make sense to assign such probabilities, and thence reassign the state of the system in this discontinuous, uncontrollable way? Certainly such amplitudes do not *always* make sense. Most commonly, this is the case when considering the likelihood of *sequences* of outcomes, as is well known, for example, in the two-slit experiment [10]. Quite generally, in experiments of this kind, probabilities may be consistently assigned *only* when there are physical mechanisms in place (such as a measuring apparatus or a suitable physical environment) which *destroy the interference* between the possible alternative histories.

Taking a rather conservative approach, this idea has been formalized into a scheme known as the "decoherent" or "consistent" histories formulation of quantum theory, in which probabilities may be consistently assigned *whenever* interference between histories vanishes, irrespective of the presence of measuring devices, external observers, or other such notions; see [2, 11, 12] for broad overviews. In fact, the principal virtue of this perspective is that it employs criteria *entirely internal* to the system to determine whether or not the assignment of probabilities makes coherent sense. [2]

The most general formulation of the consistent histories program, suitable for application to quantum cosmology, is due to J.B. Hartle [2]. In this form it is often referred to as "generalized quantum theory". We summarize very briefly its chief elements for theories with a homogeneous time here.

Particular sequences of physical outcomes ("histories") are characterized by "class operators", products of Heinseberg projections onto ranges of eigenvalues of observables of interest:

$$C_h = P^{\alpha_1}_{\Delta a_{k_1}}(t_1) P^{\alpha_2}_{\Delta a_{k_2}}(t_2) \cdots P^{\alpha_n}_{\Delta a_{k_n}}(t_n) \tag{2.2a}$$

$$= U(t_0 - t_1) P^{\alpha_1}_{\Delta a_{k_1}} U(t_1 - t_2) P^{\alpha_2}_{\Delta a_{k_2}} \cdots U(t_{n-1} - t_n) P^{\alpha_n}_{\Delta a_{k_n}} U(t_n - t_0), \tag{2.2b}$$

where $U(t) = \exp(-i\hat{H}t/\hbar)$ is the system's propagator and $t_1 \leq t_2 \leq \cdots \leq t_n$. Given the initial state of the system $|\psi\rangle$, the "branch wave function" corresponding to the state of a system following the history h is constructed from the class operator for h:

$$|\psi_h\rangle = C_h^\dagger |\psi\rangle. \tag{2.3}$$

One may see that $|\psi_h\rangle$ is exactly the state one would assign to this system in "Copenhagen" quantum theory were the observed outcomes to be the ranges Δa_{k_i}; the projections correspond to the "collapses". In a histories-oriented formulation, however, $|\psi_h\rangle$ is not the state of the system, $|\psi\rangle$ is. The apparent discontinuous evolution of the state in Copenhagen quantum theory is the result of insisting on assigning a meaning to $|\psi(t)\rangle$ *separately along every possible "branch"* of the system.

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[2] By which it is meant that the appropriate probability sum rules are satisfied.

The probability of the history h is given by the Lüders-von Neumann formula:

$$p(h) = \langle \psi_h | \psi_h \rangle. \tag{2.4}$$

Such a probability only makes sense, however – meaning $\sum_h p(h) = 1$ – when the interference between the possible histories vanishes, $\langle \psi_{h'} | \psi_h \rangle = 0$ for $h' \neq h$.

One therefore defines the "decoherence functional"

$$d(h, h') = \langle \psi_{h'} | \psi_h \rangle. \tag{2.5}$$

When the decoherence functional is diagonal on a given set of histories,[3] then that set is said to "decohere" or "be consistent". In such sets

$$d(h, h') = p(h)\, \delta_{h'h}. \tag{2.6}$$

The decoherence functional of a system thus provides an internally defined, objective measure of interference between the possible histories of a system, independent of notions of external observers or measurements – though it does reproduce the predictions of ordinary quantum theory in appropriate "measurement situations". It is the decoherence functional that determines when the quantum theory makes a prediction concerning a possible history, and, when a probability may be sensibly assigned, what that probability is.

III. MODEL COSMOLOGY & ITS QUANTIZATION

To illustrate the essential importance of the consistency of alternative histories to prediction in quantum gravity, we consider the predictions of a simple quantum cosmological model concerning the volume of the universe. We find that decoherence is *critical* to a coherent prediction, and indeed, if one ignores the role it plays, one is led to infer a conclusion which is in fact the precise *opposite* of the correct prediction. The point is not so much the applicability of the content of the prediction itself for our universe. Different models might – and indeed, do – give a different result [13]. Rather, it is to emphasize the essential role played by quantum considerations in *making* such predictions.

In this paper, we concentrate on the implications of these quantum considerations, referring to [9] for technical details.

