Consistency of Equations for the Single Scalar Field Case in Second-order Gauge-invariant Cosmological Perturbation Theory

Kouji Nakamura[1]

Optical and Infrared Astronomy Division, National Astronomical Observatory of Japan., Osawa, Mitaka, Tokyo, 181-8588, Japan.

Abstract

We derived the second-order perturbations of the Einstein equations and the Klein-Gordon equation for a generic situation in terms of gauge-invariant variables. The consistency of all the equations is confirmed. This confirmation implies that all the derived equations of the second order are self-consistent and these equations are correct in this sense. We also discuss the physical implication of these equations.

1 Introduction

The general relativistic second-order cosmological perturbation theory is one of topical subjects in the recent cosmology. Recently, the first-order approximation of our universe from a homogeneous isotropic one was revealed through the observation of the CMB by the Wilkinson Microwave Anisotropy Probe (WMAP)[1], the cosmological parameters are accurately measured, we have obtained the standard cosmological model, and the so-called "precision cosmology" has begun. These developments in observations were also supported by the theoretical sophistication of the linear order cosmological perturbation theory. To explore more detail observations, the Planck satellite was launched on the last May and its first light was reported[2]. With the increase of precision of the CMB data, the study of relativistic cosmological perturbations beyond linear order is a topical subject. The second-order cosmological perturbation theory is one of such perturbation theories beyond linear order.

In this article, we show a part of our formulation of the second-order gauge-invariant perturbation theory[3]. We give the consistency relations of the source terms in all the second-order perturbation of the Einstein equations and the Klein-Gordon equation in the single scalar field case as in the case of the perfect fluid case[4]. These consistency relations imply the all derived equations of the second order are self-consistent and these equations are correct in this sense. Further, we also discuss the physical implication of our second-order Einstein equations.

2 Metric and matter perturbations

The background spacetime for the cosmological perturbations is a homogeneous isotropic background spacetime. The background metric is given by

$$g_{ab} = a^2 \left\{ -(d\eta)_a(d\eta)_b + \gamma_{ij}(dx^i)_a(dx^j)_b \right\}, \tag{1}$$

where $\gamma_{ab} := \gamma_{ij}(dx^i)_a(dx^j)_b$ is the metric on the maximally symmetric three-space and the indices $i, j, k, ...$ for the spatial components run from 1 to 3. On this background spacetime, we consider the perturbative expansion of the metric as $\bar{g}_{ab} = g_{ab} + \lambda_{\mathcal{X}} h_{ab} + \frac{\lambda^2}{2} {}_{\mathcal{X}} l_{ab} + O(\lambda^3)$, where λ is the infinitesimal parameter for perturbation and h_{ab} and l_{ab} are the first- and the second-order metric perturbations, respectively. As shown in Refs. [3], the metric perturbations h_{ab} and l_{ab} are decomposed as

$$h_{ab} =: \mathcal{H}_{ab} + \pounds_X g_{ab}, \quad l_{ab} =: \mathcal{L}_{ab} + 2\pounds_X h_{ab} + \left(\pounds_Y - \pounds_X^2\right) g_{ab}, \tag{2}$$

[1]Email address: kouji.nakamura@nao.ac.jp

where $\mathcal{H}_{ab}$ and $\mathcal{L}_{ab}$ are the gauge-invariant parts of h_{ab} and l_{ab}, respectively. The components of $\mathcal{H}_{ab}$ and $\mathcal{L}_{ab}$ can be chosen so that

$$\mathcal{H}_{ab} = a^2\left\{-2\stackrel{(1)}{\Phi}(d\eta)_a(d\eta)_b + 2\stackrel{(1)}{\nu}_i(d\eta)_{(a}(dx^i)_{b)} + \left(-2\stackrel{(1)}{\Psi}\gamma_{ij} + \stackrel{(1)}{\chi}_{ij}\right)(dx^i)_a(dx^j)_b\right\}, \quad (3)$$

$$\mathcal{L}_{ab} = a^2\left\{-2\stackrel{(2)}{\Phi}(d\eta)_a(d\eta)_b + 2\stackrel{(2)}{\nu}_i(d\eta)_{(a}(dx^i)_{b)} + \left(-2\stackrel{(2)}{\Psi}\gamma_{ij} + \stackrel{(2)}{\chi}_{ij}\right)(dx^i)_a(dx^j)_b\right\}. \quad (4)$$

In Eqs. (3) and (4), the vector-mode $\stackrel{(p)}{\nu}_i$ and the tensor-mode $\stackrel{(p)}{\chi}_{ij}$ $(p = 1, 2)$ satisfy the properties

$$D^i\stackrel{(p)}{\nu}_i = \gamma^{ij}D_p\stackrel{(p)}{\nu}_j = 0, \quad \stackrel{(p)}{\chi}{}^i{}_i = 0, \quad D^i\stackrel{(p)}{\chi}_{ij} = 0, \quad (5)$$

where γ^{kj} is the inverse of the metric γ_{ij}.

