Topological geon black holes in Einstein-Yang-Mills theory

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Abstract

We construct topological geon quotients of two families of Einstein-Yang-Mills black holes. For Künzle's static, spherically symmetric SU(n) black holes with $n > 2$, a geon quotient exists but generically requires promoting charge conjugation into a gauge symmetry. For Kleihaus and Kunz's static, axially symmetric SU(2) black holes a geon quotient exists without gauging charge conjugation, and the parity of the gauge field winding number determines whether the geon gauge bundle is trivial. The geon's gauge bundle structure is expected to have an imprint in the Hawking-Unruh effect for quantum fields that couple to the background gauge field.

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1 Introduction

Given a stationary black hole spacetime with a bifurcate Killing horizon, it may be possible to form from it a time-orientable quotient spacetime in which the two exterior regions separated by the Killing horizon become identified. If the quotient is asymptotically flat, it is a topological geon [1, 2, 3, 4], in the sense that the spatial geometry is that of a compact manifold minus a point, with the omitted point at an asymptotically flat infinity. The showcase example is the $\mathbb{RP}^3$ geon [5, 6, 7, 8], formed as a $\mathbb{Z}_2$ quotient of Kruskal. There exist also quotients in which the geometry at the infinity is only asymptotically locally flat, and others in which the geometry at the infinity is asymptotically anti-de Sitter or asymptotically locally anti-de Sitter [9, 10, 11, 12]. We shall understand topological geons to encompass all these cases.

Topological geon black holes of the kind described above are unlikely to be created in an astrophysical star collapse, as their formation from conventional initial data would require a change in the spatial topology. However, they provide an arena for the Hawking-Unruh effect in a setting where the black hole is eternal and has nonvanishing surface gravity, but thermality for a quantum field cannot arise by the usual procedure of tracing over a causally disconnected exterior [13]. There is still thermality, in the usual Hawking temperature, but only for a limited set of observations, and the non-thermal correlations bear an imprint of the unusual geometry behind the horizons [9, 10, 11, 12, 14, 15]. In a sense, the Hawking-Unruh effect on a topological geon black hole reveals to an exterior observer features of the geometry that are classically hidden behind the horizons. A recent review can be found in [16].

When the black hole has a Maxwell field, it may be necessary to include charge conjugation in the map with which the black hole gauge bundle is quotiented into the geon gauge bundle [12]. This happens for example for the Reissner-Nordström hole, both with electric and magnetic charge; it also happens for the higher-dimensional Reissner-Nordström hole with electric charge in any dimension and with magnetic charge in even dimensions. Maxwell's theory on the geon incorporates then charge conjugation as a gauge symmetry, rather than just as a global symmetry: technically, the gauge group is no longer $\mathrm{U}(1) \simeq \mathrm{SO}(2)$ but $\mathbb{Z}_2 \ltimes \mathrm{U}(1) \simeq \mathrm{O}(2)$, where the nontrivial element of $\mathbb{Z}_2$ acts on $\mathrm{U}(1)$ by complex conjugation [17]. The presence of the charge conjugation in the quotienting map can further be verified to leave its imprint in the Hawking-Unruh effect for a quantum field that couples to the background Maxwell field [18, 19]. By contrast, spherically symmetric Einstein-SU(2) black holes admit a geon quotient without the inclusion of charge conjugation in the quotienting map, and the geon's gauge bundle is in fact trivial [12].

The purpose of this paper is to construct two new families of Einstein-Yang-Mills geon black holes. We shall specifically examine whether charge conjugation needs to be promoted into a gauge symmetry when taking the geon quotient.

In Sections 2 and 3 we consider the static, spherically symmetric Einstein-SU(n) black holes of Künzle [20] and their generalisations to a negative cosmological constant [21, 22]. The case $n = 2$ was covered in [12] as discussed above. For $n > 2$ we show that a geon quotient exists and generically requires including charge conjugation in the quotienting map: the enlarged gauge group is $\mathbb{Z}_2 \ltimes \mathrm{SU}(n)$, where the nontrivial element of $\mathbb{Z}_2$ acts on $\mathrm{SU}(n)$ by complex conjugation. A quotient without charge conjugation is possible only for certain special field configurations, of which we give a complete list, and we show that the geon gauge bundle is then trivial.

In Sections 4 and 5 we consider the static, axially symmetric Einstein-SU(2) black holes of Kleihaus and Kunz [23, 24]. We show that all these holes admit a geon quotient without the need to gauge charge conjugation. When the winding number of the gauge field configuration is odd, the geon gauge bundle is trivial; this includes as a special case the spherically symmetric geon discussed in [12]. When the winding number is even, the geon gauge bundle is nontrivial.

Section 6 summarises the results and discusses their relevance for the Hawking-Unruh effect.

The metric signature is $(-+++)$. Sections 2 and 3 use the convention of an antihermitian gauge field, common in mathematical literature. Sections 4 and 5 use the convention of a hermitian gauge field, common in physics literature. Homotopies are assumed smooth, without loss of generality [25].

2 Spherically symmetric $\mathrm{SU}(n)$ black holes

In this section we review the relevant properties of the static, spherically symmetric $\mathrm{SU}(n)$ Einstein-Yang-Mills black holes of Künzle [20] and their generalisations to a negative cosmological constant [21, 22]. We also give explicit Kruskal-type coordinates that extend these solutions across the Killing horizon. We assume $n > 2$ when not explicitly mentioned otherwise, although most of the formulas hold also for $n = 2$.

2.1 Exterior ansatz

A static, spherically symmetric metric in Schwarzschild-type coordinates (t, r, θ, ϕ) reads

$$ds^2 = -Ne^{-2\delta}\mathrm{d}t^2 + N^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2), \tag{2.1}$$

where the functions N and δ depend only on the radial coordinate r and $N > 0$. The coordinates (θ, ϕ) are the usual angle coordinates on the two-spheres on which the SO(3) isometry acts, and ∂_t is a timelike Killing vector, orthogonal to the hypersurfaces of constant t. The ansatz (2.1) does not cover spherically symmetric spacetimes in which the area of the two-sphere is constant, but this special case does not occur in the black hole spacetimes in which we are interested.

The definition of a spherically symmetric $\mathrm{SU}(n)$ gauge field on a spherically symmetric spacetime $\mathcal{M}$ builds on the observation that the double cover of the spacetime isometry group SO(3) is SU(2), which can be embedded in $\mathrm{SU}(n)$: given the embedding, the gauge field configuration is required to be invariant under the SU(2) bundle automorphisms that are compatible with the SO(3) action on $\mathcal{M}$. We briefly recall here the classification of these configurations and their description in an adapted Lie algebra basis [26].

The first part of the argument consists of determining all SU(n) principal bundles that admit an SU(2) action of the required kind. For spacetimes that are regularly foliated by the SO(3) orbits, as is the case in (2.1), this amounts to classifying all SU(n) principal bundles over S^2. The classification relies on presenting S^2 as the quotient space $\simeq$ SU(2)/U(1) of the base space and analysing the action of the isotropy subgroup U(1) $\subset$ SU(2) on the total space of the bundle. The result is that, up to isomorphisms, the bundles are in one-to-one correspondence with the conjugacy classes of group homomorphisms from U(1) to SU(n) [27].

A convenient unique representative from each conjugacy class is the map $\lambda : \mathrm{U}(1) \to \mathrm{SU}(n),\ z \mapsto \mathrm{diag}(z^{k_1}, \ldots, z^{k_n})$, where the n integers $k_1, \ldots, k_n$ satisfy $k_1 \geq k_2 \geq \cdots \geq k_n$ and sum to zero. It follows that the equivalence classes of the SU(n) principal bundles can be uniquely indexed by sets of n integers $\{k_1, \ldots, k_n\}$ that sum to zero and are ordered so that $k_1 \geq k_2 \geq \cdots \geq k_n$.