The model we consider is a flat, homoegeneous and isotropic Friedmann-Robertson-Walker (FRW) universe with line element

$$ds^2 = -dt^2 + a^2(t)\, d\mathring{q}^2, \tag{3.1}$$

where $d\mathring{q}^2$ is a fixed, flat fiducial metric on the spatial slices. Since the flat FRW universe is spatially infinite we choose a fiducial cell with unit spatial volume with respect to $d\mathring{q}^2$ and restrict all integrations to it. Choosing for matter content a free, massless, minimally coupled scalar field, the action for this model cosmology is

$$S = \frac{3}{8\pi G} \int dt\, (-a\dot{a}^2) + \frac{1}{2} \int dt\, a^3 \dot{\phi}^2. \tag{3.2}$$

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[3] Complete in the sense that it consists in a set of mutually exclusive, collectively exhaustive histories.

Canonical phase space variables for the model are $\{a, p_a, \phi, p_\phi\}$, the scale factor and scalar field and their conjugate momenta. A canonical transformation to the volume $V = a^3$ and its conjugate $\beta = -\frac{4\pi G}{3}\frac{p_a}{a^2}$ (so $\{\beta, V\} = 4\pi G$) [14] puts the Hamiltonian constraint in the form

$$-\beta^2 V^2 + \frac{4\pi G}{3}p_\phi^2 \approx 0, \tag{3.3}$$

from which one sees that β^2 measures the energy density $p_\phi^2/2V^2$ in the classical theory. Classically, p_ϕ is a constant of the motion, and

$$\phi = \pm\frac{1}{\sqrt{12\pi G}}\ln\frac{V}{V_c} + \phi_c \tag{3.4}$$

are expanding and contracting branches of the classical solution. (V_c and ϕ_c are integration constants.)

It is shown in [14, 15] that this model may be canonically quantized by setting $[\hat{\beta}, \hat{V}] = i4\pi G\hbar$ and $\hat{p}_\phi = -i\hbar\,\partial/\partial\phi$. Defining a rescaled volume variable

$$\nu = \frac{V}{(\frac{4\pi}{3})^{3/2}\,l_p^3/K}, \tag{3.5}$$

where $K = 2\sqrt{2}/3\sqrt{3\sqrt{3}}$, the constraint becomes the Wheeler-DeWitt equation

$$\partial_\phi^2\chi(\nu, \phi) = -\Theta(\nu)\chi(\nu, \phi), \tag{3.6}$$

where $\Theta(\nu) = -2\pi G\,\nu\partial_\nu(\nu\partial_\nu)$ is positive definite and self adjoint on $L^2(R, \nu^{-1}d\nu)$. General solutions of the quantum constraint may be decomposed in terms of positive and negative frequency parts, each of which satisfy the first order Schrödinger-like quantum constraint equation:

$$\mp i\,\partial_\phi\chi_\pm(\nu, \phi) = \sqrt{\Theta}\,\chi_\pm(\nu, \phi)\ . \tag{3.7}$$

The scalar field ϕ thus appears mathematically in the role of a "time" parameter,[4] and the dynamics may be expressed in terms of the propagator $U(\phi) = \exp(i\sqrt{\Theta}\phi)$.

The multiplicative volume operator $\hat{\nu}$ has eigenvalues $0 \leq \nu < \infty$ with eigenfunctions satisfying

$$\langle\nu|\nu'\rangle = \delta(\ln\nu - \ln\nu') \tag{3.8}$$

and projections on the range $d\nu$ given by

$$dP_\nu = \frac{d\nu}{\nu}|\nu\rangle\langle\nu|. \tag{3.9}$$

The scalar momentum $\hat{p}_\phi$ is a constant of the motion just as it is in the classical theory. Physical predictions may be extracted from observables which commute with the quantum constraint. These are the Dirac observables $\hat{p}_\phi$ and $\hat{\nu}|_\phi$. Here $\hat{\nu}_{|\phi^*}$, defined by $\hat{\nu}_{|\phi^*}\psi(\nu, \phi) = \exp i\sqrt{\Theta}(\phi - \phi^*)\,\hat{\nu}\,\psi(\nu, \phi^*)$, is the "relational observable" corresponding to $\hat{\nu}$ giving the volume at $\phi = \phi^*$.

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[4] This interpretation may provide a convenient way of conceptualizing mathematical results, but is by no means essential to the analysis or its conclusions.

The inner product and hence the physical Hilbert space may be found by demanding that the action of these observables be self adjoint; it turns out to be [15]

$$\langle \chi | \psi \rangle = \int_0^\infty \frac{d\nu}{\nu} \chi^*(\nu, \phi) \psi(\nu, \phi) \ . \tag{3.10}$$

The same inner product can be obtained from a more rigorous group averaging procedure [16, 17]. (See Ref. [18] for details.)