On the other hand, we also expand the scalar field as $\bar{\varphi} = \varphi + \lambda\hat{\varphi}_1 + \frac{\lambda^2}{2}\hat{\varphi}_2 + O(\lambda^3)$ and decompose $\hat{\varphi}_1$ and $\hat{\varphi}_2$ into gauge-invariant and gauge-variant parts as

$$\hat{\varphi}_1 =: \varphi_1 + \pounds_X\varphi, \quad \hat{\varphi}_2 =: \varphi_2 + 2\pounds_X\varphi_1 + \left(\pounds_Y - \pounds_X^2\right)\varphi, \quad (6)$$

respectively, where X^a and Y^a are the gauge-variant parts of the first- and the second-order metric perturbations, respectively, in Eqs. (2).

3 Equations for Perturbations

Here, we summarize the Einstein equations and the Klein-Gordon equations for the background, the first order, and the second order on the above background spacetime (1).

The background Einstein equations for a single scalar field system are given by

$$\mathcal{H}^2 + K = \frac{8\pi G}{3}a^2\left(\frac{1}{2a^2}(\partial_\eta\varphi)^2 + V(\varphi)\right), \quad 2\partial_\eta\mathcal{H} + \mathcal{H}^2 + K = 8\pi G\left(-\frac{1}{2}(\partial_\eta\varphi)^2 + a^2V(\varphi)\right), \quad (7)$$

where $\mathcal{H} := \partial_\eta a/a$, K is the curvature constant of the maximally symmetric three-space.

On the other hand, the second-order perturbations of the Einstein equation are summarized as

$$2\partial_\eta\stackrel{(2)}{\Psi} + 2\mathcal{H}\stackrel{(2)}{\Phi} - 8\pi G\varphi_2\partial_\eta\varphi = \Delta^{-1}D^k\Gamma_k, \quad \stackrel{(2)}{\Psi} - \stackrel{(2)}{\Phi} = \frac{3}{2}(\Delta + 3K)^{-1}\left\{\Delta^{-1}D^iD_j\Gamma_i{}^j - \frac{1}{3}\Gamma_k{}^k\right\}, \quad (8)$$

$$\begin{aligned}
&\left\{\partial_\eta^2 + 2\left(\mathcal{H} - \frac{\partial_\eta^2\varphi}{\partial_\eta\varphi}\right)\partial_\eta - \Delta - 4K + 2\left(\partial_\eta\mathcal{H} - \frac{\partial_\eta^2\varphi}{\partial_\eta\varphi}\mathcal{H}\right)\right\}\stackrel{(2)}{\Phi} \\
&\quad = -\Gamma_0 - \frac{1}{2}\Gamma_k{}^k + \Delta^{-1}D^iD_j\Gamma_i{}^j + \left(\partial_\eta - \frac{\partial_\eta^2\varphi}{\partial_\eta\varphi}\right)\Delta^{-1}D^k\Gamma_k \\
&\qquad - \frac{3}{2}\left\{\partial_\eta^2 - \left(\frac{2\partial_\eta^2\varphi}{\partial_\eta\varphi} - \mathcal{H}\right)\partial_\eta\right\}(\Delta + 3K)^{-1}\left\{\Delta^{-1}D^iD_j\Gamma_i{}^j - \frac{1}{3}\Gamma_k{}^k\right\}, \quad (9)
\end{aligned}$$

$$\stackrel{(2)}{\nu_i} = \frac{2}{\Delta + 2K}\left\{D_i\Delta^{-1}D^k\Gamma_k - \Gamma_i\right\}, \quad \partial_\eta\left(a^2\stackrel{(2)}{\nu_i}\right) = \frac{2a^2}{\Delta + 2K}\left\{D_i\Delta^{-1}D^kD_l\Gamma_k{}^l - D_k\Gamma_i{}^k\right\}, \quad (10)$$