For the second part of the argument, one fixes the bundle and an SU(2) action on it and considers all connections that are invariant under this action. Let the map $\lambda : \mathrm{U}(1) \to \mathrm{SU}(n)$ be as defined above, and let $\lambda' : \mathfrak{su}(2) \to \mathfrak{su}(n)$ denote the derivative of λ at the identity. A theorem of Wang [28] then states that the invariant connections are in one-to-one correspondence with the set of linear maps $\Lambda : \mathfrak{su}(2) \to \mathfrak{su}(n)$ satisfying the conditions

$$\Lambda(X) = \lambda'(X), \tag{2.2a}$$

$$\Lambda \circ \mathrm{ad}_z = \mathrm{ad}_{\lambda(z)} \circ \Lambda, \tag{2.2b}$$

for all $X \in \mathfrak{u}(1)$ and $z \in \mathrm{U}(1)$, where U(1) is again the isotropy subgroup of the SU(2) action. The curvature F of these connections takes the form

$$F(\tilde{X}, \tilde{Y}) = [\Lambda(X), \Lambda(Y)] - \Lambda\left([X, Y]\right), \tag{2.3}$$

where $X, Y \in \mathfrak{su}(2)$ and $\tilde{X}, \tilde{Y}$ are the corresponding vector fields induced by the SU(2) action on the total space.

We adopt for $\mathfrak{su}(2)$ the basis $\tau_l := -\frac{i}{2}\sigma_l$, $l = 1, 2, 3$, where σ_l are the Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.4}$$

We write $\Lambda_l := \Lambda(\tau_l)$, $l = 1, 2, 3$, and we may without loss of generality choose the isotropy subgroup U(1) to be embedded in SU(2) as $z \mapsto \left(\begin{smallmatrix} z & 0 \\ 0 & z^{-1} \end{smallmatrix}\right)$. From (2.2a) it then follows that $\Lambda_3 = -\frac{i}{2}\,\mathrm{diag}\left(k_1, \ldots, k_n\right)$. The infinitesimal version of (2.2b) reads

$$\Lambda\left([\tau_3, \tau_l]\right) = [\Lambda_3, \Lambda_l], \quad i = 1, 2, \tag{2.5}$$

which implies that Λ_1 and Λ_2 can be written as

$$\Lambda_1 = \frac{1}{2}\left(C - C^H\right), \quad \Lambda_2 = -\frac{i}{2}\left(C + C^H\right), \tag{2.6}$$

where C is a strictly upper triangular complex $n \times n$ matrix, C^H is its Hermitian conjugate, and $C_{ij} \neq 0$ if and only if $k_i = k_j + 2$.

Evaluating (2.3) on the $\mathfrak{su}(2)$ basis τ_l shows that the only non-vanishing component of the curvature form is $F(\tilde{\tau}_1, \tilde{\tau}_2) = [\Lambda_1, \Lambda_2] - \Lambda_3$. As the base space S^2 is two-dimensional, the curvature form must be proportional to the spherically symmetric volume form $\sin\theta \, \mathrm{d}\theta \wedge \mathrm{d}\phi$. The curvature form on S^2 must hence take the form

$$F = ([\Lambda_1, \Lambda_2] - \Lambda_3) \sin\theta \, \mathrm{d}\theta \wedge \mathrm{d}\phi. \tag{2.7}$$

A corresponding connection form is

$$\hat{A} := \Lambda_1 \mathrm{d}\theta + (\Lambda_2 \sin\theta + \Lambda_3 \cos\theta) \, \mathrm{d}\phi. \tag{2.8}$$

Finally, the connection form A on the four-dimensional spacetime (2.1) can be decomposed as

$$A = \tilde{A} + \hat{A}, \tag{2.9}$$

where $\hat{A}$ is as in (2.8) but the components of the matrix C in (2.6) are allowed to depend on the coordinates (t, r). The remaining part $\tilde{A}$ is an $\mathfrak{su}(n)$-valued one-form on the two-dimensional spacetime obtained by dropping the angles from (2.1), invariant under the adjoint action of the subgroup $\lambda([\mathrm{U}(1)])$ [27].

In what follows we consider only the case [20, 21, 22] where the set of n integers is $\{k_1, \ldots, k_n\} = \{n-1, n-3, n-5, \ldots, -n+3, -n+1\}$. The connection form is taken to have a vanishing Coulomb component, $A_t = 0$, and one can then choose the gauge so that also the radial component A_r is zero. This means that we consider purely magnetic configurations of the form

$$A = \Lambda_1 \mathrm{d}\theta + (\Lambda_2 \sin\theta + \Lambda_3 \cos\theta) \, \mathrm{d}\phi, \tag{2.10}$$

where the traceless antihermitian matrices Λ_1, Λ_2 and Λ_3 are given by

$$\Lambda_1 = \frac{1}{2} \begin{pmatrix} 0 & w_1 & & & & \\ -w_1 & 0 & w_2 & & & \\ & -w_2 & 0 & w_3 & & \\ & & \cdots & \cdots & \cdots & \\ & & & -w_{n-2} & 0 & w_{n-1} \\ & & & & -w_{n-1} & 0 \end{pmatrix}, \tag{2.11a}$$

$$\Lambda_2 = -\frac{i}{2} \begin{pmatrix} 0 & w_1 & & & & \\ w_1 & 0 & w_2 & & & \\ & w_2 & 0 & w_3 & & \\ & & \cdots & \cdots & \cdots & \\ & & & w_{n-2} & 0 & w_{n-1} \\ & & & & w_{n-1} & 0 \end{pmatrix}, \tag{2.11b}$$

$$\Lambda_3 = -\frac{i}{2}\begin{pmatrix} n-1 & & & & & \\ & n-3 & & & & \\ & & n-5 & & & \\ & & & \cdots & & \\ & & & & -n+3 & \\ & & & & & -n+1 \end{pmatrix}, \tag{2.11c}$$

and the real-valued functions w_j, $j = 1, \ldots, n-1$, depend only on the coordinate r.

The one-form (2.10) has a Dirac string singularity as $\theta \to 0$ and $\theta \to \pi$ [29]. The regularity of the curvature form (2.7) shows that this singularity is a gauge artefact. As the triviality of the fundamental group of $\mathrm{SU}(n)$ implies that $\mathrm{SU}(n)$ principal bundles over two-spheres are trivial [30, 31], the one-form (2.10) must therefore be a local representative of a connection one-form in the trivial $\mathrm{SU}(n)$ bundle over the spacetime. We shall explicitly remove the Dirac string singularity in Section 3.2.

The ansatz (2.10) has a residual gauge freedom in that a gauge transformation by

$$e^{ik\pi/n}\,\mathrm{diag}\big(\underbrace{-1,-1,\ldots,-1}_{k},\underbrace{1,1,\ldots,1}_{n-k}\big) \in \mathrm{SU}(n) \tag{2.12}$$

leaves w_j invariant for $j \neq k$ but changes the sign of w_k [20]. We shall use this gauge freedom to simplify the special geon configurations that will be found in Section 3.1.

2.2 Nondegenerate Killing horizon: Kruskal-like extension

The metric (2.1) and the connection form (2.10) give an ansatz that can be inserted in the Einstein-Yang-Mills field equations. We are interested in spacetimes that have a nondegenerate Killing horizon at $r = r_h > 0$, where $N(r_h) = 0$ and $N'(r_h) > 0$, the prime indicating derivative with respect to r. Initial data for integrating the field equations from $r = r_h$ towards increasing r then consists of r_h, $\delta(r_h)$ and $w_j(r_h)$, $j = 1, \ldots, n-1$. Local solutions in some neighbourhood of the horizon exist under a weak regularity restriction on $w_j(r_h)$ [20, 21]. Not all of these local solutions extend to an asymptotically flat (for a vanishing cosmological constant) or asymptotically anti–de Sitter (for a negative cosmological constant) infinity at $r \to \infty$, but for those that do, the solution is a static region of a nondegenerate black hole spacetime. Numerical results are given in [20, 21, 22, 32].

To extend the metric across the Killing horizon, we start in the exterior region and define the Kruskal-type coordinates (U, V, θ, ϕ) by

$$U := -\exp\left[-\alpha\left(t - \int_{r_0}^{r} \frac{e^{\delta(r)}}{N(r)}\mathrm{d}r\right)\right], \tag{2.13a}$$

$$V := \exp\left[\alpha\left(t + \int_{r_0}^{r} \frac{e^{\delta(r)}}{N(r)}\mathrm{d}r\right)\right], \tag{2.13b}$$

where $\alpha := \frac{1}{2}N'(r_h)e^{-\delta(r_h)}$ and the constant r_0 is chosen so that the product UV,

$$UV = -\exp\left[2\alpha\int_{r_0}^{r}\frac{e^{\delta(r)}}{N(r)}\mathrm{d}r\right], \tag{2.14}$$

has the Taylor expansion

$$UV = -\left(r - r_h\right)\left[1 + \left(\delta'(r_h) - \frac{1}{2}\frac{N''(r_h)}{N'(r_h)}\right)(r - r_h) + O\left((r - r_h)^2\right)\right] \tag{2.15}$$

as $r \to r_h$. It follows that in the exterior we have $U < 0$ and $V > 0$, and the Killing horizon is at $UV \to 0_-$. Whether UV is bounded below depends on the asymptotic behaviour of the metric at large r, but this will not affect what follows.