Because the dynamics (and hence Dirac observables) preserve the positive and negative frequency subspaces in the physical Hilbert space, we can restrict to either subspace. Here we choose to work with positive frequency solutions. Solutions to the Wheeler-DeWitt equation may then be represented as a sum of "expanding" (R for "right-moving" in a plot of ϕ vs. ν) and "contracting" (L for "left-moving") orthogonal branches,

$$\Psi(\nu, \phi) = \Psi^R(\nu_-) + \Psi^L(\nu_+), \tag{3.11}$$

where $\nu_\pm = \ln \nu \pm \sqrt{12\pi G}\, \phi$. (Negative frequency solutions can be written in a similar way).

Defining Heisenberg projections with the dynamics $U(\phi)$, class operators for histories pertaining to values of physical quantities at specific values of ϕ may be defined similarly to Eq. (2.2). The decoherence functional for this quantum cosmological model may then be defined just as in Eq. (2.5) with the inner product given by Eq. (3.10) [5].[5]

One can then show within the framework of generalized quantum theory outlined above that the probability (for the history in which) the volume ν is in a range $\Delta\nu$ when $\phi = \phi^*$ is given by

$$p_{\Delta\nu}(\phi^*) = \int_{\Delta\nu} \frac{d\nu}{\nu} |\Psi(\nu, \phi^*)|^2. \tag{3.12}$$

A comment concerning this result is in order. The operator $\hat{\nu}$ does *not* commute with the quantum constraint, and is therefore not a Dirac observable [19–21]. However, the corresponding relational observable $\hat{\nu}_{|\phi^*}$ is. It is satisfying that the probability for the history in which $\hat{\nu}$ assumes values in $\Delta\nu$ at ϕ^* gives precisely the result one would expect from considering expectation values of $\hat{\nu}_{|\phi^*}$. One thus may see how consideration of relational observables arises naturally within histories formulations of quantum theory. (Alternately, one may consider class operators constructed directly from projections onto ranges of values of the Dirac observables.) For further details see [9].[6]

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[5] The striking similarity of this formulation of generalized quantum theory for our cosmological model with that of ordinary non-relativistic quantum mechanics described in section II is a consequence of two factors. First, in this quantization a natural "clock" (namely, the scalar field ϕ) appears in the model through Eq. (3.7). Second, we are here restricting consideration to histories of alternatives defined by sequences of values of physical quantities at definite values of that clock. More general categories of alternatives, and therefore class operators, are certainly of potential physical relevance. We do not however consider them here. Previous discussions of decoherent histories formulations of quantum cosmological models may be found in [3–8]. These prior formulations generally lack, however, a well-defined underlying Hilbert space and observables. On the other hand, some of them may be better suited to models with no emergent "clock".

[6] An alternative approach demands that the class operators themselves must commute with the constraint [3, 4, 7, 8].

IV. QUANTUM SINGULARITIES

We turn now to the question of whether this model quantum universe is singular, just as the classical model is. There are many incommensurate criteria one might apply to this question. We focus on a simple one: does the universe ever achieve zero volume?

For a universe whose state is purely "left-moving" (contracting) or "right-moving" (expanding) (*cf.* Eq. (3.11)), this question is easy to answer in the affirmative. From Eq. (3.12) it is straightforward to show that if $\Delta\nu^* = [0, \nu^*]$ and $|\Psi\rangle = |\Psi^L\rangle$ or $|\Psi^R\rangle$,

$$\lim_{\phi\to-\infty} p^L_{\Delta\nu^*}(\phi) = 0 \qquad\qquad \lim_{\phi\to+\infty} p^L_{\Delta\nu^*}(\phi) = 1 \tag{4.1a}$$

$$\lim_{\phi\to-\infty} p^R_{\Delta\nu^*}(\phi) = 1 \qquad\qquad \lim_{\phi\to+\infty} p^R_{\Delta\nu^*}(\phi) = 0 \tag{4.1b}$$

for any fixed choice of ν^*, no matter how small. This is just as one may have expected: contracting universes will inevitably shrink to arbitrarily small volume, and expanding universes will equally inevitably have grown *from* arbitrarily small volume.