$$\begin{aligned}
&\left(\partial_\eta^2 + 2\mathcal{H}\partial_\eta + 2K - \Delta\right)\stackrel{(2)}{\chi_{ij}} \\
&\quad = 2\Gamma_{ij} - \frac{2}{3}\gamma_{ij}\Gamma_k{}^k - 3\left(D_iD_j - \frac{1}{3}\gamma_{ij}\Delta\right)(\Delta + 3K)^{-1}\left(\Delta^{-1}D^kD_l\Gamma_k{}^l - \frac{1}{3}\Gamma_k{}^k\right) \\
&\qquad + 4\left\{D_{(i}(\Delta + 2K)^{-1}D_{j)}\Delta^{-1}D^lD_k\Gamma_l{}^k - D_{(i}(\Delta + 2K)^{-1}D^k\Gamma_{j)k}\right\} = 0, \quad (11)
\end{aligned}$$

where $\Gamma_i{}^j := \gamma^{kj}\Gamma_{ik}$ and $\Gamma_k{}^k = \gamma^{ij}\Gamma_{ij}$. The source terms Γ_0, Γ_i, and Γ_{ij} are the collections of the quadratic terms of the linear-order perturbations in the second-order Einstein equations. Further, the second-order perturbation of the Klein-Gordon equation

$$\partial_\eta^2 \varphi_2 + 2\mathcal{H}\partial_\eta\varphi_2 - \Delta\varphi_2 - \left(\partial_\eta \overset{(2)}{\Phi} + 3\partial_\eta \overset{(2)}{\Psi}\right)\partial_\eta\varphi + 2a^2 \overset{(2)}{\Phi} \frac{\partial V}{\partial\bar{\varphi}}(\varphi) + a^2\varphi_2 \frac{\partial^2 V}{\partial\bar{\varphi}^2}(\varphi) = \Xi_{(K)}, \quad (12)$$

where the source term $\Xi_{(K)}$ is also the collections of the quadratic terms of the linear-order perturbations in the second-order Klein-Gordon equation. The explicit form of these Γ_0, Γ_i, Γ_{ij}, and $\Xi_{(K)}$ are given in Refs. [3]. The first-order perturbations of the Einstein equations are given by the replacements $\overset{(2)}{\Phi} \to \overset{(1)}{\Phi}$, $\overset{(2)}{\Psi} \to \overset{(1)}{\Psi}$, $\overset{(2)}{\nu_i} \to \overset{(1)}{\nu_i}$, $\overset{(2)}{\chi_{ij}} \to \overset{(1)}{\chi_{ij}}$, $\varphi_2 \to \varphi_1$, and $\Gamma_0 = \Gamma_i = \Gamma_{ij} = \Xi_{(K)} = 0$.

4 Consistency of equations for second-order perturbations

Now, we consider the consistency of the second-order perturbations of the Einstein equations (8) and (9) for the scalar modes, Eqs. (10) for vector mode, and the Klein-Gordon equation (12).

Since the first equation in Eqs. (10) is the initial value constraint for the vector mode $\overset{(2)}{\nu_i}$ and it should be consistent with the evolution equation, i.e., the second equation of Eqs. (10). Explicitly, these equations are consistent with each other if the equation

$$\partial_\eta \Gamma_k + 2\mathcal{H}\Gamma_k - D^l \Gamma_{lk} = 0 \quad (13)$$

is satisfied. Actually, through the first-order perturbative Einstein equations, we can directly confirm the equation (13) through the background Einstein equations, the first-order Einstein equations, and the long expressions of Γ_i and Γ_{ij} given in Refs.[3].

Next, we consider the consistency of the second-order perturbation of the Klein-Gordon equation (12) and the Einstein equations (8) and (9). From these equation, we can show that the second-order perturbation of the Klein-Gordon equation is consistent with the background and the second-order Einstein equations if the equation

$$2\left(\partial_\eta + \mathcal{H}\right)\Gamma_0 - D^k\Gamma_k + \mathcal{H}\Gamma_k{}^k + 8\pi G \partial_\eta\varphi \Xi_{(K)} = 0 \quad (14)$$

is satisfied under the background and the first-order Einstein equations. Further, we can directly confirm Eq. (14) through the background Einstein equations, the first-order perturbation of the Einstein equations, and the long expression of Γ_0, Γ_i, Γ_{ij}, and $\Xi_{(K)}$ which are given in Refs. [3].

Equation (13) comes from the consistency of the initial value constraint and evolution equation and Eq. (14) comes from the consistency between the Klein-Gordon equation and the Einstein equation. These equation should be trivially satisfied from a general viewpoint, because the Einstein equation is the first class constrained system. However, these trivial results imply that we have derived the source terms Γ_0, Γ_i, Γ_{ij}, and $\Xi_{(K)}$ are consistent with each other and are correct in this sense. We also note that these relations are independent of the details of the potential of the scalar field.