The metric in the coordinates (U, V, θ, ϕ) reads

$$ds^2 = \frac{1}{\alpha^2}\frac{N(r)e^{-2\delta(r)}}{UV}\mathrm{d}U\mathrm{d}V + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2), \tag{2.16}$$

where r is a function of UV via (2.14). Inverting (2.15) as

$$r - r_h = -UV\left[1 + \left(\delta'(r_h) - \frac{1}{2}\frac{N''(r_h)}{N'(r_h)}\right)UV + O\left((UV)^2\right)\right], \tag{2.17}$$

we find that the metric (2.16) has the near-horizon expansion

$$\begin{aligned} ds^2 = &-\frac{4}{N'(r_h)}\left[1 + \left(3\delta'(r_h) - \frac{N''(r_h)}{N'(r_h)}\right)UV + O\left((UV)^2\right)\right]\mathrm{d}U\mathrm{d}V \\ &+ r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right), \end{aligned} \tag{2.18}$$

which is regular across $UV = 0$. The metric can hence be extended from the original, 'right-hand-side' exterior to the black hole interior where $U > 0$ and $V > 0$, to the white hole interior where $U < 0$ and $V < 0$ and to the 'left-hand-side' exterior where $U > 0$ and $V < 0$. If the functions $N(r)$ and $\delta(r)$ are smooth at $r = r_h$, it further follows that the metric in the Kruskal coordinates is smooth at the horizon. Whether UV has an upper limit in the black and white hole regions, and whether there are further Killing horizons past these regions, is not relevant in what follows.

The extension of the gauge field across the horizon is given by (2.10) and (2.11), with $w_j = w_j\big(r(UV)\big)$. The extension is regular since $w_j(r_h)$ are part of the boundary data for the exterior solution, and the extension is smooth if $w_j(r)$ are smooth at $r = r_h$.

3 Geon quotient of the spherically symmetric SU(n) black hole

We wish to take a geon quotient of the SU(n) black hole of Section 2. For presentational simplicity, we take the gauge group of the black hole bundle to be SU(n) for odd n and

$\mathrm{SU}(n)/\{\pm \mathrm{Id}\}$ for even n. We denote this gauge group by G. We write equations in G as matrix equations in the defining matrix representation, understanding for even n the matrices to be defined up to overall sign. Proceeding with the gauge group $\mathrm{SU}(n)$ for all n would yield the same end results but our choice of G will shorten the analysis as will be seen in Section 3.2.

3.1 Geon quotient for special configurations

Let $\mathcal{M}$ denote the spacetime manifold of the Kruskal-type extension, and let A_{ext} denote the gauge field (2.10) on $\mathcal{M}$,

$$A_{\mathrm{ext}} := \Lambda_1 \mathrm{d}\theta + (\Lambda_2 \sin\theta + \Lambda_3 \cos\theta)\, \mathrm{d}\phi. \tag{3.1}$$

From the expression of the metric in the Kruskal-type coordinates it is clear that the map

$$J : (U, V, \theta, \phi) \mapsto (V, U, \pi - \theta, \phi + \pi) \tag{3.2}$$

is an involutive isometry on $\mathcal{M}$ without fixed points, it interchanges the two exterior regions, and it also preserves both time and space orientation. The quotient spacetime $\mathcal{M}/\{\mathrm{Id}, J\}$ is therefore a topological geon black hole, with spatial topology $\mathbb{RP}^3 \setminus \{\text{point at infinity}\}$ and an exterior region that is isometric to an exterior region of $\mathcal{M}$.

What remains to be examined is whether there exists a corresponding bundle map that leaves the gauge field invariant: does J map the gauge field A_{ext} (3.1) to a gauge-equivalent one? Denoting by A^J_{ext} the pull-back of A_{ext} by J, we thus seek a gauge function $\Omega : \mathcal{M} \to G$, such that a gauge transformation by Ω maps A^J_{ext} back to A_{ext},

$$\Omega A^J_{\mathrm{ext}} \Omega^{-1} + \Omega \mathrm{d}\Omega^{-1} = A_{\mathrm{ext}}. \tag{3.3}$$

From (3.2) we find

$$A^J_{\mathrm{ext}} = -\Lambda_1 \mathrm{d}\theta + (\Lambda_2 \sin\theta - \Lambda_3 \cos\theta)\, \mathrm{d}\phi. \tag{3.4}$$

As neither (3.1) nor (3.4) involves $\mathrm{d}U$ or $\mathrm{d}V$, we may assume Ω to depend only on the angular coordinates (θ, ϕ). Equation (3.3) is then equivalent to the pair

$$-\Omega \Lambda_1 \Omega^{-1} + \Omega \partial_\theta \Omega^{-1} = \Lambda_1, \tag{3.5a}$$

$$\Omega(\Lambda_2 \sin\theta - \Lambda_3 \cos\theta)\Omega^{-1} + \Omega \partial_\phi \Omega^{-1} = \Lambda_2 \sin\theta + \Lambda_3 \cos\theta. \tag{3.5b}$$

It suffices to consider the field strengths of A_{ext} and A^J_{ext}. These can be computed from

$$F(X, Y) = \mathrm{d}A(X, Y) + \frac{1}{2}[A(X), A(Y)], \tag{3.6}$$

with the result

$$\begin{aligned} F_{\mathrm{ext}} = {} & \partial_U \Lambda_1 \mathrm{d}U \wedge \mathrm{d}\theta + \partial_V \Lambda_1 \mathrm{d}V \wedge \mathrm{d}\theta + \partial_U \Lambda_2 \sin\theta\, \mathrm{d}U \wedge \mathrm{d}\phi \\ & + \partial_V \Lambda_2 \sin\theta \mathrm{d}V \wedge \mathrm{d}\phi + ([\Lambda_1, \Lambda_2] - \Lambda_3) \sin\theta\, \mathrm{d}\theta \wedge \mathrm{d}\phi, \end{aligned} \tag{3.7a}$$

$$\begin{aligned} F^J_{\mathrm{ext}} = {} & -\partial_U \Lambda_1 \mathrm{d}U \wedge \mathrm{d}\theta - \partial_V \Lambda_1 \mathrm{d}V \wedge \mathrm{d}\theta + \partial_U \Lambda_2 \sin\theta\, \mathrm{d}U \wedge \mathrm{d}\phi \\ & + \partial_V \Lambda_2 \sin\theta \mathrm{d}V \wedge \mathrm{d}\phi - ([\Lambda_1, \Lambda_2] - \Lambda_3) \sin\theta\, \mathrm{d}\theta \wedge \mathrm{d}\phi. \end{aligned} \tag{3.7b}$$

From (3.3) it follows that these field strengths are related by

$$\Omega F^J_{\text{ext}} \Omega^{-1} = F_{\text{ext}}. \tag{3.8}$$

Inserting (3.7) in (3.8) and using the fact that Ω only depends on the angular coordinates, (3.8) reduces to

$$\Omega \Lambda_1 \Omega^{-1} = -\Lambda_1, \tag{3.9a}$$
$$\Omega \Lambda_2 \Omega^{-1} = \Lambda_2, \tag{3.9b}$$
$$\Omega([\Lambda_1, \Lambda_2] - \Lambda_3)\Omega^{-1} = -([\Lambda_1, \Lambda_2] - \Lambda_3). \tag{3.9c}$$

Simplifying (3.9c) with the help of (3.9a) and (3.9b) shows that the set (3.9) is equivalent to

$$\Omega \Lambda_1 \Omega^{-1} = -\Lambda_1, \tag{3.10a}$$
$$\Omega \Lambda_2 \Omega^{-1} = \Lambda_2, \tag{3.10b}$$
$$\Omega \Lambda_3 \Omega^{-1} = -\Lambda_3. \tag{3.10c}$$

This is the set we need to analyse.