For this prediction, the role of decoherence was essentially trivial since decoherence for alternatives defined only at a single moment of "time" (ϕ) is automatic.[7]

Far more interesting, however, is the general case in which the state of the universe is a superposition

$$|\Psi\rangle = |\Psi^L\rangle + |\Psi^R\rangle \tag{4.2}$$

of expanding and contracting universes. Here, $p_L = \langle\Psi^L|\Psi^L\rangle$ and $p_R = \langle\Psi^R|\Psi^R\rangle$ (with $\langle\Psi^L|\Psi^R\rangle = 0$ and $p_L+p_R = 1$) measure the "amount" of each in the superposition. One may ask, what is the likelihood that a universe in such a "Schrödinger's Cat"-like superposition of (possibly macroscopic) states could "jump" from the contracting to the expanding branch *i.e.* remain peaked on a large classical solution at both "early" and "late" values of ϕ?

Indeed, such a "quantum bounce" may at first seem possible. As before, one may show that

$$p_{\Delta\nu^*}(\phi) = p^L_{\Delta\nu^*}(\phi) + p^R_{\Delta\nu^*}(\phi). \tag{4.3}$$

Now, though,

$$\lim_{\phi\to-\infty} p_{\Delta\nu^*}(\phi) = p_R \qquad \text{and} \qquad \lim_{\phi\to+\infty} p_{\Delta\nu^*}(\phi) = p_L. \tag{4.4}$$

Thus, there is in general a non-zero probability for the universe to be non-singular in *both* the "past" (p_L) and the "future" (p_R). Is this not precisely the possibility (with probability $p_L \cdot p_R$) of a "quantum bounce" we were seeking?

The answer is emphatically **no**. We are being misled because Eq. (4.3) is the answer to a specific quantum question: *What is the probability that the volume of the universe is in $\Delta\nu^*$ at a given value of ϕ?* But the question of whether a quantum universe bounces is really about *two* values of ϕ: *What is the probability that the universe is **not** in $\Delta\nu^*$ at both $\phi \to -\infty$ and $\phi \to +\infty$?* The class operator for this question is

$$C_{\text{bounce}} = \lim_{\substack{\phi_1\to-\infty \\ \phi_2\to+\infty}} P^\nu_{\overline{\Delta\nu^*_1}}(\phi_1) P^\nu_{\overline{\Delta\nu^*_2}}(\phi_2), \tag{4.5}$$

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[7] This is because the corresponding class operators are simply projections; the branch wave functions for different histories are thereby automatically orthogonal.

where the intervals $\overline{\Delta\nu^*}$ are the complements of the intervals $\Delta\nu^*$ specifying small volume. The complementary history

$$C_{\text{sing}} = 1 - C_{\text{bounce}} \tag{4.6}$$

is the one for which the universe is inevitably singular *i.e.* assumes zero volume (enters $\Delta\nu^*$) at some point in its history.

Now, however, decoherence is *not* automatic, and it is *not* trivial. In general, questions about the volume of the universe at two different values of ϕ make no more quantum sense than the question of which slit the particle passed through in two-slit interference. The question can only be given a coherent answer when the histories decohere, such as when there is a measuring device in place.

In this particular case, however, it is possible to show with complete generality that *in the given limit $\phi \to -\infty$ and $\phi \to +\infty$*, the histories $\{\text{bounce}, \text{sing}\}$ *do* decohere. Indeed, in this limit

$$|\Psi_{\text{bounce}}\rangle = C^{\dagger}_{\text{bounce}}|\Psi\rangle = 0 \qquad \text{and} \qquad |\Psi_{\text{sing}}\rangle = C^{\dagger}_{\text{sing}}|\Psi\rangle = |\Psi\rangle, \tag{4.7}$$

so that

$$p_{\text{bounce}} = 0 \qquad \text{and} \qquad p_{\text{sing}} = 1. \tag{4.8}$$

We have thus shown, within an explicit framework for computing quantum probabilities in an objective, observer-independent way, that a quantum bounce is not possible in these models: the universe is inevitably singular.

V. DISCUSSION

What is important to underscore is the absolutely *crucial* role played by decoherence in this analysis. A naive assessment of the behavior of $|\Psi(\phi)\rangle$ alone (as in Eq. (4.3)) seems to imply that a quantum bounce *is* possible. This, however, is critically misleading. The more careful analysis that recognizes that a quantum bounce is fundamentally a question concerning properties of the universe at a *sequence* of values of ϕ shows that it is inevitably a quantum question, and must have a quantum answer. Such an answer is *only* available if the corresponding histories decohere. We have shown that, in a certain limit, they indeed do,[8] and that the quantum bounce suggested by Eq. (4.3) cannot in fact occur.

Put another way, ignoring the fundamental role played by quantum decoherence would tend to lead one to an utterly *incorrect* conclusion. Quantum mechanics matters, even when applied to the universe as a whole.

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[8] In general, these histories *do not* decohere at finite values of ϕ, even though in the quantum theory the left- and right-moving sectors are orthogonal and preserved by the theory's Dirac observables [9].

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