5 Summary and discussions

In this article, we summarized the second-order Einstein equation for a single scalar field system. We derived all the components of the second-order perturbation of the Einstein equation without ignoring any types modes (scalar-, vector-, tensor-types) of perturbations. As in the case of the perfect fluid[4], we derived the consistency relation between the source terms of the second-order Einstein equation and the Klein-Gordon equation.

In our formulation, any gauge fixing is not necessary and we can obtain all equations in the gauge-invariant form, which are equivalent to the complete gauge fixing. In other words, our formulation gives complete gauge-fixed equations without any gauge fixing. Therefore, equations obtained in a gauge-invariant manner cannot be reduced without physical restrictions any more. In this sense, the equations shown here are irreducible. This is one of the advantages of the gauge-invariant perturbation theory.

The resulting Einstein equations of the second order show that the mode-couplings between different types of modes appears as the quadratic terms of the linear-order perturbations owing to the nonlinear effect of the Einstein equations, in principle. Perturbations in cosmological situations are classified into three types: scalar, vector, and tensor. In the second-order perturbations, we also have these three types of perturbations as in the case of the first-order perturbations. In the scalar field system shown in this article, the first-order vector mode does not appear due to the momentum constraint of the first-order perturbation of the Einstein equation. Therefore, we have seen that three types of mode-coupling appear in the second-order Einstein equations, i.e., scalar-scalar, scalar-tensor, and tensor-tensor type of mode coupling. Since the tensor mode of the linear order is also generated due to quantum fluctuations during the inflationary phase, the mode-couplings of the scalar-tensor and tensor-tensor types may appear in the inflation. If these mode-couplings occur during the inflationary phase, these effects will depend on the scalar-tensor ratio r. If so, there is a possibility that the accurate observations of the second-order effects in the fluctuations of the scalar type in our universe also restrict the scalar-tensor ratio r or give some consistency relations between the other observations such as the measurements of the B-mode of the polarization of CMB. This will be a new effect that gives some information on the scalar-tensor ratio r.

As the current status of the second-order gauge-invariant cosmological perturbation theory, we may say that the curvature terms in the second-order Einstein tensor, i.e., the second-order perturbations of the Einstein tensor, are almost completely derived, although there remain some problems should be clarified[3]. The next task is to clarify the nature of the second-order perturbation of the energy-momentum tensor through the extension to multi-fluid or multi-field systems. Further, we also have to extend our arguments to the Einstein Boltzmann system to discuss CMB physics, since we have to treat photon and neutrinos through the Boltzmann distribution functions. This issue is also discussed in some literature[5]. If we accomplish these extension, we will be able to clarify the Non-linear effects in CMB physics.

References

[1] C. L. Bennett et al., Astrophys. J. Suppl. Ser. **148** (2003), 1. E. Komatsu et al., Astrophys. J. Suppl. Ser. **180** (2009), 330.

[2] F. R. Bouchet, arXiv:0911.3101 [astro-ph.CO].

[3] K. Nakamura, Prog. Theor. Phys. **110** (2003), 723; K. Nakamura, Prog. Theor. Phys. **113** (2005), 481; K. Nakamura, Phys. Rev. D **74** (2006), 101301(R); K. Nakamura, Prog. Theor. Phys. **117** (2005), 17; K. Nakamura, Phys. Rev. D **80** (2009), 124021; K. Nakamura, Prog. Theor. Phys. **121** (2009), 1321; K. Nakamura, arXiv:1001.2621 [gr-qc].

[4] K. Nakamura, arXiv:0901.3638 [gr-qc].

[5] N. Bartolo, S. Matarrese and A. Riotto, JCAP **0401** (2004), 003; N. Bartolo, S. Matarrese and A. Riotto, Phys. Rev. Lett. **93** (2004), 231301; N. Bartolo, E. Komatsu, S. Matarrese and A. Riotto, Phys. Rep. **402** (2004), 103; N. Bartolo, S. Matarrese and A. Riotto, JCAP **0605** (2006), 010; N. Bartolo, S. Matarrese and A. Riotto, JCAP **0606** (2006), 024; N. Bartolo, S. Matarrese and A. Riotto, JCAP **0701** (2007), 019; D. Nitta, E. Komatsu, N. Bartolo, S. Matarrese and A. Riotto, JCAP **0905** (2009), 014; C. Pitrou, J. P. Uzan and F. Bernardeau, Phys. Rev. D **78** (2008), 063526; L. Senatore, S. Tassev and M. Zaldarriaga, JCAP **0908** (2009), 031; C. Pitrou, Class. Quantum Grav. **24** (2007), 6127; *ibid.* **26** (2009), 065006.