First, observe from (2.11c) that Λ_3 and $-\Lambda_3$ are diagonal and their diagonal elements appear in the reverse order,

$$-\Lambda_3 = -\frac{i}{2} \begin{pmatrix} -n+1 & & & & & \\ & -n+3 & & & & \\ & & -n+5 & & & \\ & & & \cdots & & \\ & & & & n-3 & \\ & & & & & n-1 \end{pmatrix}. \tag{3.11}$$

Identity (3.10c) thus implies that Ω has the form

$$\Omega = (-i)^{n-1} \begin{pmatrix} & & & & \alpha_1 \\ & & & \alpha_2 & \\ & & \cdots & & \\ & \alpha_{n-1} & & & \\ \alpha_n & & & & \end{pmatrix} \tag{3.12}$$

where α_j are complex numbers with unit magnitude and $\prod_{j=1}^n \alpha_j = 1$.

Consider then (3.10a) and (3.10b). Using $\Omega^{-1} = \overline{\Omega}^T$, where the overline denotes complex conjugation and ^{T} transposition, we find

$$\Omega \Lambda_1 \Omega^{-1} = \frac{1}{2} \begin{pmatrix} 0 & -\alpha_1 \overline{\alpha}_2 w_{n-1} & & & \\ \alpha_2 \overline{\alpha}_1 w_{n-1} & 0 & -\alpha_2 \overline{\alpha}_3 w_{n-2} & & \\ \cdots & \cdots & \cdots & & \\ & \alpha_{n-1} \overline{\alpha}_{n-2} w_2 & 0 & -\alpha_{n-1} \overline{\alpha}_n w_1 \\ & & \alpha_n \overline{\alpha}_{n-1} w_1 & 0 \end{pmatrix}. \tag{3.13}$$

By (2.11a) and (3.13), (3.10a) reduces to the set

$$\begin{aligned}
\alpha_1\overline{\alpha}_2 w_{n-1} &= w_1 = \alpha_2\overline{\alpha}_1 w_{n-1}, \\
\alpha_2\overline{\alpha}_3 w_{n-2} &= w_2 = \alpha_3\overline{\alpha}_2 w_{n-2}, \\
&\vdots \\
\alpha_{n-2}\overline{\alpha}_{n-1} w_2 &= w_{n-2} = \alpha_{n-1}\overline{\alpha}_{n-2} w_2, \\
\alpha_{n-1}\overline{\alpha}_n w_1 &= w_{n-1} = \alpha_n\overline{\alpha}_{n-1} w_1,
\end{aligned} \tag{3.14}$$

and it can be similarly verified that also (3.10b) reduces to (3.14).

As $n > 2$ by assumption, alphas satisfying (3.14) do not exist for generic gauge field configurations. There is however a special class of gauge field configurations for which such alphas exist. If w_j is vanishing, the jth line of (3.14) requires w_{n-j} to vanish. If w_j is nonvanishing, the jth line of (3.14) implies $w_{n-j} = \epsilon_j w_j$ and $\alpha_{j+1} = \epsilon_j \alpha_j$, where $\epsilon_j \in \{-1, +1\}$, and if n is even, $\epsilon_{n/2} = 1$. A necessary condition for the alphas to exist is therefore that the gauge field functions satisfy $w_{n-j} = \epsilon_j w_j$ for all j, with $\epsilon_j \in \{-1, +1\}$ and $\epsilon_j = \epsilon_{n-j}$. Note that this condition is compatible with the radial evolution equation for the gauge field functions [20]. As observed in Section 2.1, the sign of each w_j can be independently changed by a gauge transformation. The gauge can therefore be chosen so that the necessary condition for the alphas to exist reads

$$w_{n-j} = w_j, \qquad \forall j. \tag{3.15}$$

When (3.15) holds, it is immediate that (3.14) is solved by $\alpha_j = 1\ \forall j$, and Ω (3.12) then takes the form

$$\Omega = (-i)^{n-1} \begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & \cdots & & \\ & 1 & & & \\ 1 & & & & \end{pmatrix}. \tag{3.16}$$

The necessary condition (3.15) is hence also sufficient.

We summarise. The necessary and sufficient condition for a geon quotient with the gauge group G to exist is (3.15), up to gauge transformations. When (3.15) holds, the gauge transformation that compensates for J in the quotienting bundle map is given by (3.16).

We note in passing that for $n = 2$ the only gauge field function is w_1 and the equations (3.14) have the solution $\alpha_1 = \alpha_2 = 1$. This yields the purely magnetic special case of the SU(2) geon described in [12].

3.2 Triviality of the black hole bundle

Up to now we have been working in a gauge in which the gauge field A_{ext} (3.1) has Dirac string singularities at $\theta = 0$ and $\theta = \pi$. As noted at the end of Section 2.1, the gauge bundle over the Kruskal-type spacetime $\mathcal{M}$ is trivial, and a globally regular gauge on $\mathcal{M}$ must hence exist. In this section we transform A_{ext} into a globally regular gauge. This will be used in Section 3.3 to analyse the gauge bundle over the geon spacetime.

To begin, observe that gauge transformations by the functions

$$\Omega_N := \operatorname{diag}\bigl(e^{-i(n-1)\phi/2}, e^{-i(n-3)\phi/2}, \ldots, e^{-i(-n+3)\phi/2}, e^{-i(-n+1)\phi/2}\bigr), \tag{3.17a}$$

$$\Omega_S := \operatorname{diag}\bigl(e^{i(n-1)\phi/2}, e^{i(n-3)\phi/2}, \ldots, e^{i(-n+3)\phi/2}, e^{i(-n+1)\phi/2}\bigr), \tag{3.17b}$$

make the gauge field A_{ext} (3.1) regular everywhere except respectively at $\theta = \pi$ and $\theta = 0$. This is the step where taking the gauge group to be $\mathrm{SU}(n)/\{\pm \mathrm{Id}\}$ for even n shortens the discussion, as the expressions (3.17) are not single-valued in $\mathrm{SU}(n)$ for even n.

It therefore suffices to find a gauge function $H : S^2 \setminus (\{\theta = 0\} \cup \{\theta = \pi\})$ that agrees with Ω_N in some punctured neighbourhood of $\theta = 0$, agrees with Ω_S in some punctured neighbourhood of $\theta = \pi$, and interpolates in between: a transformation by H puts A_{ext} (3.1) into a globally regular gauge. We shall show that such gauge functions exist.

Let first n be odd. The formulas (3.17) for Ω_N and Ω_S define two paths in $G = \mathrm{SU}(n)$, with path parameter $\phi \in [0, 2\pi]$. These paths are closed, starting and ending at the identity. As the fundamental group of $\mathrm{SU}(n)$ is trivial [33], these paths are homotopic, and any homotopy between them, with θ as the homotopy parameter (for example with $\pi/2 \le \theta \le 3\pi/4$), provides the interpolation we need.

Let then n be even. The formulas (3.17) for Ω_N and Ω_S again define two closed paths in $G = \mathrm{SU}(n)/\{\pm \mathrm{Id}\}$, starting and ending at the identity, with path parameter $\phi \in [0, 2\pi]$. When these paths are lifted from G to its double cover $\mathrm{SU}(n)$, formulas (3.17) show that each of the lift starts at $\mathrm{Id} \in \mathrm{SU}(n)$ and ends at $-\mathrm{Id} \in \mathrm{SU}(n)$. As the fundamental group of $\mathrm{SU}(n)$ is trivial, these two lifts are homotopic to each other in $\mathrm{SU}(n)$, and this homotopy in $\mathrm{SU}(n)$ projects down into a homotopy between the original closed paths in $G = \mathrm{SU}(n)/\{\pm \mathrm{Id}\}$. Hence the homotopy between the closed paths in G provides again the interpolation we need.

Finally, note that for even n a connection in the trivial $\mathrm{SU}(n)/\{\pm \mathrm{Id}\}$ bundle lifts into a connection in the trivial $\mathrm{SU}(n)$ bundle. Using the gauge group $\mathrm{SU}(n)/\{\pm \mathrm{Id}\}$ instead of $\mathrm{SU}(n)$ for even n is hence just a presentational convenience.

3.3 Triviality of the geon bundle for the configurations of Section 3.1

In this section we show that the geons of Section 3.1 have a trivial gauge bundle.

We showed in Section 3.2 that the black hole bundle P is trivial and we can realise it as $P := \mathcal{M} \times G$. In this realisation, the geon bundle P' is the quotient of P by the $\mathbb{Z}_2$ group of bundle automorphisms whose nontrivial element K takes the form

$$\begin{aligned} K : \mathcal{M} \times G &\to \mathcal{M} \times G; \\ (x, h) &\mapsto \bigl(J(x), h \cdot \Xi(x)^{-1}\bigr), \end{aligned} \tag{3.18}$$

where $\Xi : \mathcal{M} \to G$ is the gauge function that compensates for J in a globally regular gauge. The G-multiplication denoted by a dot is matrix multiplication for odd n and matrix multiplication up to overall sign for even n.

We shall work in the globally regular gauge that is obtained from the gauge (3.15) by the procedure of Section 3.2. In this gauge we have

$$\Xi(x) = H(x)\Omega\big[H\big(J(x)\big)\big]^{-1} \tag{3.19}$$

for $0 < \theta < \pi$, where Ω is given by (3.16) and H was defined in Section 3.2. It follows from (3.16) and (3.17) that Ξ takes a constant value in sufficiently small punctured neighbourhoods of $\theta = 0$ and $\theta = \pi$. Ξ is therefore well defined on $\mathcal{M}$, by (3.19) for $0 < \theta < \pi$ and by continuity at $\theta = 0$ and $\theta = \pi$.

Recall that a principal bundle is trivial iff it admits a global section. The geon bundle P' admits a global section iff P admits a global section σ that is invariant under K. By (3.18), this invariance condition reads

$$\sigma\big(J(x)\big) = \sigma(x) \cdot \Xi(x)^{-1}, \quad \forall x \in \mathcal{M}. \tag{3.20}$$

As the gauge field depends on U and V only through the combination UV, it suffices to consider the condition (3.20) on the two-sphere at $U = V = 0$. It further suffices to consider (3.20) on the equator $\theta = \pi/2$ of the two-sphere. To see this, let γ and Ξ_{eq} denote the respective restrictions of σ and Ξ to the equator. The restriction of (3.20) to the equator then reads

$$\gamma(\phi + \pi) = \gamma(\phi) \cdot \Xi_{\mathrm{eq}}(\phi)^{-1}. \tag{3.21}$$

If σ exists, it defines a solution to (3.21) by restriction. Conversely, suppose that a solution to (3.21) exists. We can view γ equivalently as a G-valued function on S^1 or as a closed path in G, denoted by the same letter and given by $\gamma : [0, 2\pi] \to G$; $\phi \mapsto \gamma(\phi)$. When viewed as a closed path, γ is contractible. For odd n this follows because $G = \mathrm{SU}(n)$ has a trivial fundamental group. For even n the fundamental group of $G = \mathrm{SU}(n)/\{\pm \mathrm{Id}\}$ is $\mathbb{Z}_2$, but γ is contractible by the observation made in the last paragraph of Section 3.2, or alternatively by the explicit construction of γ below. Given γ, we can define σ for $0 \le \theta \le \pi/2$ by an arbitrary contraction of γ into a trivial path at $\theta = 0$. Defining σ for $\pi/2 < \theta \le \pi$ by (3.20) then gives the desired σ.

What hence remains is to show that a solution to (3.21) exists. We now proceed to construct such a solution.

Let H_{eq} denote the restriction of H to the equator. From (3.19) we have $\Xi_{\mathrm{eq}}(\phi) = H_{\mathrm{eq}}(\phi)\Omega\big[H_{\mathrm{eq}}(\phi + \pi)\big]^{-1}$. Defining

$$\tilde{\gamma}(\phi) := \gamma(\phi) \cdot H_{\mathrm{eq}}(\phi + \pi), \tag{3.22}$$

the condition (3.21) can be rearranged into

$$\tilde{\gamma}(\phi + \pi) = \tilde{\gamma}(\phi) \cdot \Omega^{-1}. \tag{3.23}$$

Without loss of generality, we may set $\tilde{\gamma}(0) = \mathrm{Id}$; then $\tilde{\gamma}(\pi) = \Omega^{-1}$.

Since Ω^{-1} is special unitary, it can be diagonalised by

$$\Omega^{-1} = UDU^{-1}, \tag{3.24}$$

where U is unitary and D is a diagonal special unitary matrix whose diagonal elements are the eigenvalues of Ω^{-1}. We need to analyse these eigenvalues.

Let n be odd. A recursive evaluation of the determinant shows that $|\Omega^{-1} - \lambda\,\mathrm{Id}| = -(\lambda^2-1)^{(n-1)/2}\big(\lambda - (-1)^{(n-1)/2}\big)$. The eigenvalues of Ω^{-1} are hence ± 1, and the multiplicity of -1 is even. We now define the ϕ-dependent matrix $\hat{D}(\phi)$ by replacing an arbitrarily-chosen half of the -1s in D by $e^{i\phi}$ and the other half by $e^{-i\phi}$. It is immediate that $\hat{D}(\phi) \in G$, $\hat{D}$ has period 2π, $\hat{D}(0) = \mathrm{Id}$ and $\hat{D}(\pi) = D$. Given $\hat{D}$, we define $\tilde{\gamma}(\phi) := U\hat{D}(\phi)U^{-1}$. Then $\tilde{\gamma}(\phi)\cdot\Omega^{-1} = U\hat{D}(\phi)U^{-1}UDU^{-1} = U\hat{D}(\phi)DU^{-1} = U\hat{D}(\phi)\hat{D}(\pi)U^{-1} = U\hat{D}(\phi+\pi)U^{-1} = \tilde{\gamma}(\phi+\pi)$, so that $\tilde{\gamma}$ satisfies (3.23) and γ satisfies (3.21).

Let then n be even. Proceeding as above, we find $|\Omega^{-1} - \lambda\,\mathrm{Id}| = (\lambda^2+1)^{n/2}$. The eigenvalues of Ω^{-1} are hence $\pm i$, each with multiplicity $n/2$. We now define $\hat{D}(\phi)$ by replacing in D the eigenvalues i by $e^{i\phi/2}$ and the eigenvalues $-i$ by $e^{-i\phi/2}$. Then $\hat{D}(0) = \mathrm{Id}$, $\hat{D}(\pi) = D$, and although $\hat{D}$ is not 2π-periodic as an $\mathrm{SU}(n)$ matrix, it is as a $G = \mathrm{SU}(n)/\{\pm\,\mathrm{Id}\}$ matrix. Defining again $\tilde{\gamma}(\phi) := U\hat{D}(\phi)U^{-1}$, the conditions (3.23) and (3.21) can be verified as for odd n.

Finally, for even n, we verify explicitly the claim that the path $\gamma : [0, 2\pi] \to G$; $\phi \mapsto \gamma(\phi)$ constructed above is contractible in G. Without loss of generality, the gauge function H can be chosen to equal Ω_N (3.17a) on the equator. In this gauge it is transparent that the lift of H_{eq} into $\mathrm{SU}(n)$ is periodic in ϕ with period 4π and changes sign after a translation in ϕ by 2π. From (3.22) it follows that the lift of γ to $\mathrm{SU}(n)$ is a closed path in $\mathrm{SU}(n)$, and the contraction of this lift in $\mathrm{SU}(n)$ projects down to a contraction of γ in G.

This completes the proof of triviality of the geon bundle.

3.4 Generic configurations: geon quotient with gauged charge conjugation

We saw in Section 3.1 that a geon quotient with gauge group G does not exist for generic gauge field configurations. A similar obstacle for the Maxwell gauge field in the Reissner-Nordström black hole [5] can be overcome by promoting U(1) charge conjugation from a global symmetry into a gauge symmetry [12]. In this section we show that a similar gauging of charge conjugation works also for the $\mathrm{SU}(n)$ black holes at hand.

In the abelian case, the usual Maxwell gauge group $\mathrm{U}(1) \simeq \mathrm{SO}(2)$ is enlarged into $\mathrm{O}(2) \simeq \mathbb{Z}_2 \ltimes \mathrm{SO}(2) \simeq \mathbb{Z}_2 \ltimes \mathrm{U}(1)$. In the $\mathbb{Z}_2 \ltimes \mathrm{U}(1)$ representation, the group multiplication law reads

$$(a_1, u_1)\cdot(a_2, u_2) = \big(a_1 a_2, u_1 \rho_{a_1}(u_2)\big), \tag{3.25}$$

where $a_i \in \mathbb{Z}_2$, $u_i \in \mathrm{U}(1)$, and $\rho : \mathbb{Z}_2 \to \mathrm{Aut}\big(\mathrm{U}(1)\big)$, $a \mapsto \rho_a$, is the group homomorphism for which the nontrivial element of $\mathbb{Z}_2$ acts on $\mathrm{U}(1)$ by complex conjugation. Writing $\mathbb{Z}_2 \simeq \{0, 1\}$, where the identity element is 0, the explicit formula for ρ is

$$\rho_0(u) = u, \tag{3.26a}$$

$$\rho_1(u) = \overline{u}. \tag{3.26b}$$

In the nonabelian case at hand, the original gauge group G is $\mathrm{SU}(n)$ for odd n and $\mathrm{SU}(n)/\{\pm \mathrm{Id}\}$ for even n. We enlarge G to $G_{\mathrm{enl}} := \mathbb{Z}_2 \ltimes G$ by (3.25) and (3.26). The group multiplication table of G_{enl} reads

$$\begin{aligned}
(0, u_1) \cdot (0, u_2) &= (0, u_1 u_2), \\
(0, u_1) \cdot (1, u_2) &= (1, u_1 u_2), \\
(1, u_1) \cdot (0, u_2) &= (1, u_1 \overline{u}_2), \\
(1, u_1) \cdot (1, u_2) &= (0, u_1 \overline{u}_2).
\end{aligned} \tag{3.27}$$

If Ω is a gauge function with values in G, it follows that the gauge function $\widetilde{\Omega} := (a, \Omega) : \mathcal{M} \to G_{\mathrm{enl}}$ transforms the gauge field by

$$A \mapsto \widetilde{\Omega} A \widetilde{\Omega}^{-1} + \widetilde{\Omega} \mathrm{d} \widetilde{\Omega}^{-1} = \begin{cases} \Omega A \Omega^{-1} + \Omega \mathrm{d} \Omega^{-1} & \text{if } \widetilde{\Omega} = (0, \Omega), \\ \Omega \bar{A} \Omega^{-1} + \Omega \mathrm{d} \Omega^{-1} & \text{if } \widetilde{\Omega} = (1, \Omega). \end{cases} \tag{3.28}$$

To find a geon, we follow Section 3.1 with Ω replaced by $\widetilde{\Omega}$. The conditions (3.10) are replaced by

$$\widetilde{\Omega} \Lambda_1 \widetilde{\Omega}^{-1} = -\Lambda_1, \tag{3.29a}$$

$$\widetilde{\Omega} \Lambda_2 \widetilde{\Omega}^{-1} = \Lambda_2, \tag{3.29b}$$

$$\widetilde{\Omega} \Lambda_3 \widetilde{\Omega}^{-1} = -\Lambda_3. \tag{3.29c}$$

It follows from (2.11) and (3.28) that the set (3.29) is solved by $\widetilde{\Omega} = (1, \Omega)$, where

$$\begin{aligned}
\Omega &= \mathrm{diag}\left(i^{-n+1}, i^{-n+3}, \ldots, i^{n-3}, i^{n-1}\right) \\
&= (-i)^{n-1} \, \mathrm{diag}\left(1, -1, 1, -1, \ldots, (-1)^{n-1}\right).
\end{aligned} \tag{3.30}$$

Hence the black hole bundle now admits a geon quotient without restrictions on the gauge field configuration.

If desired, the geon quotient can be described as in Section 3.3, by adopting in the trivial black hole bundle $\mathcal{M} \times G_{\mathrm{enl}}$ a globally regular gauge. Now, however, the geon bundle is not trivial, since the gauge transformation part of the bundle map is in the disconnected component of G_{enl}.

4 Axially symmetric SU(2) black holes

In this section we first review the static, axially symmetric Einstein-SU(2) black holes discovered by Kleihaus and Kunz [23, 24]. For a generalisation to a negative cosmological constant, see [34]. We then give Kruskal-type coordinates that extend the spacetime across the horizon.

4.1 The exterior solution of Kleihaus and Kunz

A static, axially symmetric metric can be written in the isotropic coordinates (t, r, θ, ϕ) as

$$ds^2 = -f\mathrm{d}t^2 + \frac{m}{f}\mathrm{d}r^2 + \frac{m}{f}r^2\mathrm{d}\theta^2 + \frac{l}{f}r^2\sin^2\theta\,\mathrm{d}\phi^2, \tag{4.1}$$

where the positive functions f, m and l depend only on r and θ. Here θ and ϕ are the usual angular coordinates on the (topological) S^2, with coordinate singularities at $\theta = 0$ and $\theta = \pi$; for regularity of the spacetime at these coordinate singularities, we need $m \to 1$ as $\theta \to 0$ and as $\theta \to \pi$. The spacetime is static, with the timelike hypersurface-orthogonal Killing vector ∂_t. The Killing vector of axial symmetry is ∂_ϕ, with the symmetry axis at $\theta = 0$ and $\theta = \pi$.

The ansatz for the gauge field is

$$A = \frac{1}{2er}\Big[\tau^n_\phi\big(H_1\mathrm{d}r + (1 - H_2)\,r\mathrm{d}\theta\big) - n\big(\tau^n_r H_3 + \tau^n_\theta\,(1 - H_4)\big)r\sin\theta\,\mathrm{d}\phi\Big], \tag{4.2}$$

where e is the coupling constant, the functions H_i depend only on r and θ, and

$$\tau^n_r := \sin\theta\cos n\phi\;\tau^x + \sin\theta\sin n\phi\;\tau^y + \cos\theta\,\tau^z, \tag{4.3a}$$

$$\tau^n_\theta := \cos\theta\cos n\phi\;\tau^x + \cos\theta\sin n\phi\;\tau^y - \sin\theta\,\tau^z, \tag{4.3b}$$

$$\tau^n_\phi := -\sin n\phi\;\tau^x + \cos n\phi\;\tau^y, \tag{4.3c}$$

where n is a positive integer and, to conform to the notation of [23, 24], τ^x, τ^y and τ^z denote respectively the Pauli matrices σ_1, σ_2 and σ_3 (2.4). This ansatz is purely magnetic, with no term proportional to dt. The ansatz is static, containing no dependence on t, and it is axially symmetric, in the sense that the rotation $\phi \mapsto \phi+\alpha$ can be undone by a gauge transformation with $\exp\big[-in(\alpha/2)\tau^z\big]$. With a 2π rotation in ϕ, the ansatz undergoes a $2\pi n$ rotation in $\mathfrak{su}(2)$: we hence refer to n as the *winding number*.

Finally, we require both the metric and the gauge field to be invariant, in an appropriate sense, under the north-south reflection $\theta \mapsto \pi - \theta$. For the metric the sense is that of isometry, implying that f, m and l are even under $\theta \mapsto \pi-\theta$. For the gauge field the sense is [24] that H_1 and H_3 and are odd and H_2 and H_4 are even under $\theta \mapsto \pi - \theta$.

We are interested in solutions to the Einstein-SU(2) field equations with a nondegenerate Killing horizon of the Killing vector ∂_t at $r = r_h > 0$. The boundary conditions at the horizon and at the symmetry axis and the integration of the field equations into the exterior region $r > r_h$ were discussed in [23, 24, 35, 36, 37], and numerical evidence was found that solutions exist, including solutions that have an asymptotically flat infinity at $r \to \infty$. The defining properties of the nondegenerate horizon are $f(r_h, \theta) = 0$ and $f'(r_h, \theta) > 0$, where the prime indicates derivative with respect to r. Working in the dimensionless variable $\delta := (r/r_h - 1)$, it follows that the near-horizon Taylor expansions of the metric functions and the gauge field functions begin

$$f(\delta, \theta) = f_2(\theta)\delta^2 \left[1 - \delta + \frac{1}{24}\delta^2 F(\theta) + O(\delta^3)\right], \tag{4.4a}$$

$$m(\delta, \theta) = m_2(\theta)\delta^2 \left[1 - 3\delta + \frac{1}{24}\delta^2 M(\theta) + O(\delta^3)\right], \tag{4.4b}$$

$$l(\delta, \theta) = l_2(\theta)\delta^2 \left[1 - 3\delta + \frac{1}{12}\delta^2 L(\theta) + O(\delta^3)\right], \tag{4.4c}$$

$$H_1(\delta, \theta) = \delta \left(1 - \frac{\delta}{2}\right) H_{11}(\theta) + O(\delta^3), \tag{4.5a}$$

$$H_2(\delta, \theta) = H_{20}(\theta) + \frac{1}{4}\delta^2 H_{21}(\theta) + O(\delta^3), \tag{4.5b}$$

$$H_3(\delta, \theta) = H_{30}(\theta) + \frac{1}{8}\delta^2 H_{31}(\theta) + O(\delta^3), \tag{4.5c}$$

$$H_4(\delta, \theta) = H_{40}(\theta) + \frac{1}{8}\delta^2 H_{41}(\theta) + O(\delta^3), \tag{4.5d}$$

where the O-terms may depend on θ and the field equations yield various relations among the coefficient functions [24]. One of these relations is

$$\frac{1}{m_2}\frac{\mathrm{d}m_2}{\mathrm{d}\theta} - \frac{2}{f_2}\frac{\mathrm{d}f_2}{\mathrm{d}\theta} = 0, \tag{4.6}$$

from which it follows that f_2^2/m_2 is independent of θ, implying that the horizon has constant surface gravity [24]. The gauge field can further be chosen regular everywhere, including $\theta = 0$ and $\theta = \pi$ [36, 37]. The SU(2) bundle is thus trivial and the field is expressed in a globally regular gauge.

In the special case $n = 1$ the field equations imply that $l = m$, $H_1 = H_3 = 0$, $H_2 = H_4$ and all the metric and gauge field functions are independent of θ. The metric and the gauge field are then spherically symmetric, and the solution reduces to that of [38, 39].

4.2 Kruskal-like extension

A complication with finding Kruskal-type coordinates that cover a neighbourhood of the full bifurcate Killing horizon is that the null geodesics with constant ϕ generically have nontrivial evolution in both r and θ. However, because of the discrete isometry $\theta \mapsto \pi - \theta$, the submanifold at $\theta = \pi/2$ is totally geodesic, and Kruskal-type coordinates that extend this submanifold across the horizon can be found as in the spherically symmetric case of Section 2.2. We shall show that the Kruskal-type coordinates adapted to the $\theta = \pi/2$ submanifold can be extended to other values of θ to give a C^0 extension across the horizon. This C^0 extension will suffice for taking the geon quotient in Section 5.

We start at $r > r_h$ and define the coordinates (U, V, θ, ϕ) by

$$U := -\exp\left[-\alpha\left(t - \int_{r_0}^{r} \frac{\sqrt{m(r,\pi/2)}}{f(r,\pi/2)}\mathrm{d}r\right)\right], \tag{4.7a}$$

$$V := \exp\left[\alpha\left(t + \int_{r_0}^{r} \frac{\sqrt{m(r,\pi/2)}}{f(r,\pi/2)}\mathrm{d}r\right)\right], \tag{4.7b}$$

where

$$\alpha := \frac{f_2(\pi/2)}{r_h\sqrt{m_2(\pi/2)}} \tag{4.8}$$

and r_0 is chosen so that

$$\int_{r_0}^{r} \frac{\sqrt{m(r,\pi/2)}}{f(r,\pi/2)}\mathrm{d}r = \frac{1}{\alpha}\left[\ln\delta - \frac{1}{2}\delta + O(\delta^2)\right] \tag{4.9}$$

as $r \to r_h$. The region $r > r_h$ is at $U < 0$ and $V > 0$, and the Killing horizon is at $UV \to 0_-$. The metric in the coordinates (U, V, θ, ϕ) reads

$$\begin{aligned} ds^2 =& f(r,\theta)\frac{1}{2\alpha^2}\frac{1}{UV}\left[\frac{m(r,\theta)}{f(r,\theta)^2}\frac{f(r,\pi/2)^2}{m(r,\pi/2)} + 1\right]\mathrm{d}U\mathrm{d}V \\ &+ f(r,\theta)\frac{1}{4\alpha^2}\frac{1}{(UV)^2}\left[\frac{m(r,\theta)}{f(r,\theta)^2}\frac{f(r,\pi/2)^2}{m(r,\pi/2)} - 1\right]\left(V^2\mathrm{d}U^2 + U^2\mathrm{d}V^2\right) \\ &+ \frac{m(r,\theta)}{f(r,\theta)}r^2\mathrm{d}\theta^2 + \frac{l(r,\theta)}{f(r,\theta)}r^2\sin^2\theta\,\mathrm{d}\phi^2, \end{aligned} \tag{4.10}$$

where r is a function of UV by

$$\begin{aligned} UV &= -\exp\left[2\alpha\int_{r_0}^{r} \frac{\sqrt{m(r,\pi/2)}}{f(r,\pi/2)}\mathrm{d}r\right] && (4.11\text{a}) \\ &= -\delta^2\left[1 - \delta + O(\delta^2)\right], \quad r \to r_h. && (4.11\text{b}) \end{aligned}$$

Inverting the near-horizon expansion (4.11b) and substituting in (4.10) yields

$$\begin{aligned} ds^2 = &-\frac{1}{\alpha^2}f_2(\theta)\left[1 + O(UV)\right]\mathrm{d}U\mathrm{d}V + \frac{1}{96\alpha^2}f_2(\theta)\times \\ &\times\left[M(\theta) - M(\pi/2) - 2F(\theta) + 2F(\pi/2) + O\big(\sqrt{-UV}\,\big)\right]\left(V^2\mathrm{d}U^2 + U^2\mathrm{d}V^2\right) \\ &+ \frac{1}{\alpha^2}f_2(\theta)\left[1 + O(UV)\right]\mathrm{d}\theta^2 + \frac{l_2(\theta)}{f_2(\theta)}\left[1 + O(UV)\right]\sin^2\theta\,\mathrm{d}\phi^2. \end{aligned} \tag{4.12}$$

Similarly, the near-horizon expansion of the gauge field (4.2) reads

$$\begin{aligned}A =&\frac{1}{2e}\Bigg\{\tau^n_\phi\bigg[-\frac{1}{2}\big(1+O(UV)\big)H_{11}(\theta)(V\mathrm{d}U+U\mathrm{d}V)\\&\qquad\qquad+\big(1-H_{20}(\theta)+O(UV)\big)\mathrm{d}\theta\bigg]\\&\qquad-n\Big[\tau^n_r\big(H_{30}(\theta)+O(UV)\big)+\tau^n_\theta\big(1-H_{40}+O(UV)\big)\Big]\sin\theta\,\mathrm{d}\phi\Bigg\}.\end{aligned} \tag{4.13}$$

The components of the metric (4.12) and the gauge field (4.13) are well defined at the horizon, $UV \to 0_-$, but the components of the metric are not guaranteed to be differentiable because of the $O\big(\sqrt{-UV}\,\big)$ error term. Our Kruskal coordinates therefore give a C^0 extension of the spacetime into a neighbourhood of the bifurcate Killing horizon, but they are not sufficiently regular for discussing the field equations across the horizon. Coordinates that allow a smooth extension are discussed in [40, 41], but at the expense of rendering the discrete isometry that we wish to utilise less transparent. In this paper we shall work with the above C^0 extension.

5 Geon quotient of the axially symmetric $\mathrm{SU}(2)$ black hole

In this section we show that the SU(2) black hole of Section 4 has a geon quotient.

Let $\mathcal{M}$ denote the spacetime manifold of the Kruskal-type (C^0) extension in the coordinates (U, V, θ, ϕ). Let A_{ext} denote the gauge field on $\mathcal{M}$, given in the right-hand-side exterior by (4.2) and having the near-horizon form (4.13). An involutive isometry J on $\mathcal{M}$ with the required properties is given by (3.2). What remains to be shown that there is a corresponding bundle map that leaves the gauge field invariant. As in Section 3, this reduces to examining whether A_{ext} is invariant under J up to a gauge transformation.

From the evenness of the gauge field functions H_2 and H_4 and the oddness of the gauge field functions H_1 and H_3 under $\theta \mapsto \pi-\theta$, and from the properties of the matrices (4.3) under J, it follows that the cases of odd and even n require separate treatment.

Let first n be odd. A_{ext} is then clearly invariant under J, and the bundle map can be chosen to be

$$\begin{aligned}K_{\text{odd}} : \mathcal{M}\times\mathrm{SU}(2) &\to \mathcal{M}\times\mathrm{SU}(2);\\(U,V,\theta,\phi,h) &\mapsto (V,U,\pi-\theta,\phi+\pi,h).\end{aligned} \tag{5.1}$$

K_{odd} is involutive, and the quotient bundle is the trivial SU(2) bundle over $\mathcal{M}_g := \mathcal{M}/\{\mathrm{Id}, J\}$. As the gauge field is invariant under a gauge transformation by $-\mathrm{Id} \in \mathrm{SU}(2)$, the geon bundle can be alternatively taken to be the trivial $\mathrm{SO}(3) \simeq \mathrm{SU}(2)/\{\pm\mathrm{Id}\}$ bundle over $\mathcal{M}_g$.

Let then n be even, and let $A^J_{\rm ext}$ denote the pull-back of $A_{\rm ext}$ by J. In the right-hand-side exterior covered by the coordinates (t,r,θ,ϕ), $A^J_{\rm ext}$ takes the form

$$\begin{aligned} A^J_{\rm ext} = \frac{1}{2er}\Big\{ &(\tau^x \sin n\phi - \tau^y \cos n\phi)\big[H_1 {\rm d}r + (1-H_2) r {\rm d}\theta\big] \\ &- n\big[(-\tau^x \sin\theta\cos n\phi - \tau^y \sin\theta \sin n\phi + \tau^z \cos\theta) H_3 \\ &\quad + (-\tau^x \cos\theta\cos n\phi - \tau^y \cos\theta\sin n\phi - \tau^z \sin\theta)(1-H_4)\big] r\sin\theta\, {\rm d}\phi\Big\}. \end{aligned} \tag{5.2}$$

Comparison with (4.2) shows that $A^J_{\rm ext}$ and $A_{\rm ext}$ do not coincide. They are however taken to each other by $(\tau^x,\tau^y,\tau^z) \mapsto (-\tau^x,-\tau^y,\tau^z)$, which is a gauge transformation: defining

$$g_0 := \exp\left(i\frac{\pi}{2}\tau^z\right) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \in {\rm SU}(2), \tag{5.3}$$

we have

$$A_{\rm ext} = g_0 A^J_{\rm ext} g_0^{-1}, \tag{5.4}$$

and (5.4) is a gauge transformation because the inhomogeneous term involving ${\rm d}g_0$ vanishes. The bundle map can thus be chosen to be

$$\begin{aligned} &K_{\rm ev} : \mathcal{M} \times {\rm SU}(2) \to \mathcal{M} \times {\rm SU}(2); \\ &\quad (U,V,\theta,\phi,h) \mapsto \big(V,U,\pi-\theta,\phi+\pi, h\cdot g_0^{-1}\big)\,, \end{aligned} \tag{5.5}$$

where the dot denotes matrix multiplication in SU(2). $K_{\rm ev}$ generates the cyclic group of order four, $\bar\Gamma := \{{\rm Id}, K_{\rm ev}, K^2_{\rm ev}, K^3_{\rm ev}\}$, and the geon bundle is the quotient $\big(\mathcal{M}\times{\rm SU}(2)\big)/\bar\Gamma$. As the normal subgroup $\{{\rm Id}, K^2_{\rm ev}\} \subset \bar\Gamma$ identifies points in $\mathcal{M}\times{\rm SU}(2)$ by the position-independent gauge transformation by $g_0^2 = -\,{\rm Id} \in {\rm SU}(2)$, and as this gauge transformation leaves the gauge field invariant, the geon bundle can be equivalently presented as a $\mathbb{Z}_2$ quotient of the trivial ${\rm SO}(3) \simeq {\rm SU}(2)/\{\pm\,{\rm Id}\}$ bundle over $\mathcal{M}$. Explicitly, we may realise the projection ${\rm SU}(2) \to {\rm SO}(3)$, $g \mapsto \hat g$, in the defining matrix representations so that $g\tau^i g^{-1} = \sum_j \hat g^i{}_j \tau^j$. Note that $\hat g_0 = {\rm diag}(-1,-1,1)$. The involutive bundle map then reads

$$\begin{aligned} &\hat K_{\rm ev} : \mathcal{M} \times {\rm SO}(3) \to \mathcal{M} \times {\rm SO}(3); \\ &\quad (U,V,\theta,\phi,\hat h) \mapsto \Big(V,U,\pi-\theta,\phi+\pi, \hat h\cdot \hat g_0^{-1}\Big)\,, \end{aligned} \tag{5.6}$$

where the dot denotes matrix multiplication in SO(3).

The geon bundle for even n is not trivial. To see this, we view the geon bundle as the quotient $\big(\mathcal{M}\times{\rm SO}(3)\big)/\big\{{\rm Id}, \hat K_{\rm ev}\big\}$. Suppose this bundle is trivial. Proceeding as in the discussion of Section 3.3 leading to (3.21), we see that there then exist a continuous 2π-periodic function $\gamma : \mathbb{R} \to {\rm SO}(3)$ such that

$$\gamma(\phi+\pi) = \gamma(\phi)\cdot \hat g_0^{-1} \tag{5.7}$$

and the closed path $\gamma_0 : [0, 2\pi] \to \mathrm{SO}(3);\ \phi \mapsto \gamma(\phi)$ is contractible. We may assume without loss of generality that $\gamma(0) = \mathrm{Id} \in \mathrm{SO}(3)$. The condition (5.7) then implies that γ_0 is homotopic to the path $\gamma_1 : [0, 2\pi] \to \mathrm{SO}(3);\ \phi \mapsto \gamma_1(\phi)$, where

$$\gamma_1(\phi) := \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.8}$$

But as the lift of γ_1 to SU(2) is not closed, γ_1 is not contractible, and hence neither is γ_0. This is a contradiction and implies that the assumed triviality of the geon bundle cannot hold.

6 Conclusions

We have shown that the static, spherically symmetric SU(n) black hole solutions of Künzle [26, 20] and the static, axially symmetric SU(2) black hole solutions of Kleihaus and Kunz [23, 24] admit topological geon quotients. These constructions extend the family of known non-abelian Einstein-Yang-Mills geon-type black holes from the static, spherically symmetric SU(2) geon-type black hole [12] to include geons with a more general Yang-Mills gauge group and to geons with less symmetry.

For Künzle's static, spherically symmetric SU(n) black holes with $n > 2$, we showed that a geon quotient generically requires an extension of the gauge group from SU(n) to $\mathbb{Z}_2 \ltimes \mathrm{SU}(n)$, where the nontrivial element of $\mathbb{Z}_2$ acts on SU(n) by complex conjugation. This means that the SU(n) charge conjugation must be treated as a gauge symmetry, rather than just as a global symmetry. This gauging is very similar to the U(1) charge conjugation gauging that is necessary for taking a geon quotient of the Reissner-Nordström black hole [12]. By contrast, static, spherically symmetric SU(2) black holes were known to admit a geon quotient without the need to gauge the SU(2) charge conjugation [12], and we showed that the same holds for the static, axially symmetric SU(2) black holes of Kleihaus and Kunz [23, 24].

In the cases where gauging the charge conjugation is not required, we showed that the geons built from Künzle's black holes have a trivial gauge bundle, whereas those built from the black holes of Kleihaus and Kunz have a trivial (respectively nontrivial) gauge bundle for odd (even) winding number of the gauge field configuration. We have not investigated whether this phenomenon reflects some deeper geometric property.

Our results on the axially symmetric solutions have a technical limitation in that the extension across the Killing horizon was C^0 but was not guaranteed to be differentiable. We suspect that this limitation is an artefact of a non-optimal coordinate choice and the results continue to hold within extensions of higher differentiability. It should be possible to examine this question with the techniques of Rácz and Wald [40, 41].

The topological geon black holes that we have found should provide an interesting arena for investigating the Hawking-Uhruh effect for quantum fields coupled to the background Yang-Mills field. How does the geon's charge show up in the Hawking-Unruh effect, compared with the Hawking-Unruh effect on the conventional Kruskal-type extension? In particular, does the Hawking-Unruh effect feel the gauging of SU(n) charge conjugation, as it does feel the gauging of U(1) charge conjugation [18, 19]? When the charge conjugation is not gauged, does the Hawking-Unruh effect feel the triviality versus nontriviality of the geon's gauge bundle? A technically simple test field with which to address these questions might be a multiplet of charged scalars minimally coupled to the Yang-Mills field. A more interesting case might be a neutrino multiplet, for which the additional issue of inequivalent spin structures arises [15].

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